

CURVATURE INTEGRAL ESTIMATES FOR COMPLETE HYPERSURFACES

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday.

ABSTRACT. We consider the integrals of the r -mean curvatures S_r of a complete hypersurface M in the space form \mathbb{Q}_c^{n+1} . Among other results, we prove that $\int_M S_r dM = \infty$ for a complete properly immersed hypersurfaces in a space form with $S_r \geq 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \leq n - 1$.

1. Introduction

Let M^n be a complete orientable hypersurface immersed in the space form \mathbb{Q}_c^{n+1} of constant sectional curvature c . We denote by A and $\lambda_1, \dots, \lambda_n$ the second fundamental operator and the eigenvalues of A , respectively. It is well known that the r -mean curvature at a point p is defined by

$$H_r(p) = \frac{1}{\binom{n}{r}} \sum_{i_1 < \dots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r(p),$$

where S_r is the r -symmetric function of $\lambda_1, \dots, \lambda_n$, for $1 \leq r \leq n$, and H_0 is defined to be zero and $H_r = 0$, for all $r \geq n + 1$. In particular, for $r = 1$, $H_1 = H$ is the mean curvature.

We define the r -area of a domain $D \subset M$ by

$$\mathcal{A}_r(D) = \int_D S_r(p) dM.$$

Then, when $r = 0$, \mathcal{A}_0 is the volume of D .

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In this paper, we are interested in r -areas estimates. When $r = 0$, it is well known that a complete properly immersed minimal hypersurface in \mathbb{R}^{n+1} has, at least, polynomial volume growth. In fact, infinity volume results hold for more general ambient spaces. Precisely, we have the following result of K. Frensel [9].

THEOREM ([9], Theorem 1). *Let M^m be a complete, noncompact manifold and let $x : M^m \rightarrow N^n$ be an isometric immersion with mean curvature vector field bounded in norm. If N^n has sectional curvature bounded from above and injective radius bounded from below by a positive constant, then the volume of M^m is infinite.*

It is also true that each end of M has infinite volume under the same conditions (see [4]). These estimates have been used in studying the topology and geometric properties of minimal hypersurfaces and hypersurfaces with constant mean curvature (see for example [4], [9], [7]). It is natural to ask the following.

QUESTION. *Let M^n be a complete noncompact manifold and let $x : M^n \rightarrow N^{n+1}$ be an isometric immersion such that there is a positive constant C satisfying*

$$|S_{r+1}| \leq CS_r$$

for some $r = 0, 1, \dots, n - 1$. Is the r -area of M^n infinite?

When $r = n$, $S_{r+1} = 0$, one can find a negative answer to this question. For example, if M is a complete noncompact surface in \mathbb{R}^3 with positive Gaussian curvature, then the total curvature is finite by a theorem of Cohn-Vossen. When $r < n$ we obtain a r -area estimate and give positive answers to this question in some interesting cases.

In order to state our results we need the r th Newton transformation, $P_r : T_p M \rightarrow T_p M$, which is defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A \circ P_{r-1}, \quad r > 1. \end{aligned}$$

THEOREM A (Theorem 2.8). *Let \mathcal{Q}_c^{n+1} be a Riemannian manifold with constant sectional curvature c and let M^n be a complete noncompact properly immersed hypersurface of \mathcal{Q}_c^{n+1} . Assume that there exists a nonnegative constant α such that*

$$(r + 1)|S_{r+1}| \leq (n - r)\alpha S_r$$

for some $r \leq n - 1$. If P_r is positive semidefinite, then for any $q \in M$ such that $S_r(q) \neq 0$ and any $\mu_0 > 0$ there exists a positive constant C depending on μ_0 , q and M such that for every $\mu > \mu_0$,

$$A_r(\overline{B}_\mu(q) \cap M) = \int_{\overline{B}_\mu(q) \cap M} S_r dM \geq \int_{\mu_0}^\mu C e^{-\alpha\tau} d\tau,$$

where $\overline{B}_\mu(q)$ is the ball of radius μ and center q in \mathcal{Q}_c^{n+1} . For the case $c > 0$, we assume $\mu \leq \frac{\pi}{2\sqrt{c}}$.

As a consequence of this result we obtain the following.

THEOREM B (Corollary 2.9). *Let \mathcal{Q}_c^{n+1} be a Riemannian manifold with constant sectional curvature $c \leq 0$ and let M^n be a complete noncompact properly immersed hypersurface of \mathcal{Q}_c^{n+1} . Assume that $S_r \geq 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \leq n - 1$. Then $\int_M S_r dM = \infty$.*

REMARK 1.1. The cases when r is even and r is odd are different. If r is odd and $S_r \leq 0$, we can change the orientation so that $S_r \geq 0$. But when r is even, S_r is independent of the choice of orientation. It has been proved by Gromov and Lawson that the existence of a complete metric with nonpositive scalar curvature ($r = 2$) implies some topological obstructions, which is called enlargeable (see Corollary A in [11]). Enlargeable manifolds cannot carry metrics of positive scalar curvature.

Topping [18] used Sobolev inequality to get a diameter estimate in terms of the mean curvature integral. In Section 4, using his estimate we get a global estimate of the mean curvature integral.

THEOREM C (Theorem 4.1). *Let M^m be an m -dimensional complete noncompact Riemannian manifold isometrically immersed in \mathbb{R}^n . Then there exists a positive constant δ depending on m such that if*

$$\limsup_{r \rightarrow +\infty} \frac{V(x, r)}{r^m} < \delta,$$

where $V(x, r)$ denotes the volume of the geodesic ball $B_r(x)$, then

$$\limsup_{R \rightarrow +\infty} \frac{\int_{B_R(x)} |H|^{m-1} dM}{R} > 0.$$

In particular, $\int_M |H|^{m-1} dM = +\infty$.

For a complete noncompact surface M with finite total curvature, Cohn-Vossen theorem says that (see Theorem 6 in [6])

$$\int_M K dM \leq 2\pi\chi(M).$$

A special case of Corollary 4.3 says that if $\int_M K dM = 2\pi\chi(M)$, then $\int_M |H| dM = +\infty$.

The rest of the paper is organized as follows. In Section 2, we obtain the formulas relating the distance function and the r -mean curvature. The estimate obtained in Section 2 can be improved when $r = 0$ and this is proved in Section 3. In Section 4, we give the proof of Theorem C.

2. r -area estimate

Let $x : M^n \rightarrow N^{n+1}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N .

In [15], Reilly showed that P_r satisfies the following

PROPOSITION 2.1 ([15], p. 224, see also [2], Lemma 2.1). *Let $x : M^n \rightarrow N^{n+1}$ be an isometric immersion between two Riemannian manifolds and let A be the second fundamental form of x . The r th Newton transformation P_r associated to A satisfies:*

$$(2.1) \quad \text{trace}(P_r) = (n - r)S_r,$$

$$(2.2) \quad \text{trace}(A \circ P_r) = (r + 1)S_{r+1}.$$

For hypersurfaces with bounded mean curvature, the Laplacian of the intrinsic distance to a fixed point of M played an important role in the proof of Frensel's estimate of the volume of M . In the case of H_r bounded, with $r > 1$, we used another second order differential operator defined on M , which seems to be natural for this problem. Associated to each Newton transformation P_r , if $f : M \rightarrow \mathbb{R}$ is a smooth function, we define

$$L_r(f) = \text{trace}(P_r \circ \text{Hess } f).$$

These operators are, in a certain sense, generalizations of the Laplace operator since $L_0(f) = \text{trace}(\text{Hess } f) = \Delta f$. They were introduced by Voss [19] in connection with variational arguments. In general, these operators are not elliptic and some conditions are necessary to ensure the ellipticity. For completeness, we include here some useful facts.

PROPOSITION 2.2 ([8], Lemma 3.10). *Let N^{n+1} be an $(n + 1)$ -dimensional oriented Riemannian manifold and let M^n be a connected n -dimensional orientable Riemannian manifold. Suppose $x : M \rightarrow N$ is an isometric immersion. If $H_2 > 0$, then the operator L_1 is elliptic.*

PROPOSITION 2.3 ([5], Proposition 3.2). *Let N^{n+1} be an $(n + 1)$ -dimensional oriented Riemannian manifold and let M^n be a connected n -dimensional orientable Riemannian manifold (with or without boundary). Suppose $x : M \rightarrow N$ is an isometric immersion with $H_r > 0$ for some $1 \leq r \leq n$. If there exists an interior point p of M such that all the principal curvatures at p are non-negative, then for all $1 \leq j \leq r - 1$, the operator L_j is elliptic, and the j -mean curvature H_j is positive.*

We need the following proposition which is essentially the content of Lemma 1.1 and equation (1.3) of [12]. We include here with a direct proof.

PROPOSITION 2.4. *Let $M^n \rightarrow N^{n+1}$ be an isometric immersion. Suppose that $S_{r+1}(p) = 0$, for some r , $0 \leq r < n$. Then P_r is semidefinite at p .*

Proof. Consider $S_r = S_r(\lambda_1, \dots, \lambda_n)$. Then $\frac{\partial S_r}{\partial \lambda_i}$ are the eigenvalues of P_r . Let $(\lambda_1^0, \dots, \lambda_n^0)$ be the principal curvatures of M at p . Hence

$$S_{r+1}(\lambda_1^0, \dots, \lambda_n^0) = 0.$$

We choose $\epsilon = \min_{\lambda_i^0 \neq 0} \{1, |\lambda_i^0|\}$. Then, for all $(\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_i \in (0, \epsilon)$, $S_{r+1}(\lambda_1^0 + \epsilon_1, \dots, \lambda_n^0 + \epsilon_n)$ does not change sign. This implies that $\frac{\partial S_r}{\partial \lambda_i} \geq 0$ for all $i = 1, \dots, n$ or $\frac{\partial S_r}{\partial \lambda_i} \leq 0$ for all $i = 1, \dots, n$. Thus P_r is semidefinite at p . \square

Let M^n and N^{n+1} be Riemannian manifolds and let $x : M^n \rightarrow N^{n+1}$ be an isometric immersion. Henceforth, we shall tacitly make the usual identification of $X \in T_p M$ with $dx_p(X)$. In particular, if $F : N \rightarrow \mathbb{R}$ is smooth and we consider the composition $f = F \circ x$, then we have at $p \in M$, for every $X \in T_p M$:

$$\langle \text{grad}_M f, X \rangle = df(X) = dF(X) = \langle \text{grad}_N F, X \rangle,$$

where grad_M and grad_N denote the gradient on M and the gradient on N , respectively. So that

$$(2.3) \quad \text{grad}_N F = \text{grad}_M f + (\text{grad}_N F)^\perp,$$

where $(\text{grad}_N F)^\perp$ is perpendicular to $T_p M$. Let $F : N \rightarrow \mathbb{R}$ be a C^2 function and denote $f : M \rightarrow \mathbb{R}$ the function induced by F by restriction, that is $f = F \circ x$. We have the following.

LEMMA 2.5. *Let $x : M^n \rightarrow N^{n+1}$ be an isometric immersion. Let $F : N \rightarrow \mathbb{R}$ a smooth function and consider $f = F \circ x : M \rightarrow \mathbb{R}$. For an orthonormal frame $\{e_i\}$ on M , we have*

$$(2.4) \quad L_r f = \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) + (r + 1)S_{r+1} \langle \text{grad}_N F, \eta \rangle,$$

where η denotes the normal vector field of the immersion and grad_N is the gradient of N .

Proof. Let ∇ and $\bar{\nabla}$ be the connections of M and N , respectively. If α denotes the second fundamental form of the immersion, Gauss' equation and equations (2.2) and (2.3) imply that

$$\begin{aligned} L_r f &= \sum_{i=1}^n \langle \nabla_{e_i}(\text{grad}_M f), P_r(e_i) \rangle \\ &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_M f) - [\bar{\nabla}_{e_i}(\text{grad}_M f) - \nabla_{e_i}(\text{grad}_M f)], P_r(e_i) \rangle \\ &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_M f) - \alpha(e_i, \text{grad}_M f), P_r(e_i) \rangle \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_M f), P_r(e_i) \rangle \\
 &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_N F - (\text{grad}_N F)^\perp), P_r(e_i) \rangle \\
 &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \text{grad}_N F, P_r(e_i) \rangle - \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_N F)^\perp, P_r(e_i) \rangle \\
 &= \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) - \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\langle \text{grad}_N F, \eta \rangle \eta), P_r(e_i) \rangle \\
 &= \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) - \sum_{i=1}^n \langle \langle \text{grad}_N F, \eta \rangle \bar{\nabla}_{e_i} \eta, P_r(e_i) \rangle \\
 &= \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) - \langle \text{grad}_N F, \eta \rangle \sum_{i=1}^n \langle -A(e_i), P_r(e_i) \rangle \\
 &= \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) + \langle \text{grad}_N F, \eta \rangle \sum_{i=1}^n \langle e_i, AP_r(e_i) \rangle \\
 &= \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) + \langle \text{grad}_N F, \eta \rangle \text{trace}(AP_r) \\
 &= \sum_{i=1}^n \text{Hess}_N(F)(e_i, P_r(e_i)) + (r + 1)S_{r+1} \langle \text{grad}_N F, \eta \rangle.
 \end{aligned}$$

□

Let $c \in \mathbb{R}$. Define the function:

$$\theta_c(t) := \int_0^t s_c(u) du,$$

where

$$(2.5) \quad s_c(t) = \begin{cases} \frac{\sin \sqrt{c}t}{\sqrt{c}}, & \text{if } c > 0; \\ t, & \text{if } c = 0; \\ \frac{\sinh \sqrt{|c|}t}{\sqrt{|c|}}, & \text{if } c < 0. \end{cases}$$

Let ρ denotes the distance function to the point Q in N^{n+1} , and $F : N^{n+1} \rightarrow \mathbb{R}$ given by $F(p) = \theta_c(\rho(p))$. In this case, Lemma 2.5 with $f = F \circ x$ and $F = \theta_c \circ \rho$ implies the following corollary.

COROLLARY 2.6. *Let \mathcal{Q}_c^{n+1} be a Riemannian manifold with constant sectional curvature c . Let M be an immersed hypersurface in \mathcal{Q}_c^{n+1} . Then, for*

all $p \in M$,

$$(2.6) \quad L_r(\theta_c(\rho(p))) = (n-r)s'_c(\rho(p))S_r(p) + (r+1)S_{r+1}(p)s_c(\rho(p))\langle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho(p), \eta \rangle.$$

In particular, when $c = 0$,

$$\frac{1}{2}L_r(\rho^2(p)) = (n-r)S_r(p) + (r+1)S_{r+1}(p)\rho(p)\langle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho(p), \eta \rangle.$$

Proof. First observe that

$$(2.7) \quad \text{Hess}_{\mathcal{Q}_c^{n+1}} F(X, Y) = s'_c(\rho)\langle X, Y \rangle,$$

where $X, Y \in T_{x(p)}\mathcal{Q}_c^{n+1}$. In fact,

$$\begin{aligned} \text{Hess}_{\mathcal{Q}_c^{n+1}} F(X, Y) &= \text{Hess}_{\mathcal{Q}_c^{n+1}}(\theta_c(\rho)) \\ &= \langle \bar{\nabla}_X \text{grad}_{\mathcal{Q}_c^{n+1}}(\theta_c(\rho)), Y \rangle \\ &= \langle \bar{\nabla}_X s_c(\rho) \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y \rangle \\ &= s_c(\rho) \text{Hess}_{\mathcal{Q}_c^{n+1}} \rho(X, Y) \\ &\quad + s'_c(\rho) \langle \langle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, X \rangle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y \rangle. \end{aligned}$$

On the other hand, see [1], p. 6,

$$\begin{aligned} \text{Hess}_{\mathcal{Q}_c^{n+1}} \rho(X, Y) &= \langle \bar{\nabla}_X \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y \rangle \\ &= \frac{s'_c(\rho)}{s_c(\rho)} [\langle X, Y \rangle - \langle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, X \rangle \langle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y \rangle]. \end{aligned}$$

This concludes the proof of (2.7). Now, by using equation (2.4), we have

$$\begin{aligned} L_r f &= \sum_{i=1}^n s'_c(\rho) \langle e_i, P_r(e_i) \rangle + (r+1)S_{r+1} \langle \text{grad}_{\mathcal{Q}_c^{n+1}}(\theta_c \circ \rho), \eta \rangle \\ &= s'_c(\rho) \text{trace } P_r + (r+1)S_{r+1}s_c(\rho) \langle \text{grad}_{\mathcal{Q}_c^{n+1}} \rho, \eta \rangle. \end{aligned}$$

Finally, by using equation (2.1), we conclude the proof of equation (2.6). The case $c = 0$ follows immediately. \square

It follows from Codazzi equation (see Rosenberg [16], p. 225) that L_r is a divergent form operator, that is,

$$L_r(f) = \text{div}_M(P_r \nabla f)$$

for all smooth functions $f : M \rightarrow \mathbb{R}$. Denote by $B_r(Q)$ the geodesic ball of \mathcal{Q}_c^{n+1} with radius r centered at $Q \in \mathcal{Q}_c^{n+1}$, and by $\bar{B}_r(Q)$ its closure. We will use the following proposition.

PROPOSITION 2.7. *Let \mathcal{Q}_c^{n+1} be a Riemannian manifold with constant sectional curvature c and let $x : M^n \rightarrow \mathcal{Q}_c^{n+1}$ be an isometric immersion. For*

$Q \in \mathcal{Q}_c^{n+1}$, we denote by $\rho(x)$ the distance to the point $Q \in \mathcal{Q}_c^{n+1}$ and $\rho(x(p))$, $p \in M$, its restriction to M . If for some $r \leq n - 1$, $S_r \geq 0$, then

$$(2.8) \quad \int_{\partial D} s_c(\rho(q)) \langle P_r(\text{grad}_M \rho(q)), \nu \rangle dA \geq (n - r) \int_D \left(s'_c(\rho(q)) S_r(p) - \frac{r+1}{n-r} |S_{r+1}(p)| s_c(\rho(q)) \right) dM,$$

where $q = x(p)$, $D \subset M$ is a bounded domain with nonempty boundary ∂D and ν is the conormal vector along ∂D . In the case $c > 0$, we assume that $D \subset \overline{B}_{\frac{\pi}{2\sqrt{c}}}(Q)$.

Proof. Since $|\text{grad}_{\mathcal{Q}_c^{n+1}} \rho(x(p))| \leq 1$ and $s'_c(\rho(x(p))) \geq 0$, from (2.6) we have

$$L_r(\theta_c(\rho(x))) \geq (n - r) \left[s'_c(\rho) S_r - \frac{r+1}{n-r} |S_{r+1}| s_c(\rho) \right].$$

Integrating this inequality, we get

$$(2.9) \quad \int_D L_r(\theta_c(\rho(x))) dM \geq (n - r) \int_D \left[s'_c(\rho(x)) S_r - \frac{r+1}{n-r} |S_{r+1}| s_c(\rho(x)) \right] dM.$$

On the other hand, we have that

$$\begin{aligned} \int_D L_r(\theta_c(\rho(x))) dM &= \int_D \text{div } P_r(\text{grad}_M(\theta_c(\rho(x(p)))) dM \\ &= \int_D \text{div}(s_c \rho(x(p)) P_r(\text{grad}_{\mathcal{Q}_c^{n+1}} \rho)^\top) dM \\ &= \int_{\partial D} s_c(\rho(x)) \langle P_r((\text{grad}_{\mathcal{Q}_c^{n+1}} \rho)^\top), \nu \rangle dA, \end{aligned}$$

where ν denotes the outward unit normal vector field on ∂D . Therefore, if $q = x(p)$,

$$\begin{aligned} &\int_{\partial D} s_c(\rho(q)) \langle P_r((\text{grad}_{\mathcal{Q}_c^{n+1}} \rho(q))^\top), \nu \rangle dA \\ &\geq (n - r) \int_D \left[s'_c(\rho(x)) S_r - \frac{r+1}{n-r} |S_{r+1}| s_c(\rho(x)) \right] dM, \end{aligned}$$

and the proposition is proved. □

We would like to point out that the above proposition is valid for a more general class of domains. For instance, it is valid in the setting of Gauss–Green Theorem (see [10], p. 478). In particular, if we take D to be the intersection of the extrinsic ball with M i.e. $D = \overline{B}_\mu \cap M$ in Proposition 2.7, we have the following

THEOREM 2.8. *Let \mathcal{Q}_c^{n+1} be a Riemannian manifold with constant sectional curvature c and let M^n be a complete noncompact properly immersed hypersurface of \mathcal{Q}_c^{n+1} . Assume that there exists a nonnegative constant α such that*

$$(2.10) \quad (r + 1)|S_{r+1}| \leq (n - r)\alpha S_r$$

for some $r \leq n - 1$. If P_r is positive semidefinite, then for any $q \in M$ such that $S_r(q) \neq 0$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, q and M such that for every $\mu > \mu_0$,

$$A_r(\overline{B}_\mu(q) \cap M) = \int_{\overline{B}_\mu(q) \cap M} S_r dM \geq \int_{\mu_0}^\mu C e^{-\alpha\tau} d\tau,$$

where $\overline{B}_\mu(q)$ is the ball of radius μ and center q in \mathcal{Q}_c^{n+1} . For the case $c > 0$, we assume $\mu \leq \frac{\pi}{2\sqrt{c}}$.

Proof. We use the notation introduced in Proposition 2.7. Take $D_\tau = \overline{B}_\tau(q) \cap M$, $\mu \leq 2\pi/\sqrt{c}$, if $c > 0$. Since the immersion is proper, we have that $\partial D_\tau \neq \emptyset$, for all $0 < \tau < \mu$. Thus, by using (2.10) in equation (2.8), we obtain that

$$(2.11) \quad \begin{aligned} & \int_{\partial D_\mu} s_c(\rho(x)) \langle P_r(\text{grad}_M \rho), \nu \rangle dA \\ & \geq (n - r) \int_{D_\mu} (s'_c(\rho(x)) - \alpha s_c(\rho(x))) S_r dM \\ & = (n - r) \int_0^\mu \int_{\partial D_\tau} \frac{s'_c(\rho(x)) - \alpha s_c(\rho(x))}{s_c(\rho(x))} \\ & \quad \times s_c(\rho(x)) |\text{grad}_M \rho|^{-1} S_r dA d\tau, \end{aligned}$$

where we have used the co-area formula (see [3], p. 80). Observe that the conormal vector ν to ∂D_τ is parallel to $\text{grad}_M \rho$. This fact together with the fact that P_r is positive semidefinite, imply the following:

$$\langle P_r(\text{grad}_M \rho), \nu \rangle \leq \text{trace}(P_r) |\text{grad}_M \rho| = (n - r) S_r |\text{grad}_M \rho|.$$

Using the above equation and the fact that along ∂D_τ , $\rho(x) = \tau$, we get

$$(2.12) \quad \begin{aligned} & \int_{\partial D_\mu} s_c(\rho(x)) |\text{grad}_M \rho| S_r dA \\ & \geq \int_0^\mu \frac{s'_c(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \int_{\partial D_\tau} s_c(\rho(x)) |\text{grad}_M \rho|^{-1} S_r dA d\tau. \end{aligned}$$

Now we define

$$\varphi(\tau) = \int_{\partial D_\tau} s_c(\rho(x)) |\text{grad}_M \rho|^{-1} S_r dA.$$

Since $|\text{grad}_M \rho| \leq 1$, equation (2.12) implies

$$\varphi(\mu) \geq \int_0^\mu \frac{s'_c(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau) d\tau.$$

By writing

$$\phi(\mu) = \int_0^\mu \frac{s'_c(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau) d\tau,$$

one finds

$$\phi'(\mu) \geq \frac{s'_c(\mu) - \alpha s_c(\mu)}{s_c(\mu)} \phi(\mu).$$

Thus, by integrating from $\mu_0 > 0$ to μ , the above differential inequality arises

$$\ln \frac{\phi(\mu)}{\phi(\mu_0)} \geq \ln \left(\frac{s_c(\mu)}{s_c(\mu_0)} \right) - \alpha(\mu - \mu_0) = \ln \left(\left(\frac{s_c(\mu)}{s_c(\mu_0)} \right) e^{-\alpha(\mu - \mu_0)} \right).$$

Hence,

$$\phi(\mu) \geq \frac{\phi(\mu_0)}{s_c(\mu_0)} s_c(\mu) e^{-\alpha\mu}.$$

Define

$$f(\mu) = \int_{D_\mu} S_r dM.$$

Again, by the co-area formula, it follows that

$$f(\mu) = \int_0^\mu \left(\int_{\partial D_\tau} |\text{grad}_M \rho|^{-1} S_r dA \right) d\tau.$$

Since

$$f'(\mu) = \int_{\partial D_\mu} |\text{grad}_M \rho|^{-1} S_r dA = \frac{1}{s_c(\mu)} \varphi(\mu) \geq \frac{\phi(\mu_0)}{s_c(\mu_0)} e^{-\alpha\mu},$$

then for $\mu > \mu_0$,

$$f(\mu) \geq \int_{\mu_0}^\mu \frac{\phi(\mu_0)}{s_c(\mu_0)} e^{-\alpha\tau} d\tau. \quad \square$$

COROLLARY 2.9. *Let \mathcal{Q}_c^{n+1} be a Riemannian manifold with constant sectional curvature $c \leq 0$ and let M^n be a complete noncompact properly immersed hypersurface of \mathcal{Q}_c^{n+1} . Assume that $S_r \geq 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \leq n - 1$. Then $\int_M S_r dM = \infty$.*

Proof. Since the immersion is proper, we have $\partial(M \cap \overline{B}_\mu(q))$ is nonempty for all $\mu > \mu_0$. By using Proposition 2.4, since $S_{r+1} = 0$, we have that P_r is semidefinite. Now, the condition $S_r \geq 0$ implies that P_r is positive semidefinite. Therefore, using Theorem 2.8, with $\alpha = 0$, for all $\mu > \mu_0$,

$$\int_{\overline{B}_\mu \cap M} S_r dM \geq \int_{\mu_0}^\mu C e^{-\alpha\tau} d\tau = C(\mu - \mu_0).$$

Then

$$\int_M S_r dM = \infty. \quad \square$$

REMARK 2.10. When r is odd, the condition $S_r \geq 0$ can be obtained by choosing the right orientation.

The condition of semi-positiveness of P_2 is satisfied when M is a hypersurface immersed in \mathbb{R}^{n+1} with $S_3 = 0$ (which is called 2-minimal) and $S_2 > 0$. In fact, in this case P_2 is positive definite, since L_2 is elliptic (see Proposition 2.2). So we have

COROLLARY 2.11. *Let M^n be a complete 2-minimal noncompact properly immersed hypersurface of \mathbb{R}^{n+1} with nonnegative scalar curvature. Then either the scalar curvature is zero or the total scalar curvature is infinite.*

REMARK 2.12. When $n = 3$ the corollary can be proved by using Theorem III in [13] without the assumption that the immersion is proper. In this case, M^n has to be a cylinder and the conclusion of the above corollary follows immediately.

REMARK 2.13. The condition of semi-positiveness of P_r is also satisfied when M is a hypersurface in \mathbb{R}^{n+1} with nonnegative sectional or positive Ricci curvature, $\text{Ric}_M > 0$. Indeed when $\text{Ric}_M > 0$, for each point in M , the principal curvatures can be arranged as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i < 0 < \lambda_{i+1} \leq \dots \leq \lambda_n$. The positivity of the Ricci curvature implies

$$\text{Ric}_M(e_j) = \lambda_j \left(\sum_{k \neq j} \lambda_k \right) > 0 \quad \forall j = 1, \dots, n.$$

If $i \in \{1, \dots, n - 1\}$, it follows from the above equation that

$$(2.13) \quad \sum_{k \neq j} \lambda_k < 0, \quad \text{when } j \leq i,$$

and

$$(2.14) \quad \sum_{k \neq j} \lambda_k > 0, \quad \text{when } j > i.$$

From (2.13), we have for $j_1 \leq i$,

$$\sum_{k \neq j_1} \lambda_k = \left(\sum_{k=1}^i \lambda_k - \lambda_{j_1} \right) + \sum_{k=i+1}^n \lambda_k < 0.$$

Thus

$$-\sum_{k=1}^i \lambda_k > \sum_{k=1}^i \lambda_k + \lambda_{j_1} > \sum_{k=i+1}^n \lambda_k.$$

On the other hand, using (2.14), for $j_2 > i$, we find

$$\sum_{k \neq j_2} \lambda_k = \left(\sum_{k=1}^i \lambda_k - \lambda_{j_2} \right) + \sum_{k=i+1}^n \lambda_k > 0,$$

hence

$$-\sum_{k=1}^i \lambda_k < \sum_{k=1}^i \lambda_k + \lambda_{j_1} < \sum_{k=i+1}^n \lambda_k,$$

which is a contradiction. Thus, all λ_i has the same sign (we are indebted to F. Fontenele for this observation). So we can choose an orientation such that P_r is positive definite and $S_r > 0$.

Thus we have the following consequence.

COROLLARY 2.14. *Let M^n be a complete noncompact properly immersed hypersurface of \mathbb{R}^{n+1} with positive Ricci curvature. Assume that there exists a positive constant α such that*

$$(r + 1)|S_{r+1}| \leq (n - r)\alpha S_r$$

for some $r \leq n - 1$. Then, for any $q \in M$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, Q and M such that

$$\int_{\overline{B}_{\mu}(q) \cap M} S_r dM \geq \int_{\mu_0}^{\mu} C e^{-\alpha\tau} d\tau,$$

where $\overline{B}_{\mu}(q)$ is the geodesic ball in \mathbb{R}^{n+1} centered at q .

The following is a direct consequence of Theorem 2.8 and Proposition 2.3.

COROLLARY 2.15. *Let M^n be a complete noncompact properly immersed hypersurface of \mathcal{Q}_c^{n+1} . Assume that S_r is positive and there exists a positive constant α such that*

$$(r + 1)|S_{r+1}| \leq (n - r)\alpha S_r$$

for some $r \leq n - 1$. If there exists a point such that all principal curvatures are nonnegative, then, for any $q \in M$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, q and M such that

$$\int_{\overline{B}_{\mu}(q) \cap M} S_r dM \geq \int_{\mu_0}^{\mu} C e^{-\alpha\tau} d\tau,$$

where $\overline{B}_{\mu}(q)$ is the geodesic ball in \mathcal{Q}_c^{n+1} centered at q . For the case $c > 0$, we assume $\mu \leq \frac{\pi}{2\sqrt{c}}$.

3. Volume estimates in general manifolds

In this section we consider N^{n+p} a Riemannian manifold with sectional curvature bounded from above by a constant c . Let M^n be a submanifold isometrically immersed in $N = N^{n+p}$.

Let $F : N \rightarrow \mathbb{R}$ be a C^2 function and denote $f : M \rightarrow \mathbb{R}$ the function induced by F by restriction. Essentially, following the steps involved in the proof of Lemma 2.5, we obtain

$$\Delta f = \sum_{i=1}^n \text{Hess}_N F(e_i, e_i) + n \langle \text{grad}_N F, \mathbf{H} \rangle,$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal frame along M and \mathbf{H} is the mean curvature vector. Similar to Proposition 2.7, we have

PROPOSITION 3.1. *Let N be a Riemannian manifold with sectional curvature bounded from above by a constant c and M^n an immersed connected submanifold of N . We denote by $\bar{\rho}(x)$ the distance in N between x and $Q \in N^{n+p}$ and $\rho(x)$ the induced function of $\bar{\rho}$ by restriction. Then*

$$(3.1) \quad \int_{\partial D} s_c(\rho(x)) \langle \text{grad}_M \rho, \nu \rangle dA \geq n \int_D (s'_c(\rho(x)) - |\mathbf{H}| s_c(\rho(x))) dM,$$

where $q = x(p)$, $D \subset M$ is a bounded domain with nonempty boundary ∂D and $D \cap C_N(Q) = \emptyset$, where $C_N(Q)$ is the cut locus of the point Q in N , and ν is the conormal vector along ∂D .

Proof. Let $V = s_c(\bar{\rho}) \text{grad}_N \bar{\rho}$ and V^\top the orthogonal projection of V into the tangent space of M . Then we have $V^\top = s_c(\rho) \text{grad}_M \rho$, where $\rho(x)$ is the induced function of $\bar{\rho}$ to M by restriction. Thus, Lemma 2.5 of [14], p. 713, implies, when $\bar{\rho} < \text{inj}_N(Q)$,

$$(3.2) \quad \text{Hess}_N F(X, X) \geq s'_c(\bar{\rho}) \langle X, X \rangle.$$

Then

$$\langle \bar{\nabla}_{e_i} V, e_i \rangle \geq s'_c(\bar{\rho})$$

for all $\bar{\rho}$ when $c \leq 0$, and $\rho \leq \frac{\pi}{\sqrt{c}}$, when $c > 0$. We find that

$$\Delta(\theta_c(\rho(x))) \geq n[s'_c(\rho) - s_c(\rho)|\mathbf{H}|].$$

Integrating this inequality and applying Stokes' formula, we get

$$\int_{\partial D} s_c \langle (\text{grad}_N \bar{\rho})^\top, \nu \rangle dA \geq n \int_D [s'_c(\rho(x)) - s_c(\rho(x))|\mathbf{H}|] dM,$$

and the proposition follows. □

Similar to Proposition 2.7, the above result is valid in a more general setting, such as extrinsic geodesic balls. Using this fact, we arrive at

THEOREM 3.2. *Let M be a Riemannian manifold isometrically immersed in a geodesic ball $\overline{B}(O, \rho_0) \subset N^{n+p}$ with codimension p . Assume that the sectional curvature of N in $\overline{B}(O, \rho_0)$ is bounded from above by c and moreover that there exists a positive constant α such that*

$$|\mathbf{H}| \leq \alpha.$$

Then

$$\text{vol}(B_\mu(q)) \geq n\omega_n \int_0^\mu s_c(t)^{n-1} e^{-n\alpha s} dt,$$

where ω_n is the volume of the unit ball in \mathbb{R}^n and $B_\mu(q)$ is the intrinsic geodesic ball in M with center $q \in M$ and radius $\mu < \text{inj}_N(q)$.

Proof. By taking $D = B_\tau(q)$ in Proposition 3.1, we obtain

$$\langle \text{grad}_M \rho, \nu \rangle \leq |\text{grad}_M \rho|.$$

Thus,

$$\begin{aligned} (3.3) \quad & \int_{\partial B_\tau(q)} \frac{s_c(\rho(x))}{n} |\text{grad}_M \rho| dA \\ & \geq \int_{B_\tau(q)} (s'_c(\rho(x)) - \alpha s_c(\rho(x))) dM \\ & = \int_0^\mu \int_{\partial B_\tau(q)} \frac{s'_c(\rho(x)) - \alpha s_c(\rho(x))}{s_c(\rho(x))} s_c(\rho(x)) |\text{grad}_M \rho|^{-1} dA d\tau, \end{aligned}$$

where we have used the co-area formula (see [3], p. 80). Since the intrinsic distance is not less than the extrinsic one and

$$\left(\frac{s'_c}{s_c}\right)' \leq 0,$$

then

$$\begin{aligned} (3.4) \quad & \frac{1}{n} \int_{\partial B_\mu(q)} s_c(\rho(x)) |\text{grad}_M \rho| dA \\ & \geq \int_0^\mu \frac{s'_c(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \int_{\partial B_\tau(q)} s_c(\rho(x)) |\text{grad}_M \rho|^{-1} dA d\tau. \end{aligned}$$

Now we define

$$\varphi(\tau) = \int_{\partial B_\tau(q)} s_c(\rho(x)) |\text{grad}_M \rho|^{-1} dA.$$

Equation (3.4) implies that

$$\frac{1}{n} \varphi(\mu) \geq \int_0^\mu \frac{s'_c(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau) d\tau.$$

By writing

$$\phi(\mu) = \int_0^\mu \frac{s'_c(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau) d\tau,$$

we have

$$\phi'(\mu) \geq \frac{n(s'_c(\mu) - \alpha s_c(\mu))}{s_c(\mu)} \phi(\mu).$$

Thus, by integrating from $\varepsilon > 0$ to μ , with $\mu \leq \min\{\text{inj}_N(q), \frac{\pi}{2\sqrt{c}}\}$ when $c > 0$, the above differential inequality arises

$$\frac{1}{n} \ln \frac{\phi(\mu)}{\phi(\varepsilon)} \geq \ln \left(\frac{s_c(\mu)}{\varepsilon} \right) - \alpha(\mu - \varepsilon) = \ln \left[\left(\frac{s_c(\mu)}{\varepsilon} \right) e^{-\alpha(\mu - \varepsilon)} \right].$$

Hence,

$$(3.5) \quad \frac{\phi(\mu)}{\phi(\varepsilon)} \geq \left[\left(\frac{s_c(\mu)}{\varepsilon} \right) e^{-\alpha(\mu - \varepsilon)} \right]^n.$$

Observe that by the mean value theorem,

$$\lim_{\varepsilon \rightarrow 0} \frac{\phi(\varepsilon)}{\varepsilon^n} = \omega_n.$$

Then

$$\phi(\mu) \geq \omega_n s_c(\mu)^n e^{-n\alpha\mu}.$$

Now, define

$$f(\mu) = \int_{B_\mu(q)} dM = \text{vol}(B_\mu(q)).$$

Again, by the co-area formula, we can write $f(\mu)$ as

$$f(\mu) = \int_0^\mu \left(\int_{\partial B_\tau(q)} |\text{grad}_M \rho|^{-1} dA \right) d\tau.$$

Hence

$$f'(\mu) = \int_{\partial B_\mu(q)} |\text{grad}_M \rho|^{-1} dA.$$

This equality together with $|\text{grad}_M \rho| \leq 1$, and equation (3.3) imply that

$$\frac{s_c(\mu)}{n} f'(\mu) \geq \int_{\partial B_\mu(q)} \frac{s_c(\rho(x))}{n} |\text{grad}_M \rho| dA \geq \int_0^\mu (s'_c(\tau) - \alpha s_c(\tau)) f'(\tau) d\tau.$$

Since

$$f'(\mu) \geq \frac{\varphi(\mu)}{s_c(\mu)},$$

then

$$f(\mu) \geq \int_0^\mu \omega_n n s_c(\tau)^{n-1} e^{-n\alpha\tau} d\tau,$$

which concludes the proof. □

The following corollary follows immediately.

COROLLARY 3.3. (i) *Let M^n be an immersed minimal hypersurface of the Euclidean space \mathbb{R}^{n+p} . Then*

$$\text{vol}(B_\mu(q)) \geq \omega_n \mu^n,$$

where ω_n is the volume of the unit ball in \mathbb{R}^n and $B_\mu(q)$ is the intrinsic geodesic ball in M centered at $q \in M$.

(ii) *Let M^n be an immersed hypersurface of the hyperbolic space $\mathbb{H}^{n+p}(-1)$. Assume that there exists a positive constant α such that*

$$|H| \leq \alpha < \frac{n-1}{n}.$$

Then, there exists a constant $C > 0$ so that, for $\mu \geq 1$,

$$\text{vol}(B_\mu(q)) \geq C e^{(n-1-n\alpha)\mu},$$

where $B_\mu(q)$ is the intrinsic geodesic ball in M with center $q \in M$.

4. Mean curvature integral

In this section, inspired by a recent work of Topping [18], we prove a type of mean curvature integral estimate for complete submanifold in a Euclidean space \mathbb{R}^n and we apply it to surfaces in \mathbb{R}^n .

THEOREM 4.1. *Let M^m be a m -dimensional complete noncompact Riemannian manifold isometrically immersed in \mathbb{R}^n . Then there exists a positive constant δ depending on m such that if*

$$(4.1) \quad \limsup_{r \rightarrow +\infty} \frac{V(x,r)}{r^m} < \delta,$$

where $V(x,r)$ denotes the volume of the geodesic ball $B_r(x)$, then

$$(4.2) \quad \limsup_{R \rightarrow +\infty} \frac{\int_{B_R(x)} |H|^{m-1} dM}{R} > 0.$$

In particular, $\int_M |H|^{m-1} dM = +\infty$.

We need the following lemma of Topping [18].

LEMMA 4.2 ([18], Lemma 1.2). *Let M^m be a m -dimensional complete Riemannian manifold isometrically immersed in \mathbb{R}^n . Then a positive constant δ depending on m exists, such that for any $x \in M$ and $R > 0$, at least one of the following statements is true:*

- (i) $\sup_{r \in (0,R]} r^{-\frac{1}{m-1}} [V(x,r)]^{-\frac{m-2}{m-1}} \int_{B(x,r)} |H|^{m-1} dM > \delta,$
- (ii) $\inf_{r \in (0,R]} \frac{V(x,r)}{r^m} > \delta.$

Proof of Theorem 4.1. We can choose L large enough so that $V(z, L) \leq \delta L^m$ for all $z \in M$. Then, from Lemma 4.2, we have

$$\sup_{r \in (0, L]} r^{-\frac{1}{m-1}} [V(z, r)]^{-\frac{m-2}{m-1}} \int_{B_r(z)} |H|^{m-1} dM > \delta.$$

Since

$$\int_{B_r(z)} |H| dM \leq \left(\int_{B_r(z)} |H|^{m-1} dM \right)^{\frac{1}{m-1}} \cdot (V(z, r))^{\frac{m-2}{m-1}}$$

for any $z \in M$, there exists a $r(z) \in (0, R]$ such that

$$\int_{B_{r(z)}} |H|^{m-1} dM > \delta^{m-1} r(z).$$

Fix a point $o \in M$, and let $\gamma : [0, +\infty) \rightarrow M$ be a ray parametrized by an arclength with $\gamma(0) = o$. For any fixed $R > 0$,

$$\gamma([0, R]) \subset \bigcup_{t \in [0, R]} B_{r(\gamma(t))}(\gamma(t)).$$

From a covering argument used in Theorem 1.1 of [18], we can find an at most countable sequence $t_1, t_2, \dots, t_q, \dots \in [0, R]$ such that $\sum_i r(\gamma(t_i)) \geq \frac{1}{4}R$. Thus, when $i \neq j$,

$$B_{r(\gamma(t_i))}(\gamma(t_i)) \cap B_{r(\gamma(t_j))}(\gamma(t_j)) = \emptyset.$$

Then

$$\begin{aligned} \int_{B_{2R}(o)} |H|^{m-1} dM &\geq \sum_i \int_{B_{r(\gamma(t_i))}(\gamma(t_i))} |H|^{m-1} dM \\ &\geq \delta^{m-1} \sum_i r(\gamma(t_i)) \\ &\geq \delta^{m-1} \frac{1}{4}R. \end{aligned}$$

And the result is proved. □

For complete surfaces in \mathbb{R}^n that satisfy the Gauss–Bonnet relation, we obtain the following result.

COROLLARY 4.3. *Let δ be as in Theorem 4.1. If M is a complete noncompact surface in \mathbb{R}^n satisfying*

$$(4.3) \quad 2\pi\chi(M) - \int_M K dM < 2\delta,$$

where $\chi(M)$ is the Euler characteristic of M , then

$$\int_M |H| dM = +\infty.$$

Proof. From Theorem A of Shiohama [17], for any $q \in M$, we find that

$$\lim_{r \rightarrow \infty} \frac{2V(B_r(q))}{r^2} = 2\pi\chi(M) - \int_M K dM.$$

It should be noted here that there is a misprint in the denominator of this expression in Shiohama's paper. So,

$$\lim_{r \rightarrow \infty} \frac{V(B_r(q))}{\pi r^2} < \delta.$$

Thus, Theorem 4.1 implies the result. \square

REMARK 4.4. The flat plane embedded in \mathbb{R}^n shows that the condition (4.3) is necessary.

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