REAL ANALYTICITY OF HAUSDORFF DIMENSION OF JULIA SETS OF PARABOLIC POLYNOMIALS
$$f_\lambda(z) = z(1 - z - \lambda z^2)$$

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ABSTRACT. We prove that D_*, the set of all parameters $\lambda \in \mathbb{C} \setminus \{0\}$ for which the cubic polynomial f_λ is parabolic and has no other parabolic or finite attracting periodic cycles, contains a deleted neighborhood D_0 of the origin 0. Our main result is that if D_0 is sufficiently small then the function $D_0 \ni \lambda \mapsto \mathrm{HD}(J(f_\lambda)) \in \mathbb{R}$ is real-analytic. This function ascribes to the polynomial f_λ the Hausdorff dimension of its Julia set $J(f_\lambda)$. The theory of parabolic and hyperbolic graph directed Markov systems with infinite number of edges is used in the proofs.

1. Introduction

Hausdorff dimension as a function of subsets of a given metric space usually behaves extremely irregularly. For example if $n \geq 1$ and $\mathcal{K}(\mathbb{R}^n)$ denotes the space of all non-empty compact subsets of of the Euclidean space $\mathbb{R}^n$, then the function $\mathcal{K}(\mathbb{R}^n) \ni \mathcal{K} \mapsto \mathrm{HD}(K) \in \mathbb{R}$, ascribing to the compact set K its Hausdorff dimension $\mathrm{HD}(K)$, is discontinuous at every point. It is therefore surprising indeed that the function $c \mapsto \mathrm{HD}(J_c)$ is continuous, where c belongs to M_0, the main cardioid of the Mandelbrot set $\mathcal{M}$, and J_c denotes the Julia set of the polynomial $\mathbb{C} \ni z \mapsto z^2 + c$. This is a relatively straightforward consequence of (classical) Bowen's formula which states that the Hausdorff dimension of a conformal expanding repeller is given by the unique zero of the corresponding pressure function. Bowen's formula was proved in [1] for

limit sets of quasi-Fuchsian groups, and it was the first application of thermo-
dynamic formalism to fractal geometry. Its extension to conformal expanding
repellers is rather straightforward; see [9] for the proof and related issues. As
a matter of fact Bowen's formula can be used to prove much more. Namely,
that the function $M_0 \ni c \mapsto \mathrm{HD}(J_c)$ is real-analytic. This fact was proved in
[11] based on considerations involving dynamical zeta-functions. Going be-
yond the classical (finite-to-one) conformal expanding case, real analyticity
of the Hausdorff dimension was proved in [17] for Julia–Lavours maps and in
[16] for the hyperbolic family of exponential maps. The proofs in both papers
are based on a different idea than in [11]; their point is to exploit complex an-
alyticity of the corresponding (generalized) Perron–Frobenius operators and
to prove applicability of the Kato–Rellich perturbation theorem. Further re-
sults in this direction (for expanding systems) and simplifications of the proof
can be found in [8] and [10], see also [7] and [14]. Real analyticity for still
expanding though random systems is proven in [6] with the use of [13], comp.
[12].

Going beyond the expanding case, up to our knowledge, the first real ana-
lyticity result is proved in [15] for analytic families of semi-hyperbolic general-
ized polynomial-like mappings. In this realm, the Julia set is allowed to con-
tain critical points but their forward orbits are assumed to be non-recurrent.
This allows us to associate with such a family an analytic family of conformal
graph directed Markov systems (in the sense of [5]) with infinite number of
edges and to reduce the problem of real analyticity of Hausdorff dimension
of limit sets of this family. In the current paper we investigate another im-
portant case where the expanding property breaks down, this time because
of presence of parabolic points. We choose to deal with this phenomenon by
working with a concrete but representative family of cubic polynomials

$$f_\lambda(z) = z\bigl(1 - z - \lambda z^2\bigr).$$

Note that a simpler, allegedly more natural, family $z \mapsto z(1 - \lambda z)$ is too trivial
since all its members are conjugate via Möbius transformations, and therefore
all their Julia sets have the same Hausdorff dimension. On the other hand,
what concerns the above family $\{f_\lambda\}$, we prove that they are generally not
even bi-Lipschitz conjugate on their Julia sets.

A rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ was called in [2] parabolic if its restriction to
the Julia set $J(f)$ is expansive but not expanding, equivalently, if the Julia set
contains no critical points but it contains at least one rationally indifferent
(parabolic) periodic point. We prove in Section 2 (Theorem 2.4) that D_0,
the set of all parameters $\lambda \in \mathbb{C}$ for which the cubic polynomial f_λ is para-
bolic and has no other parabolic or finite attracting periodic points, contains
a deleted neighborhood of the origin 0. Our main result is that the func-
tion $D_0 \ni \lambda \mapsto \mathrm{HD}(J(f_\lambda)) \in \mathbb{R}$ is real analytic. As in [15], the general idea
is to associate to the family $\{f_\lambda\}_{\lambda \in D_0}$ an analytic family, call it $\{S_\lambda\}_{\lambda \in D_0}$,

of conformal graph directed Markov systems with infinite number of edges in order to reduce the problem of real analyticity of Hausdorff dimension for this family to prove the corresponding statement for the family $\{S_\lambda\}_{\lambda \in D_0}$. The basic steps of this approach are these. In Section 2, The Family $\mathcal{P}_3$, we prove basic facts about polynomials f_λ, $\lambda \in \mathbb{C}$. In Section 3, Parabolic Graph Directed Markov Systems, we introduce the class of parabolic graph directed Markov systems (PGDMS) and provide the reader with their basic properties. In particular, we associate to each PGDMS S the canonical hyperbolic system $\hat{S}$. The concept of parabolic graph directed Markov System generalizes slightly the notion of parabolic iterated function systems introduced in [3], further investigated in [4], and treated at length in the book [5]. In Section 4, Analytic Families of PGDMS, we first generalize a theorem from [15] about real analyticity of the Hausdorff dimension for regularly analytic families of conformal (hyperbolic) graph directed Markov Systems. Then we introduce the concept of a holomorphic family of holomorphic parabolic graph directed Markov systems, and the central part of the section is a rather long proof that a holomorphic family $\{S_\lambda\}_{\lambda \in \Lambda}$ of holomorphic parabolic graph directed Markov systems gives rise to a locally regular analytic family $\{\hat{S}_\lambda^l\}_{\lambda \in \Lambda}$ (with some $l \geq 1$) of corresponding conformal (hyperbolic) graph directed Markov Systems. These considerations are so long since they require a detailed analysis of local behavior of families of parabolic maps around their common parabolic fixed points. This permits us to conclude, see Corollary 4.10, the section with the theorem that the Hausdorff dimension of limit sets of a holomorphic family of holomorphic parabolic graph directed Markov systems is real analytic. In Section 5, PGDMS Associated With f_λ, $\lambda \in D_0$, which is the last section of the paper, we apply the machinery developed in the previous sections to study the family of polynomials f_λ, $\lambda \in D_0$. The idea is to associate to this family of polynomials a holomorphic family of holomorphic parabolic graph directed Markov systems whose limit sets coincide with the Julia sets of polynomials f_λ up to a countable set. Then to apply Corollary 4.10.

2. The family $\mathcal{P}_3$

By definition, the family $\mathcal{P}_3$ consists of all cubic polynomials of the form

$$(2.1) \qquad f_\lambda(z) = z\big(1 - z - \lambda z^2\big), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Note that

$$(2.2) \qquad f_\lambda(0) = 0$$

and

$$(2.3) \qquad f_\lambda'(z) = 1 - 2z - 3\lambda z^2.$$

Hence $f'_\lambda(0) = 1$, and therefore (looking also at (2.1)), we get the following proposition.

PROPOSITION 2.1. *The number 0 is a parabolic fixed point of f_λ with multiplicity equal to 1 and with one petal. The ray $[0, +\infty)$ forms its attracting direction and the ray $(-\infty, 0]$ forms its repelling direction.*

The other finite fixed point of f_λ is the non-zero solutions to the equation $1 - z - \lambda z^2 = 1$, that is,

$$z = -\frac{1}{\lambda}.$$

We have,

$$(2.4) \qquad f'_\lambda\left(-\frac{1}{\lambda}\right) = 1 + \frac{2}{\lambda} - \frac{3}{\lambda} = 1 - \frac{1}{\lambda}.$$

Since any two polynomials bi-Lipschitz conjugate on their Julia sets have the same moduli of multipliers at corresponding periodic points, (2.4) yields the following.

THEOREM 2.2. *If $\lambda, \gamma \in (-\infty, 1/2)$, $\lambda\gamma > 0$, and $\lambda \neq \gamma$, then f_λ and f_γ are not bi-Lipschitz conjugate on their Julia sets.*

The critical points of f_λ are the solutions to the equation $1 - 2z - 3\lambda z^2 = 0$, i.e.,

$$(2.5) \qquad c_\lambda^{(1)} = \frac{-1 + \sqrt{1 + 3\lambda}}{3\lambda} \quad \text{and} \quad c_\lambda^{(2)} = \frac{-1 - \sqrt{1 + 3\lambda}}{3\lambda},$$

and we take the convention that $\sqrt{1} = 1$. We shall prove the following.

LEMMA 2.3. *For all $\lambda \in \mathbb{C} \setminus \{0\}$ sufficiently small in modulus,*

$$\lim_{n\to\infty} f_\lambda^n\left(c_\lambda^{(2)}\right) = \infty.$$

Proof. It follows from (2.5) that for all $\lambda \in \mathbb{C} \setminus \{0\}$ sufficiently small in modulus, say

$$\lambda \in B^*(0, R_1) := B(0, R_1) \setminus \{0\},$$

$$\frac{7}{12|\lambda|} \leq |c_\lambda^{(2)}| \leq \frac{3}{4|\lambda|}.$$

So,

$$(2.6) \qquad \begin{aligned} \left|f_\lambda^n\left(c_\lambda^{(2)}\right)\right| &= \left|c_\lambda^{(2)}\right|\left|1 - c_\lambda^{(2)} - \lambda\left(c_\lambda^{(2)}\right)^2\right| \\ &\geq \left|c_\lambda^{(2)}\right|\left(\left|c_\lambda^{(2)}\right| - |\lambda|\left|c_\lambda^{(2)}\right|^2 - 1\right) \\ &\geq \frac{7}{12|\lambda|}\left(\frac{7}{12|\lambda|} - \frac{9}{16|\lambda|} - 1\right) \\ &= \frac{7}{12|\lambda|}\left(\frac{28}{48|\lambda|} - \frac{27}{48|\lambda|} - 1\right) \end{aligned}$$

$$= \frac{7}{12|\lambda|}\left(\frac{1}{48|\lambda|} - 1\right) \geq \frac{7}{12|\lambda|}\frac{1}{96|\lambda|}$$

$$\geq \frac{7}{1200}\frac{1}{|\lambda|^2} \geq \frac{1}{2^9|\lambda|^2},$$

where writing the second last inequality we assumed that $R_1 \leq \frac{1}{96}$ which implies that,

$$\frac{1}{48|\lambda|} - 1 - \frac{1}{96|\lambda|} = \frac{1}{96|\lambda|} - 1 \geq \frac{1}{96R_1} - 1 \geq 0.$$

Now, if $|z| \geq (2^9|\lambda|^2)^{-1}$ and $\lambda \in B^*(0, R_2)$ with $0 < R_2 \leq R_1$ sufficiently small, we get

$$(2.7) \qquad \begin{aligned} |f_\lambda(z)| &= |z||1 - z - \lambda z^2| \geq |z|(|\lambda||z^2| - |z| - 1) \\ &= |z|(|z|(|\lambda||z| - 1) - 1) \\ &\geq |z|((2^9|\lambda|^2)^{-1}((2^9|\lambda|)^{-1} - 1) - 1) \\ &\geq |z|(2^9|\lambda|^2)^{-1}(2^{10}|\lambda|)^{-1} - 1 \\ &\geq |z|(2^9|\lambda|^2)^{-1}(2^{11}|\lambda|)^{-1} \\ &= 2^{-20}|\lambda|^{-3}|z| \geq 2|z|. \end{aligned}$$

Combining this with (2.6), we get by a straight forward induction, for all $\lambda \in B^*(0, R_2)$ that

$$|f_\lambda^{n+1}(c_\lambda^{(2)})| \geq 2^n(2^9|\lambda|^2)^{-1}.$$

We are therefore done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Let

$$D_0 = \left\{\lambda \in \mathbb{C}\setminus\{0\} : f_\lambda \text{ is a parabolic polynomial } \lim_{n\to\infty} f_\lambda^n(c_\lambda^{(2)}) = \infty \text{ and}\right.$$

$$\left. f_\lambda \text{ has no other parabolic or finite attracting periodic points}\right\}$$

and let

$$\mathcal{P}_3^0 = \{f_\lambda : \lambda \in D_0\}.$$

Now, the following theorem immediately follows from Lemma 2.3 and Fatou–Sullivan's classification of Fatou components.

THEOREM 2.4. *There exists $R > 0$ such that $B^*(0, R) \subseteq D_0$.*

3. Parabolic graph directed Markov systems

In the paper [3], the class of parabolic iterated function systems has been introduced, in [4] their finer fractal properties were investigated, and in the book [5] this class has been studied at length. In this paper, we need a slight generalization of this concept, namely parabolic graph directed Markov system. We define them now. Suppose we are given an oriented multigraph $\langle E, V\rangle$ consisting of countably many edges E and finitely many vertices V.

Suppose also that an incidence matrix $A : E \times E \to \{0,1\}$ is given. Any finite word $\omega \in E^* = \bigcup_{n=0}^{\infty} E^n$ is called A-admissible provided that $A_{\omega_i \omega_{i+1}} = 1$ for all $1 \le i \le |\omega| - 1$, where $|\omega|$ is the length of ω. The set of all finite A-admissible words is denoted by E_A^* and the set of all words of some length $0 \le n \le \infty$ is denoted by E_A^n. The matrix A is called finitely irreducible if there exists a finite set $\Lambda \subseteq E_A^*$ such that for all $\alpha, \beta \in E_A^*$ there exists $\gamma \in \Lambda$ such that $\alpha\beta\gamma \in E_A^*$. The matrix A is called finitely primitive if the set Λ can be chosen to consist of the words with the same length. Assume further that an integer number $d \ge 1$ is fixed and for every $v \in V$ a compact connected set $X_v \subseteq \mathbb{R}^d$ is given, and an open connected set $W_v \supseteq X_v$ is also given. Assume also that two functions $i, t : E \to V$ are given with the property that $A_{ab} = 1$ whenever $t(a) = i(b)$. In most known natural examples, this implication goes in fact in both directions, but we do not assume this. Assume lastly that for every $e \in E$ a continuous injective map $\varphi_e : W_{t(e)} \to \mathbb{R}^d$ is given. Fix also a non-empty finite set $\Omega \subseteq E$ such that $t(e) = i(e)$ for all $e \in \Omega$. Call a word $\omega \in E_A^*$ hyperbolic if either $\omega_{|\omega|} \notin \Omega$ or $\omega_{|\omega|-1} \ne \omega_{|\omega|}$ and $\omega_{|\omega|} \in \Omega$. All the objects introduced above are required to satisfy the following conditions.

(2a) $X_v = \overline{\mathrm{Int}\, X_v}$ for all $v \in V$.

(2b) $\varphi_e(X_{t(e)}) \subseteq X_{i(e)}$ for all $e \in E$. This enables us to define for every $\omega \in E_A^*$, say $\omega \in E_A^n$, the map $\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}$. Put also $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_1)$.

(2c) (Open Set Condition) $\varphi_a(\mathrm{Int}\, X_{t(a)}) \cap \varphi_b(\mathrm{Int}\, X_{t(b)}) = \emptyset$ whenever $a, b \in E$ and $a \ne b$.

(2d) (Cone Property) There exists $\gamma > 0$ such that for every $v \in V$ and for every $x \in X_v$ there exists an open cone $\mathrm{Cone}(x, \gamma) \subseteq \mathrm{Int}\, X_v$ with vertex x, central angle γ and some altitude l which may depend on x.

(2e) If $\omega \in E_A^*$ is a hyperbolic word, then $\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)}$ extends to a C^2-conformal map from $W_{t(\omega)}$ to $W_{i(\omega)}$. This conformal map is defined by the same symbol φ_ω.

(2f) (Bounded Distortion Property) There exists $K \ge 1$ such that for every hyperbolic word $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$,

$$\frac{|\varphi_\omega'(y)|}{|\varphi_\omega'(x)|} \le K.$$

Here and in the sequel for any conformal mapping φ, $|\varphi'(z)|$ denotes the similarity factor (equivalently its norm as a linear map from $\mathbb{R}^d$ into $\mathbb{R}^d$) of the differential $\varphi'(z) : \mathbb{R}^d \to \mathbb{R}^d$. In addition, if $\varphi : W_v \to \mathbb{R}^d$ for some $v \in V$, then

$$\|\phi'\| := \sup\{|\phi'(x)| : x \in W_{t(\omega)}\}.$$

(2g) There are constants $\alpha > 0$ and $L \ge 1$ such that

$$\left| |\varphi_e'(y)| - |\varphi_e'(x)| \right| \le \|\varphi_e'\| \|y - x\|^\alpha$$

for all $e \in E$ and all $x, y \in W_{t(e)}$.

(2h) For every hyperbolic word, $\omega \in E_A^*$, $\|\varphi_\omega'\| < 1$.

(2i) For every $e \in \Omega$, $t(e) = i(e)$ and there exists a unique fixed point x_e of the map $\varphi_e : X_{t(e)} \to X_{i(e)}$. In addition, $|\varphi_e'(x_e)| = 1$.

(2j) For every $e \in \Omega$,

$$\lim_{n \to \infty} \text{diam}\big(\varphi_{e^n}(X_{t(e)})\big) = 0.$$

This implies that

$$\bigcap_{n=0}^{\infty} \varphi_{e^n}(X_{t(e)}) = \{x_e\}.$$

Any system S satisfying the above conditions is called a parabolic (conformal) graph directed Markov system. It is abbreviated to as a PGDMS. The set Ω is referred to as the set of parabolic vertices, the maps φ_e, $e \in \Omega$, are called parabolic maps, and x_e, $e \in \Omega$, are called parabolic fixed points. We could have in principle provided a somewhat less restrictive definition of a PGDMS allowing finitely many parabolic periodic points (fixed points of φ_ω, $\omega \in E_A^*$) that are not necessarily fixed points, but then passing to a sufficiently large iterate $S^n = \{\varphi_\omega : \omega \in E_A^n\}$ we would end up in a parabolic system as described above. Notice also that our assumptions imply each map $\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)}$ such that $i(\omega) = t(\omega)$ to have a unique fixed point, call it x_ω, and that the diameters $\text{diam}(\varphi_\omega^n(X_{t(\omega)}))$ converge to zero exponentially fast unless $\omega \in \Omega^*$. It is not difficult to prove (the same argument as in the proof of Lemma 8.1.2 in [5] goes through) that

(3.1) $$\lim_{n \to \infty} \sup_{\omega \in E_A^n} \big\{ \text{diam}\big(\varphi_\omega(X_{t(\omega)})\big) \big\} = 0.$$

Since for every $\omega \in E_A^\infty$, $\{\varphi_{\omega|_n}(X_{t(\omega)})\}_{n=1}^\infty$ is a descending sequence of compact sets, this implies that the intersection $\bigcap_{n=1}^\infty \varphi_{\omega|_n}(X_{t(\omega_n)})$ is a singleton. Call its only element $\pi(\omega)$. We thus have a well-defined map

$$\pi : E_A^\infty \to X := \bigcup_{v \in V} X_v.$$

Fixing $s > 0$ and endowing E_A^∞ with the metric $d_s(\omega, \tau) = \exp(-s|\omega \wedge \tau|)$, where $\omega \wedge \tau$ is the longest common initial subword of ω and τ, the map $\pi : E_A^\infty \to X$ becomes uniformly continuous. Its image, $\pi(E_A^\infty)$, is called the limit set of the attractor of the PGDMS S, and is denoted by J_s or simply by J if only one system is under consideration. It satisfies the equation

$$J = \bigcup_{e \in E} \varphi_e(J \cap X_{t(e)}).$$

Now, define the function $\zeta : E_A^\infty \to \mathbb{R}$ by the formula

$$\zeta(\omega) = \log\big|\varphi_{\omega_1}'\big(\pi(\sigma\omega)\big)\big|.$$

It can be proved in the same way as Proposition 8.2.1 in [5] that the function ζ is acceptable in the sense of [5]. This implies that for every $t \in \mathbb{R}$ the topological pressure $P(\sigma, t\zeta)$ makes a meaningful sense as introduced in [5], and all versions of the variational principle established in [5] hold. One can also define the topological pressure without involving symbolic dynamics. Namely, see Lemma 2.1.2 in [5], for all $t \geq 0$

$$P(\sigma, t\zeta) = P(t) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \|\varphi'_\omega\|_\infty^t.$$

Observe that if the set of edges E is infinite and the matrix A contains sufficiently many 1s, for example, if A is finitely irreducible, then $P(0) = +\infty$ and it may happen that $P(t) = +\infty$ for some positive t. It is therefore natural to introduce the parameter

$$\theta = \theta_s := \inf\{t \geq 0 : P(t) < +\infty\}.$$

Given an exponent $t \geq 0$, a Borel probability measure m on X is said to be t-conformal provided that $m(J) = 1$ and the following two conditions are satisfied.

(1) $m(\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)})) = 0$ for all $a, b \in E$ with $a \neq b$.
(2) $m(\varphi_e(A)) = \int_A |\varphi'_e|^t \, dm$ for every Borel set $A \subseteq X_{t(e)}$ and $e \in E$.

It is easy to prove by induction that conditions (1) and (2) above continue to hold with E replaced by E_A^*.

Assume from now on that the system S is finitely irreducible, i.e., the incidence matrix A is finitely irreducible. Let

$$h = h_s := \mathrm{HD}(J_s) \quad \text{and let} \quad \beta = \sup\{\mathrm{HD}(\mu \circ \pi^{-1})\},$$

where the supremum is taken over all ergodic invariant probability shift-invariant measures on E_A^∞ with finite entropy, and let e be the minimum of all exponents t of all t-conformal measures on J_s. With only minor modifications, one can prove in the same way as Theorem 8.3.6, the following version of Bowen's formula.

THEOREM 3.1. $h = \beta = e =$ the minimal zero of the pressure function $t \mapsto P(t)$.

In order to get a better appreciation of the right-hand side of this theorem, let us formulate the following proposition describing the shape of the graph of the pressure function. Its proof, up to minor modifications, is the same as the proof of the Proposition 8.2.5 in [5].

PROPOSITION 3.2. The pressure function $P(t)$ has the following properties:

(1) $P(t) \geq 0$ for all $t \geq 0$,
(2) $P(t) > 0$ for all $0 \leq t < h$,

(3) $P(t) = +\infty$ *for all $0 \leq t < \theta$,*
(4) $P(t) < +\infty$ *for all $t > 0$,*
(5) $P(t) = 0$ *for all $t \geq h$,*
(6) $P(t)$ *is non-increasing,*
(7) $P(t)$ *is strictly decreasing on $[\theta, h]$,*
(8) $P(t)$ *is convex and continuous on (θ, ∞).*

The main tool to study PGDMS is the associated (hyperbolic) conformal graph directed Markov system in the sense from [3]. Following [3] and Section 8.4 from [5], we will do it now. So, given a PGDMS S, the corresponding hyperbolic system $\hat{S}$ is defined as follows.

The set of vertices $\hat{V} = V$. The set of edges

$$\hat{E} = \left\{ a^n b : n \geq 1, a \in \Omega, b \neq a, A_{ab} = 1 \right\} \cup (E \setminus \Omega).$$

The incidence matrix $\hat{A} : \hat{E} \times \hat{E} \to \{0, 1\}$ is naturally defined by requiring that $\hat{A}_{st} = 1$ if and only if $A_{s_{|s|} t_1} = 1$, where $|s|$ and t_1 are understood here in the sense of the set of edges E. The functions t and i are defined on $\hat{E}$ by their restrictions to $\hat{E}$ treated as a subset of E_A^* and by the same procedure the maps $\varphi_e, e \in \hat{E}$, are defined. A finitely irreducible parabolic system S is called properly finitely (pf) irreducible if and only if for every two letters $a, c \in E \setminus \Omega$ there exists $\beta \in \Lambda$, Λ resulting from finite irreducibility of S, such that $a\beta c \in E_A^*$ and $\{\beta_1, \beta_{|\beta|}\} \cap (E \setminus \Omega) \neq \emptyset$. Two basic facts about the system $\hat{S}$ that make them useful in study the system S are these.

THEOREM 3.3. *If S is a PGDMS, then $\hat{S}$ is a CGDMS in the sense of Chapter 4 in [5]. If S is pf-irreducible, then $\hat{S}$ is finitely irreducible.*

Proof. The proof that $\hat{S}$ is a CGDMS is a minor modification of the proof of Theorem 8.4.2 in [5]. So, suppose that S is pf-irreducible and let Λ be the corresponding finite set contained in E_A^*. Shortening the words of Λ if necessary, we may assume without loss of generality that no word of Λ contains a subword of the form $e^2, e \in E$. Call all such words reduced. If a word in E_A^* can be split into blocks such that it becomes a member of $\hat{E}_{\hat{A}}^*$, slightly abusing terminology, we say that this word is in $\hat{E}_{\hat{A}}^*$. Now notice that any reduced word $\omega \in E_A^*$ with $\omega_{|\omega|} \in E \setminus \Omega$ is in $\hat{E}_{\hat{A}}^*$. Notice also that for every reduced word $\gamma \in E_A^*$ at least one of the words γ or $\gamma|_{|\gamma|-1}$ is in $\hat{E}_{\hat{A}}^*$. In order to show that $\hat{S}$ is finitely irreducible, consider arbitrary two elements $\alpha, \beta \in \hat{E}$. If both $\alpha, \beta \in E \setminus \Omega$, then by pf-irreducibility of S there exists a word $\gamma \in \Lambda$ such that $\alpha\gamma\beta \in E_A^*$ and $\gamma_{|\gamma|} \in E \setminus \Omega$. But then, by the first of the above observations $\gamma \in \hat{E}_{\hat{A}}^*$, and we are done in this case. So, suppose that $\beta = a^n b$, where $n \geq 1, a \in \Omega$, and $b \neq a$. By finite irreducibility of S there exists $\gamma \in \Lambda$ such that $\alpha\gamma a \in E_A^*$. If γ ends with $a^q, q \geq 1$, remove from γ the last block a^q.

If $\gamma \in \hat{E}_{\hat{A}}^*$, then we are done. Otherwise, $\gamma = \hat{\gamma}c$, where $\hat{\gamma} \in \hat{E}_{\hat{A}}^*$ and $c \in \Omega \setminus \{a\}$. But then $\alpha\hat{\gamma}(ca)a^n b \in \hat{E}_{\hat{A}}^*$ and $\hat{\gamma}(ca) \in \hat{E}_{\hat{A}}^*$. So, we are also done in this case. Finally, suppose that $\alpha = ab^n$, $n \geq 1$, $a \in \Omega$, $b \neq a$, and $\beta \in E \setminus \Omega$. If $b \in E \setminus \Omega$, then by pf-irreducibility of S, there exists $\rho \in \hat{E}_{\hat{A}}^*$ such that $b\rho\beta \in E_A^*$. But then $a^n b^n \rho\beta \in \hat{E}_{\hat{A}}^*$, and we are done. So, we may suppose that $b \in \Omega$. By finite irreducibility of S there exists $\gamma \in \Lambda$ such that $b\gamma\beta \in E_A^*$. If $\gamma \in \hat{E}_{\hat{A}}^*$, we are done. Otherwise, $\gamma = \hat{\gamma}c$, where $\hat{\gamma} \in \hat{E}_{\hat{A}}^*$ and $c \in \Omega$. If $\hat{\gamma}$ is the empty word and $c = b$, then $ab^n \beta \in \hat{E}_{\hat{A}}^*$ and we are done (the empty word joins $\alpha = ab^n$ and β). If $c \neq b$, then $(ab^n)(bc)\beta \in \hat{E}_{\hat{A}}^*$ and we are also done. So, suppose that $\hat{\gamma}$ is not empty. Write $\hat{\gamma} = \gamma|_{|\gamma|-1}d$, where $d \in E$. If $d \in E \setminus \Omega$, then by pf-irreducibility of S there exists $\rho \in \hat{E}_{\hat{A}}^* \cap \Lambda$ such that $d\rho\beta \in E_A^*$. But then $\hat{\gamma}\rho \in \hat{E}_{\hat{A}}^*$ and $(a^n b)(\hat{\gamma}\rho)\beta \in \hat{E}_{\hat{A}}^*$. If $d \in \Omega$, then, as $d \neq c$, we have $\hat{\gamma}(dc) \in \hat{E}_{\hat{A}}^*$ and $ab^n(\hat{\gamma}(dc))\beta \in \hat{E}_{\hat{A}}^*$. We are done in this case as well. In order to end the proof notice that all the words in $\hat{E}_{\hat{A}}^*$ we have constructed above to join all α and β in E led from Λ to a finite set, say $\hat{\Lambda}$. □

THEOREM 3.4. *The limit sets J_s and $J_{\hat{s}}$ of the systems S and $\hat{S}$, respectively differ only by a countable set. In fact, $J_{\hat{s}} \subseteq J_s$ and $J_s \setminus J_{\hat{s}} \subseteq \pi_s(\{\omega e^\infty \in E_A^\infty : e \in \Omega\})$.*

We call a parabolic system S finite if and only if the set of edges E is finite. We call a parabolic system S holomorphic if $d = 2$ and all maps φ_e, $e \in E$, are holomorphic, and $\varphi_e'(x_e) = 1$ for all $e \in \Omega$. Then for every $e \in \Omega$, we have the following power series expansion about x_e. Namely,

$$(3.2) \qquad \varphi_e(z) = z + a_e(z - x_e)^{1+p_e} + \sum_{n=2}^{\infty} a_n(e)(z - x_i)^{n+p_i}, \quad p_i \geq 1.$$

Hence (see [4]),

$$\left|\varphi_{e^n}'(z)\right| \asymp n^{-(p_e+1)/p_e}$$

uniformly on compact subsets of $X_{t(e)} \setminus \{x_i\}$. So, looking at the series,

$$\sum_{n=1}^{\infty} \left\|\varphi_{a^n b}'\right\| \asymp \sum_{n=1}^{\infty} n^{-((p_e+1)/p_e)t}, \quad a \in \Omega, b \neq a,$$

we immediately get the following.

THEOREM 3.5. *If S is a finite holomorphic parabolic graph directed system, then the associated hyperbolic system $\hat{S}$ is cofinitely (= hereditarily) regular.*

4. Analytic families of PGDMS

We want first to recall from [15] a result about real analyticity of Hausdorff dimension of limit sets. The key idea is the concept of regularly analytic families of conformal graph directed Markov systems. We also want to weaken the assumptions of Section 4 from [15] at some important points. Let $\{\phi^\lambda\}_{\lambda\in\Lambda}$ be a family of CGDMS with the same set of vertices V, the same set of edges E, the same finitely irreducible matrix A, and the same sets $\{W_v\}_{v\in V}$ with all $W_v \subseteq \mathbb{C}$. Unlike [15], we do not assume the compact spaces $\{X_v^\lambda\}_{\lambda\in\Lambda}$ to be all equal. Fix $\lambda_0 \in \Lambda$ and for every $\omega \in E_A^\infty$ consider the function $\psi_\omega : \Lambda \to \mathbb{C}$ given by the formula

$$\psi_\omega(\lambda) = \frac{(\varphi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma\omega))}{(\varphi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))},$$

where $\pi_\lambda := \pi_\phi(\lambda) : E_A^\infty \to J_{\phi_\lambda}$ is the coding map induced by the CGDMS ϕ^λ. The family $\{\phi^\lambda\}_{\lambda\in\Lambda}$ is said to be analytic if

(a) For every $e \in E$ and every $x \in W_{t(e)}$ the function $\Lambda \ni \lambda \mapsto \varphi_e^\lambda(x) \in W_{t(e)} \subseteq \mathbb{C}$ is holomorphic.

Furthermore, the family $\{\phi^\lambda\}_{\lambda\in\Lambda}$ is called regularly analytic if

(b) the system $\{\phi^{\lambda_0}\}$ is strongly regular

and

(c) there exists a constant $D > 0$ such that

$$\sup\{|\psi_\omega(\lambda)| : \omega \in E_A^\infty, \lambda \in \Lambda\} \leq D.$$

The basic fact resulting from this kind of analyticity is providing by the following.

LEMMA 4.1. *If $\{\phi^\lambda\}_{\lambda\in\Lambda}$ is an analytic family, then the family $\{\Lambda \ni \lambda \mapsto \pi_\lambda(\omega) \in \mathbb{C} : \omega \in E_A^\infty\}$ consists of holomorphic maps and is normal.*

Proof. For every $v \in V$, choose a point $x_v \in W_v$. Since all the maps $\Lambda \times W_{t(e)} \ni (\lambda, z) \mapsto \phi_e^\lambda(z)$, $e \in E$, are holomorphic, all the maps $\Lambda \ni \lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})$, $\omega \in E_A^*$, are also holomorphic. Since their ranges are all contained in the bounded set $\bigcup_{v\in V} W_v$, the family $\{\Lambda \ni \lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})\}_{\omega\in E_A^*}$ is normal. Therefore, since for every $\omega \in E_A^\infty$, the sequence of functions $(\Lambda \ni \lambda \mapsto \phi_{\omega|_n}^\lambda(x_{t(\omega|_n)}))_{n=1}^\infty$ converges pointwise to $\pi_\lambda(\omega)$, we conclude that each function $\Lambda \ni \lambda \mapsto \pi_\lambda(\omega)$, is holomorphic. Since the range of all these functions is contained in the bounded set $\bigcup_{v\in V} W_v$, the family $\{\Lambda \ni \lambda \mapsto \pi_\lambda(\omega)\}_{\omega\in E_A^\infty}$ is normal. We are done. $\square$

As an immediate consequence of this Lemma 4.1, Hartog's theorem, and item (c), we get the following.

LEMMA 4.2. *If $\{\phi^\lambda\}_{\lambda\in\Lambda}$ is a regularly analytic family, then for every $\omega \in E_A^\infty$ the map $\Lambda \ni \lambda \mapsto (\varphi_{\omega_1}^\lambda)'(\pi_\lambda(\omega)) \in \mathbb{C}$ is holomorphic.*

Combining this lemma and Lemma 4.1 we conclude that for every $\omega \in E_A^\infty$, the map $\Lambda \ni \lambda \mapsto \psi_\omega(\lambda) \in \mathbb{C}$ is holomorphic. We shall prove the following.

LEMMA 4.3. *Suppose* $\{\phi^\lambda\}_{\lambda \in \Lambda}$ *is a regular analytic family of holomorphic systems. Then for every* $\omega \in E_A^\infty$ *there is a well-defined* $\log \psi_\omega : B(\lambda_0, R) \to \mathbb{C}$, *the unique holomorphic branch of logarithm of* ψ_ω *such that* $\log \psi_\omega(\lambda_0) = 0$. *In addition, the family of functions* $\{\log \psi_\omega\}_{\omega \in E_A^\infty}$ *is bounded.*

Proof. Indeed, fix $R_2 > 0$ such that $B(\lambda_0, R_2) \subseteq \Lambda$. Fix $\omega \in E_A^\infty$. Since for all $\lambda \in B(\lambda_0, \frac{R_2}{2})$ and all $0 < r \le \frac{R_2}{2}$, we have

$$\psi_\omega'(\lambda) = \frac{1}{2\pi i} \int_{\partial B(\lambda_0, r)} \frac{\psi_\omega(\gamma)}{(\gamma - \lambda)^2} \, d\gamma,$$

we thus obtain from (c) the following:

$$\left| \psi_\omega'(\lambda) \right| \le \frac{1}{2\pi} \int_{\partial B(\lambda_0, r)} \frac{D}{r^2} |d\gamma| = \frac{D}{r}.$$

Since $\psi_\omega(\lambda_0) = 1$, we therefore get for all $\lambda \in B(\lambda_0, r)$ that

$$\left| \psi_\omega(\lambda) - 1 \right| = \left| \psi_\omega(\lambda) - \psi_\omega(\lambda_0) \right| = \left| \int_{\lambda_0}^\lambda \psi_\omega'(\gamma) \, d\gamma \right| \le \int_{\lambda_0}^\lambda \left| \psi_\omega'(\gamma) \right| |d\gamma| \le \frac{D}{r} |\lambda - \lambda_0|.$$

So, if we take $r = \frac{R_2}{2}$, then for all $\lambda \in B(\lambda_0, R_3)$ with $R_3 = \frac{R_2}{8D}$, we get

$$\left| \psi_\omega(\lambda) - 1 \right| \le \frac{1}{4}.$$

Hence, for each $\omega \in E_A^\infty$ there is a well-defined $\log \psi_\omega : B(\lambda_0, R_3) \to \mathbb{C}$, the unique holomorphic branch of the logarithm of ψ_ω such that $\log \psi_\omega(\lambda_0) = 0$, and the family of functions $\{\log \psi_\omega\}_{\omega \in \hat{E}_{\hat{A}}^\infty}$ is bounded. The proof of Lemma 4.3 is complete. $\qquad\qquad\square$

Setting $\kappa(\omega_1) = 1$ in the proof of Theorem 4.2 in [15] and having Lemma 4.1, Lemma 4.2 and Lemma 4.3, the proof of Theorem 4.2 in [15] goes verbatim to result in the following.

THEOREM 4.4. *If* $\{\phi^\lambda\}_{\lambda \in \Lambda}$ *is a regular analytic family of holomorphic conformal graph directed Markov systems, then the function* $\Lambda \ni \lambda \mapsto \mathrm{HD}(J(\phi^\lambda)) \in \mathbb{R}$ *is real-analytic.*

An analytic family $\{\phi^\lambda\}_{\lambda \in \Lambda}$ of holomorphic CGDMS is called locally regularly analytic if for every $\lambda_0 \in \Lambda$ there is $R_0 > 0$ such that the family $\{\phi^\lambda\}_{\lambda \in B(\lambda_0, R_0)}$ is regularly analytic. As an immediate consequence of the Theorem 4.4, we obtain the following.

THEOREM 4.5. *If* $\{\phi^\lambda\}_{\lambda \in \Lambda}$ *is a locally regularly analytic family of holomorphic CGDMS, then the function* $\Lambda \ni \lambda \mapsto \mathrm{HD}(J(\phi^\lambda))$ *is real-analytic.*

Suppose E, V, A, Ω and $W_v \subset \mathbb{C}, v \in V$, are given so that all the requirements imposed on them by the definition of PGDMS are met. We assume in addition that A is pf-irreducible and that the set of edges E is finite. Suppose Λ is an open connected subset of $\mathbb{C}$. A family $\{S_\lambda\}_{\lambda \in \Lambda}$ of holomorphic PGDMS, each of which is built with the help of the above block $E, V, A, \Omega, \{W_v\}_{v \in V}$, is called holomorphic if and only if:

(a) The functions $\Lambda \ni \lambda \mapsto x_e^\lambda \in \mathbb{C}$, $e \in \Omega$, are constant for all $\lambda \in \Lambda$; call their common values by x_e,

(b) The family $\{\hat{S}_\lambda^\lambda\}_{\lambda \in \Lambda}$ is analytic,

(c) For every $e \in \Omega$ there exists $R_e > 0$ such that $B(x_e, R_e) \subseteq W_{t(e)}$, and the map $\Lambda \times B(x_e, R_e) \ni (\lambda, z) \mapsto \varphi_e^\lambda(z) \in \mathbb{C}$ is holomorphic,

and

(d) For every $v \in V$ there exists a compact set $Y_v \subseteq W_v$ such that $X_v^\lambda \subseteq Y_v$ for all $\lambda \in \Lambda$.

THEOREM 4.6. *If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of holomorphic PGDMS, then there exists $l \geq 1$ such that the family $\{\hat{S}_\lambda^l\}_{\lambda \in \Lambda}$ is locally regularly analytic.*

Proof. In virtue of Proposition 9.3.9 from [5], there exists $l \geq 1$ such that $(\varphi_\lambda^l)'(x_e) = 1$ for every $e \in \Omega$. This is the integer l claimed in our theorem. For the ease of exposition we replace S_λ by S_λ^l and assume without loss of generality that $l = 1$. The family $\{\hat{S}_\lambda\}_{\lambda \in \Lambda}$ is analytic by assumption. Condition (b) of regular analyticity of $\{\hat{S}_\lambda\}_{\lambda \in \Lambda}$ is satisfied by Theorem 3.5. So, we are only left to verify condition (c) of regular analyticity of the family $\{\hat{S}_\lambda\}_{\lambda \in \Lambda}$. Towards this end a detailed analysis of parabolic maps φ_a^λ, $a \in \Omega, \lambda \in \Lambda$, is needed. If we dealt with a one single parabolic system the analysis done in [4] (comp. Section 9.3 in [5]) would suffice. But we want the big $\mathcal{O}$ constant in (9.4) in [5] to be independent of λ lying in a sufficiently small neighborhood of some arbitrarily chosen and then fixed parameter $\lambda_0 \in \Lambda$. So, fix $e \in \Omega$. Then with $R = R_e$, we have that

$$\varphi_e^\lambda(z) = z - a_e^\lambda(z - x_e)^{p+1} + \sum_{n=2}^{\infty} a_n^\lambda(e)(z - x_e)^{n+p}$$

for all $\lambda \in \Lambda$ and all $z \in B(x_e, R)$, where $p = p_e$. It follows from condition (c) that all the functions $\lambda \mapsto a_e^\lambda$ and $a_n^\lambda(e)$, $n \geq 2$, are analytic. Translating and rotating the plane, we may assume without loss of generality that $x_e = 0$ and one of the contracting directions of $\varphi_e^{\lambda_0}$ coincides with $(0, +\infty)$, the positive ray emanating from 0, meaning that $a_e^{\lambda_0} \in \mathbb{R}$ and $a_e^{\lambda_0} > 0$. Further on, making a homothetic change of variables, we may assume that

(4.1) $$a_{\lambda_0} = \frac{1}{p}.$$

Further, rotating the plane again, we may of course assume that the contracting direction associating with $X_{t(e)}^{\lambda_0}$ coincides with $(0, +\infty)$. Since in the rest of this proof all iterates involving parabolic maps are of the form $\phi_{a^n b}^{\lambda}$, where $a \in \Omega$, and $b \neq \omega$ (b can be an empty word), enlarging the sets $X_{t(e)}^{\lambda_0}$, $Y_{t(e)}$, and $W_{t(e)}^{\lambda_0}$ appropriately, we may further assume without loss of generality that some initial segment of $[0, +\infty)$ is contained in $X_{t(e)}^{\lambda_0} \subseteq Y_{t(e)} \subseteq W_{t(e)}^{\lambda_0}$. We also skip for simplicity the dependence on e. The power series expansion above takes then the following form

$$\varphi_\lambda(z) = z - a_\lambda z^{p+1} + \sum_{n=2}^{\infty} a_n(\lambda) z^{n+p}, \quad z \in B(0, R),$$

where $a_\lambda := a_e^\lambda, \varphi_\lambda := \varphi_e^\lambda$.

Now, let $\sqrt[p]{z}$ be the holomorphic branch of the p-th radical defined on $\mathbb{C} \setminus (-\infty, 0]$ and sending 1 to 1. Define then $H : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by the formula

$$H(z) = \frac{1}{\sqrt[p]{z}},$$

and consider the conjugate maps

$$\tilde{\varphi}_\lambda = H^{-1} \circ \varphi_\lambda \circ H : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C},$$

where $H^{-1}(\omega) = \frac{1}{\omega^p}$; in fact $\tilde{\varphi}_\lambda$ is defined on $U = H^{-1}(B(0, R)) \setminus (-\infty, 0]$. For all $z \in U$ we have

(4.2) $\quad \tilde{\varphi}_\lambda(z) = H^{-1}\big(\varphi_\lambda\big(H(z)\big)\big)$

$$= H^{-1}\left(H(z) - a_\lambda H(z)^{p+1} + \sum_{n=2}^{\infty} a_n(\lambda) H(z)^{n+p} \right)$$

$$= H^{-1}\left(\frac{1}{\sqrt[p]{z}} - a_\lambda z^{-(p+1)/p} + \sum_{n=2}^{\infty} a_n(\lambda) z^{-(p+n)/p} \right)$$

$$= H^{-1}\left(\frac{1}{\sqrt[p]{z}} \left(1 - a_\lambda z^{-1} + \sum_{n=2}^{\infty} a_n(\lambda) z^{-(p+n-1)/p} \right) \right)$$

$$= \frac{z}{(1 - a_\lambda z^{-1} + \sum_{n=2}^{\infty} a_n(\lambda) z^{-(p+n-1)/p})^p}.$$

Set $w = H(z) = z^{-1/p}$, put

$$g_\lambda(w) = 1 - a_\lambda w^p + \sum_{n=2}^{\infty} a_n(\lambda) w^{p+n-1}$$

and $\hat{g}_\lambda(w) = (g_\lambda(w))^{-p}$. Then $(\lambda, w) \mapsto \hat{g}_\lambda(w)$ is a holomorphic function of λ and z, and

$$(4.3) \qquad \hat{g}_\lambda(0) = 1, \qquad \left.\frac{\partial^k \hat{g}_\lambda(w)}{\partial w^k}\right|_{(\lambda,0)} = 0$$

$$\text{for all } k = 1, 2, \ldots, p-1, \text{ and } \left.\frac{\partial^p \hat{g}_\lambda}{\partial w^p}\right|_{(\lambda,0)} = pa_\lambda.$$

Therefore, we have the following power series expansion

$$\hat{g}_\lambda(w) = 1 + b_\lambda w^p + \sum_{n=2}^\infty b_n(\lambda) w^{p+n}$$

for $(\lambda, z) \in D_2((\lambda_0, 0); R)$ with some $R > 0$ sufficiently small, where $D_2(a; r) \subseteq \mathbb{C}^2$ is the polydisk centered at a and of radius r. Going back to the variable $z = w^{-p}$, we thus get from (4.2) that

$$(4.4) \qquad \tilde{\varphi}_\lambda(z) = z\left(1 + b_\lambda \frac{1}{z} + \frac{1}{z}\sum_{n=1}^\infty b_n(\lambda) H(z)^n\right)$$

$$= z + b_\lambda + \sum_{n=1}^\infty b_n(\lambda) H(z)^n$$

for all $\lambda \in B(\lambda_0, R)$ and all $z \in U$. Note that because of (4.1) and (4.3), $b_{\lambda_0} = 1$. Since the series $\sum_{n=1}^\infty b_n(\lambda) w^n$ converges absolutely uniformly on compact subsets of $D_2((\lambda_0, 0); R)$, the number

$$M = \sup\left\{\sum_{n=1}^\infty |b_n(\lambda)||w|^n : (\lambda, w) \in D_2((\lambda_0, 0); R/2)\right\}$$

is finite. Hence, for all $\lambda \in B(\lambda_0, \frac{R}{2})$ and all

$$z \in U_1 := H^{-1}\left(B\left(0, \frac{R}{8}\min\{1, M^{-1}\}\right)\right) \setminus (-\infty, 0] \subseteq U$$

we get

$$(4.5) \qquad \left|\sum_{n=1}^\infty b_n(\lambda) H(z)^n\right| \le \sum_{n=1}^\infty |b_n(\lambda)||H(z)|^n \le \sum_{n=1}^\infty |b_n(\lambda)||z|^{-n/p}$$

$$= \sum_{n=1}^\infty |b_n(\lambda)|\left(\frac{R}{2}\right)^n\left(\left(\frac{R}{2}\right)^p|z|\right)^{-n/p}$$

$$\le M\left(\left(\frac{R}{2}\right)^p|z|\right)^{-1/p} = \frac{2M}{R}|z|^{-1/p} \le \frac{1}{4}.$$

Combining this estimate with (4.4), we get that if $\mathrm{Re}(z) > (\frac{R}{8}\max\{1,M\})^p$, then

$$(4.6) \qquad \mathrm{Re}\big(\tilde\varphi_\lambda(z) - (z+b_\lambda)\big) = \mathrm{Re}\left(\sum_{n=1}^\infty b_n(\lambda)H(z)^n\right)$$

$$\geq -\left|\sum_{n=1}^\infty b_n(\lambda)H(z)^n\right| \geq -\frac{1}{4}.$$

Since $b_{\lambda_0} = 1$, there exists $R_1 \in (0, R/2)$ so small that $\frac{1}{2} < \mathrm{Re}\,b_\lambda < \frac{3}{2}$ for all $\lambda \in B(\lambda_0, R_1)$. It then follows from (4.6) that

$$(4.7) \qquad \mathrm{Re}\big(\tilde\varphi_\lambda(z)\big) \geq \mathrm{Re}(z+b_\lambda) - \frac{1}{4} = \mathrm{Re}(z) + \mathrm{Re}(b_\lambda) - \frac{1}{4} > \mathrm{Re}(z) + \frac{1}{4}$$

for all $\lambda \in B(\lambda_0, R_1)$ and all

$$z \in U_2 := \big\{z \in \mathbb{C}: \ \mathrm{Re}(z) > \big((R/8)\max\{1,M\}\big)^p\big\} \subseteq U_1.$$

Analogously,

$$(4.8) \qquad\qquad\qquad \big|\tilde\varphi_\lambda(z)\big| \leq |z| + 2.$$

Hence, $\tilde\varphi_\lambda(U_2) \subseteq U_2$ for all $\lambda \in B(\lambda_0, R_1)$, and we get by a straightforward induction that

$$(4.9) \qquad \mathrm{Re}(z) + \frac{n}{4} \leq \mathrm{Re}\big(\tilde\varphi_\lambda^n(z)\big) \leq \big|\tilde\varphi_\lambda^n(z)\big| \leq |z| + 2n$$

for all $\lambda \in B(\lambda_0, R_1)$, all $z \in U_2$ and all $n \geq 0$. It follows from (4.4) that

$$(4.10) \quad \tilde\varphi_\lambda'(z) = 1 + \sum_{n=1}^\infty b_n(\lambda)nH(z)^{n-1}H'(z) = 1 - \frac{1}{p}z^{-1}\sum_{n=1}^\infty b_n(\lambda)nH(z)^n.$$

Now notice that there exists a constant $Q \geq 1$ such that $n(\frac{R}{4})^n \leq Q(\frac{R}{2})^n$ for all $n \geq 0$. Proceeding as in (4.5), we thus get for all $\lambda \in B(\lambda_0, \frac{R}{4})$ and all $z \in U_2$ that

$$(4.11) \quad \left|\sum_{n=1}^\infty nb_n(\lambda)H(z)^n\right| \leq \sum_{n=1}^\infty |b_n(\lambda)|n\left(\frac{R}{4}\right)^n\left(\left(\frac{R}{4}\right)^p|z|\right)^{-n/p}$$

$$\leq Q\sum_{n=1}^\infty |b_n(\lambda)|\left(\left(\frac{R}{4}\right)^p|z|\right)^{-n/p}$$

$$\leq MQ\left(\left(\frac{R}{4}\right)^p|z|\right)^{-1/p}$$

$$= 4MQR^{-1}|z|^{-1/p},$$

where writing the last inequality ("$\leq$") sign we were assuming that $|z| \geq (\frac{4}{R})^p$. Assume from now on that in the definition of U_2, the number $T > 0$ is taken

to be larger than $(4/R)^p$. Inserting (4.11) to (4.10), we get that

$$(4.12) \qquad \left|\varphi'_\lambda(z) - 1\right| \le 4MQ(pR)^{-1}|z|^{-(p+1)/p}.$$

Write $q_\lambda(z) = \tilde{\varphi}'_\lambda(z) - 1$. By the Chain Rule we have,

$$(4.13) \qquad \left(\tilde{\varphi}^n_\lambda\right)'(z) = \prod_{j=0}^{n-1} \tilde{\varphi}'_\lambda\left(\tilde{\varphi}^j_\lambda(z)\right) = \prod_{j=0}^{n-1}\left(1 + q_\lambda\left(\tilde{\varphi}^j_\lambda(z)\right)\right).$$

But, with $Q_1 = 4MQ(pR)^{-1}$, combining (4.12) and (4.9), we get

$$\left|q_\lambda\left(\tilde{\varphi}^j_\lambda(z)\right)\right| \le Q_1\left|\tilde{\varphi}^j_\lambda(z)\right|^{-(p+1)/p} \le Q_1\left(T + \frac{j}{4}\right)^{-(p+1)/p}.$$

Since the series $\sum_{j=0}^\infty (T + \frac{j}{4})^{-(p+1)/p}$ converges, taking $T > 0$ sufficiently large and looking at (4.13), we get the following.

LEMMA 4.7. *There exists a constant $Q_2 \ge 1$ such that*

$$Q_2^{-1} \le \left|\left(\tilde{\varphi}^n_\lambda\right)'(z)\right| \le Q_2$$

for all $(\lambda, z) \in B(\lambda_0, R_1) \times U_2$ and all $n \ge 0$.

Using the Chain Rule and the definition of $\tilde{\varphi}_\lambda$, we obtain

$$\begin{aligned}
\left|\left(\varphi^n_\lambda\right)'\left(H(z)\right)\right| &= \left|\left(H \circ \tilde{\varphi}^n_\lambda \circ H^{-1}\right)'\left(H(z)\right)\right| \\
&= \left|H'\left(\tilde{\varphi}^n_\lambda(z)\right)\right| \cdot \left|\left(\tilde{\varphi}^n_\lambda\right)'(z)\right| \cdot \left|\left(H^{-1}\right)'\left(H(z)\right)\right| \\
&= \frac{1}{p}\left|\tilde{\varphi}^n_\lambda(z)\right|^{-(p+1)/p}\left|\left(\tilde{\varphi}^n_\lambda\right)'(z)\right| \cdot \left|H'(z)\right|^{-1} \\
&= |z|^{-(p+1)/p}\left|\left(\tilde{\varphi}^n_\lambda\right)'(z)\right| \cdot \left|\tilde{\varphi}^n_\lambda(z)\right|^{-(p+1)/p}.
\end{aligned}$$

Combining this with Lemma 4.7 and (4.9) yields

$$(4.14) \quad Q_2^{-1}|z|^{-(p+1)/p}\left(|z| + 2n\right)^{-1} \le \left|\left(\varphi^n_\lambda\right)'\left(H(z)\right)\right|$$

$$\le Q_2|z|^{-(p+1)/p}\left(\operatorname{Re}(z) + \frac{n}{4}\right)^{-(p+1)/p}$$

for all $(\lambda, z) \in B(\lambda_0, R_1) \times U_2$ and all $n \ge 0$. Now, for every $\alpha \in (0, \pi)$ let

$$S_\alpha = \left\{z \in \mathbb{C} \setminus \{0\} : \left|\operatorname{Arg}(z)\right| < \alpha\right\}.$$

Then for every $\alpha \in (0, \pi/p)$, we have $H^{-1}(S_\alpha) = S_{\alpha p}$ and $H^{-1}(0) = \infty$. Now, fix $\alpha \in (0, \pi/(2p))$. Since $0 < p\alpha < \pi/2$, we conclude from the above that there exists $r_2 > 0$ so small that $\operatorname{Re}(H^{-1}(z)) \ge T$ for all $z \in S_\alpha \cap B(0, r_2)$. We therefore get from (4.14) the following.

LEMMA 4.8. *For every compact set $\Gamma \subseteq S_\alpha \cap B(0, r_2)$, where $\alpha \in (0, \pi/(2p))$, there exists a constant $Q_\Gamma \ge 1$ such that*

$$Q_\Gamma^{-1} n^{-(p+1)/p} \le \left|\left(\varphi^n_\lambda\right)'(z)\right| \le Q_\Gamma n^{-(p+1)/p}$$

for all $\lambda \in B(\lambda_0, R_1)$, all $z \in \Lambda$, and all $n \ge 1$.

As the essentially last step in the process of the verifying condition (c) of the regular analyticity of the family $\{\hat{S}_\lambda\}_{\lambda\in\Lambda}$ we prove first the following.

LEMMA 4.9. *Suppose that $\{S_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of holomorphic PGDMS. Fix $\lambda_0 \in \Lambda$. Then there exist a constant $Q \geq 1$ and radius $R_2 > 0$ such that*

$$Q^{-1}n^{-(p+1)/p} \leq \left|\left(\varphi^\lambda_{a^n b}\right)'(z)\right| \leq Qn^{-(p+1)/p}$$

for all $a \in \Omega$, all $b \in E \setminus \{a\}$ such that $A_{ba} = 1$, all $\lambda \in B(\lambda_0, R_2)$, all $z \in X^\lambda_{t(b)}$, and all $n \geq 1$.

Proof. Since the set E is finite it suffices to produce Q and R_2 for a fixed pair $(a, b) \in \Omega \times (E \setminus \{a\})$ such that $A_{ba} = 1$. Indeed, in virtue of Lemma 9.3.8 and Proposition 9.4.1 from [5], there exists $k \geq 1$ so large that $\varphi^{\lambda_0}_{a^n b}(X^{\lambda_0}_{t(b)}) \subseteq S_{\frac{\alpha}{4}} \cap B(x_a, r_2/4)$ for all $n \geq k$. By the Bounded Distortion Property, we may farther assume with $k \geq 1$ sufficiently large, and $r_3 \in (0, r_2]$, sufficiently small, that

$$\varphi^{\lambda_0}_{a^n b}(W_{t(b)}) \subseteq S_{\frac{\alpha}{3}} \cap B(x_a, r_2/3) \quad \text{and} \quad \varphi^{\lambda_0}_{a^k b}(W_{t(b)}) \cap B(x_a, 2r_3) = \emptyset$$

for all $n \geq k$. It then follows from analyticity of the function

$$\Lambda \times W_{t(b)} \ni (\lambda, z) \mapsto \varphi^\lambda_{a^k b}(z)$$

(since the family $\{\hat{S}_\lambda\}_{\lambda\in\Lambda}$ is analytic) and from the compactness of the set $Y_{t(b)}$, along with condition (d) of analyticity of S_λ, that there exists $R_2 \in (0, R_1)$ so small that

$$\varphi^\lambda_{a^k b}(X^\lambda_{t(b)}) \subseteq \varphi^\lambda_{a^k b}(Y_{t(b)}) \subseteq \left(S_{\alpha/2} \cap B(x_a, r_2/2)\right) \setminus B(x_a, r_3)$$

for all $\lambda \in B(\lambda_0, R_2)$. But then

$$\Gamma := \overline{\bigcup_{\lambda \in B(\lambda_0, R_2)} \hat{\varphi}^\lambda_{a^k b}(X^\lambda_{t(b)})} \subseteq \overline{S_{\alpha/2} \cap B(a, r_2/2)} \setminus B(a, r_3) \subseteq S_\alpha \cap B(a, r_2).$$

Since the middle set above is compact, so is Γ. Hence, applying Lemma 4.8, we conclude that

$$(4.15) \qquad Q_\Gamma^{-1}n^{-(p+1)/p} \leq \left|\left(\varphi^\lambda_{a^n}\right)'\left(\varphi^\lambda_{a^k b}\right)(z)\right| \leq Q_\Gamma n^{-(p+1)/p}$$

for all $z \in X^\lambda_{t(b)}$, all $n \geq 1$, and all $\lambda \in B(\lambda_0, R_2)$. Since, clearly,

$$0 < \inf\left\{\left|\left(\hat{\varphi}^\lambda_{a^j b}\right)'(z)\right| : 0 \leq j \leq k,\ z \in Y_{t(b)},\ \lambda \in B(\lambda_0, R_2)\right\}$$
$$\leq \sup\left\{\left|\left(\hat{\varphi}^\lambda_{a^j b}\right)'(z)\right| : 0 \leq j \leq k,\ z \in Y_{t(b)},\ \lambda \in B(\lambda_0, R_2)\right\} < +\infty,$$

using the Chain Rule, formula (4.15) yields the lemma. □

Since the set E is finite and since for every $\lambda \in \Lambda$ and every $\omega \in \hat{E}^\infty_{\hat{A}}$, we have $\pi_\lambda(\sigma(\omega)) \in X^\lambda_{t(\omega_1)}$, Lemma 4.9 yields immediately condition (c) of regular analyticity for the family $\{\hat{S}_\lambda\}_{\lambda\in B(\lambda_0, R_2)}$. The proof of the Theorem 4.6 is complete. □

Combining Theorem 4.6 with Theorem 4.5 we get the following.

COROLLARY 4.10. *If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of holomorphic PGDMS, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}(J_{S_\lambda})$ is real-analytic.*

5. PGDMS associated with f_λ, $\lambda \in D_0$

In this section, we apply the machinery developed in the previous sections to study the family of polynomials $f_\lambda(z) = z(1 - z - \lambda z^2)$, $\lambda \in D_0$, described in Section 2. The idea is to associate to this family a holomorphic family of holomorphic parabolic graph directed Markov systems whose limit sets coincide with the Julia sets of polynomials f_λ up to a countable set. Then to apply Corollary 4.10. Fix $\lambda \in D_0$. Let J_λ be the Julia set of f_λ and let K_λ be the corresponding filled in Julia set. Let $A_\lambda(\infty)$ be the basin of attraction to ∞ and let G_λ be Green's function for $A_\lambda(\infty)$ with the pole at ∞. It has the following properties.

$$(5.1) \qquad \begin{aligned} G_\lambda\big(f_\lambda(z)\big) &= 3G_\lambda(z), \quad z \in \mathbb{C}, \\ G_\lambda &\geq 0, \end{aligned}$$

and

$$(5.2) \qquad K_\lambda = G_\lambda^{-1}(0).$$

Let

$$\rho_\lambda = G_\lambda\big(c_\lambda^{(2)}\big).$$

Fix any $t_\lambda \in (0, \rho_\lambda)$ sufficiently close to ρ_λ. Then the set $G_\lambda^{-1}([0, t_\lambda])$ consists of two connected components. Denote by $\hat{W}_\lambda^0$ the component containing 0 and by $\hat{W}_\lambda^1$ the other one. It follows from (5.1) that $\{f_\lambda^n(c_\lambda^{(2)}) : n \geq 0\} \cap (\hat{W}_\lambda^0 \cup \hat{W}_\lambda^1) = \emptyset$, and from (5.2) that

$$\big\{f_\lambda^n\big(c_\lambda^{(1)}\big) : n \geq 0\big\} \subseteq \hat{W}_\lambda^0.$$

Consequently,

$$\big\{f_\lambda^n\big(c_\lambda^{(1)}\big), f_\lambda^n\big(c_\lambda^{(2)}\big) : n \geq 0\big\} \cap \hat{W}_\lambda^1 = \emptyset.$$

Starting a rather lengthy process of the definition of a PGDMS associated to f_λ, set

$$V = \{1, 2, 3\}.$$

Let $f_{\lambda,0}^{-1}$ be a maximal holomorphic continuation of the holomorphic inverse branch of f_λ defined on a sufficiently small neighborhood of 0 and sending 0 back to 0. Let Δ_λ^r be the repelling ray (emanating from 0) of f_λ. It follows from local behavior around parabolic points (see Section 4 for example) that there exists a triangle $T_\lambda^r \subseteq \hat{W}_\lambda^0$ symmetric with respect to Δ_λ^r with one vertex 0 and such that $f_{\lambda,0}^{-1}(T_\lambda^r) \subseteq T_\lambda^r$. Let ω_λ be the only point on $\Delta_\lambda \cap \partial T_\lambda^r$ different from 0. Then $f_{\lambda,0}^{-1}(\omega_\lambda) \in T_\lambda^r$ and let β_λ be the closed line segment (contained

in T) with end points ω_λ and $f_{\lambda,0}^{-1}(\omega_\lambda)$. Since the diameter of $f_{\lambda,0}^{-n}(\beta_\lambda)$ is of magnitude n^{-2} and $\lim_{n\to\infty} f_{\lambda,0}^{-n}(\omega_\lambda) = 0$, we conclude that

$$\beta_\lambda^\infty := \{0\} \cup \bigcup_{n=0}^\infty f_{\lambda,0}^{-n}(\beta_\lambda)$$

is a piecewise smooth (with countably many pieces) closed topological arc with end points 0 and β_λ. In addition, β_λ^∞ is tangent to Δ_λ^r at the point 0. We have

$$(5.3) \qquad f_\lambda(\beta_\lambda^\infty) = \{0\} \cup \bigcup_{n=1}^\infty f_{\lambda,0}^{-n}(\beta_\lambda) \subseteq \beta_\lambda^\infty.$$

Let $A_\lambda(0)$ be the basin of immediate attraction of f_λ to the rationally indifferent fixed point 0. Like above, let Δ_λ^c be the contracting ray (emanating from 0) of f_λ. Again as above, there exists a triangle $T_\lambda^c \subseteq A_\lambda(0) \cup \{0\}$ symmetric with respect to Δ_λ^c with vertex 0 and such that

$$(5.4) \qquad f_\lambda(T_\lambda^c) \subseteq \mathrm{Int}(T_\lambda^c) \cup \{0\}.$$

Let b_λ be the edge of the triangle T_λ^c not containing 0, i.e., the edge perpendicular to Δ_λ^c. We may require in addition that there exists $k \geq 1$ such that

$$(5.5) \qquad f_\lambda^k(c_\lambda^{(1)}) \in b_\lambda \quad \text{and} \quad \{c_\lambda^{(1)}, f_\lambda(c_\lambda^{(1)}), \dots, f_\lambda^{(k-1)}(c_\lambda^{(1)})\} \cap T_\lambda^c = \emptyset.$$

Take now a little open ball B_λ^1 centered at 0 and disjoint from the set $\{c_\lambda^{(1)}, f_\lambda(c_\lambda^{(1)}), \dots, f_\lambda^k(c_\lambda^{(1)})\}$. Take also an open topological disk $D_\lambda \supseteq B_\lambda^1 \cup T_\lambda^c$ which is disjoint from the set $\{c_\lambda^{(1)}, f_\lambda(c_\lambda^{(1)}), \dots, f_\lambda^{(k-1)}(c_\lambda^{(1)})\}$. Then, for every $j = 1, \dots, k-1$ there exists a unique holomorphic inverse branch $f_{\lambda,0}^{-j} : D_\lambda \to A_\lambda(0)$ sending 0 to 0. There also exists a unique holomorphic inverse branch $f_{\lambda,0}^{-k} : B_\lambda^1 \cup \mathrm{Int}(T_\lambda^c) \to A_\lambda(0)$ sending 0 to 0. Note that for all $j = 1, \dots, k$

$$(5.6) \qquad f_\lambda \circ f_{\lambda,0}^{-j} = f_{\lambda,0}^{-(j-1)}$$

and, by (5.4), for all $1 \leq j \leq k-1$,

$$(5.7) \qquad f_{\lambda,0}^{-j}(T_\lambda^c) \supseteq f_{\lambda,0}^{-(j-1)}(T_\lambda^c).$$

If $j = k$, then

$$(5.8) \qquad f_{\lambda,0}^{-j}(\mathrm{Int}(T_\lambda^c)) \supseteq f_{\lambda,0}^{-(j-1)}(\mathrm{Int}(T_\lambda^c)).$$

In particular, $f_{\lambda,0}^{-(k-1)}(f_\lambda^k(c_\lambda^{(1)})) = f_\lambda(c_\lambda^{(1)})$, and, as $f_\lambda^{-1}(f_\lambda(c_\lambda^{(1)})) \cap A_\lambda(0) = \{c_\lambda^{(1)}\}$, it follows from (5.5) that $c_\lambda^{(1)} \in \partial f_{\lambda,0}^{-k}(\mathrm{Int}(T_\lambda^c))$. Note also that $f_{\lambda,0}^{-k} = \tilde{f}_{\lambda,0}^{-1} \circ f_{\lambda,0}^{-(k-1)}$, where $\tilde{f}_{\lambda,0}^{-1}$ is the extension of $f_{\lambda,0}^{-1}$ on $f_{\lambda,0}^{-(k-1)}(B_\lambda^1 \cup \mathrm{Int}(T_\lambda^c))$. But, there also exists a second holomorphic inverse branch $\tilde{f}_{\lambda,1}^{-1}$ of f_λ defined

on $f_{\lambda,0}^{-(k-1)}(B_\lambda \cup \text{Int}(T_\lambda^c))$ and mapping $\text{Int}(T_\lambda^c)$ into $A_\lambda(0)$. Put $f_{\lambda,1}^{-k} = \tilde{f}_{\lambda,1}^{-1} \circ f_{\lambda,0}^{-(k-1)}$. As above $c_\lambda^{(1)} \in \partial f_{\lambda,1}^{-k}(\text{Int}(T_\lambda^c))$. We thus have

$$f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big) \cap f_{\lambda,1}^{-k}\big(\text{Int}(T_\lambda^c)\big) = \emptyset \quad \text{and}$$
$$\overline{f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big)} \cap \overline{f_{\lambda,1}^{-k}\big(\text{Int}(T_\lambda^c)\big)} = \{c_\lambda^{(1)}\}.$$

Put $\alpha_\lambda = f_{\lambda,1}^{-k}(1)$. By continuity of $f_{\lambda,1}^{-k}$, we get that $\alpha_\lambda \in f_{\lambda,1}^{-1}((T_\lambda^c) \subseteq \overline{f_{\lambda,1}^{-1}(\text{Int}(T_\lambda^c))} \subseteq \overline{A_\lambda(0)}$, and since $\alpha_\lambda \in J(f_\lambda)$, we obtain that

$$\alpha_\lambda \in \partial A_\lambda(0).$$

In virtue of (5.4), (5.5) and (5.6), $f_\lambda^2(c_\lambda^{(1)}) \in f_{\lambda,0}^{-(k-1)}(\text{Int}(T_\lambda^c))$. Thus, there exists a little open disk B_λ centered at $c_\lambda^{(1)}$ such that

(5.9) $$f_\lambda^2(B_\lambda) \subseteq f_{\lambda,0}^{-(k-1)}\big(\text{Int}(T_\lambda^c)\big).$$

Set

$$H_\lambda = f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big) \cup f_{\lambda,1}^{-k}\big(\text{Int}(T_\lambda^c)\big) \cup B_\lambda \cup f_\lambda(B_\lambda) \cup f_\lambda^2(B_\lambda) \subset A_\lambda(0).$$

We have, by (5.6), (5.7), and (5.8), that

$$f_\lambda\big(f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big) \cup f_{\lambda,1}^{-k}\big(\text{Int}(T_\lambda^c)\big)\big)$$
$$= f_\lambda\big(f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big)\big) \cup f_\lambda\big(f_{\lambda,1}^{-k}\big(\text{Int}(T_\lambda^c)\big)\big)$$
$$= f_{\lambda,0}^{-(k-1)}\big(\text{Int}(T_\lambda^c)\big) \cup f_{\lambda,0}^{-(k-1)}\big(\text{Int}(T_\lambda^c)\big)$$
$$= f_{\lambda,0}^{-(k-1)}\big(\text{Int}(T_\lambda^c)\big) \subseteq f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big)$$
$$\subset H_\lambda.$$

By (5.9) and (5.8), we get

$$f_\lambda^2(B_\lambda) \subseteq f_{\lambda,0}^{-(k-1)}\big(\text{Int}(T_\lambda^c)\big) \subseteq f_{\lambda,0}^{-k}\big(\text{Int}(T_\lambda^c)\big) \subseteq H_\lambda.$$

Thus,

(5.10) $$f_\lambda(H_\lambda) \subseteq H_\lambda \quad \text{and} \quad f_\lambda(\overline{H}_\lambda) \subseteq \overline{H}_\lambda.$$

Let

(5.11) $$\tilde{\beta}_\lambda^\infty = \tilde{f}_{\lambda,1}^{-1}\big(f_\lambda(\beta_\lambda^\infty)\big) \quad \text{and} \quad \tilde{\omega}_\lambda = \tilde{f}_{\lambda,1}^{-1}\big(f_\lambda(\omega_\lambda)\big).$$

Set

$$s_\lambda = G_\lambda(\omega_\lambda) = G_\lambda(\tilde{\omega}_\lambda) < t_\lambda.$$

Define X_1^λ to be the connected component of $G_\lambda^{-1}([0, s_\lambda])$ not containing 0. Then X_1^λ is simply connected and, consequently, containing $\overline{A_\lambda(1)}$. Let Z^λ be the other connected component of $G_\lambda^{-1}([0, s_\lambda])$, i.e., the one containing 0. The set Z^λ is connected and simply connected (closed topological disk with smooth boundary). By its construction the set $\overline{H}_\lambda$ is connected and simply connected

too. Since, in addition, $\beta_\lambda^\infty \cap \overline{H}_\lambda = \{0\}$, $\tilde{\beta}_\lambda^\infty \cap \overline{H}_\lambda = \{\alpha_\lambda\}$, $\beta_\lambda^\infty \cap \tilde{\beta}_\lambda^\infty = \emptyset$, and since both β_λ^∞ and $\tilde{\beta}_\lambda^\infty$ are closed arcs, the closed set

$$(5.12) \qquad F_\lambda := \beta_\lambda^\infty \cap \overline{H}_\lambda \cap \tilde{\beta}_\lambda^\infty \subseteq Z^\lambda$$

is connected and simply connected. We have

$$F_\lambda \cap \partial Z^\lambda = \{\omega_\lambda, \tilde{\omega}_\lambda\}.$$

In consequence, the set $Z^\lambda \setminus F_\lambda$ has two connected components. Label their closures by X_2^λ and X_3^λ. By the construction,

$$(5.13) \qquad X_3^\lambda \cap \overline{\bigcup_{n=0}^\infty f_\lambda^n \left(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}\right)} = \emptyset,$$

$$(5.14) \qquad \left(X_2^\lambda \cup X_3^\lambda\right) \cap \overline{\bigcup_{n=0}^\infty f_\lambda^n \left(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}\right)} = \{0\},$$

and all these sets X_1^λ, X_2^λ and X_3^λ are simply connected. Hence, all the three holomorphic inverse branches of f_λ are well defined on each set X_1^λ, X_2^λ and X_3^λ. Since the polynomial f_λ is of degree 3, for every $a \in \{1, 2, 3\}$ there are three holomorphic inverse branches $f_{\lambda,(a,1)}^{-1}$, $f_{\lambda,(a,2)}^{-1}$ and $f_{\lambda,(a,3)}^{-1}$ of f_λ defined on X_1^λ, X_2^λ and X_3^λ, respectively if $a = 1, 2$, or 3. Consider first the case when $a = 3$. Then

$$f_\lambda^{-1}\left(X_1^\lambda\right) = G_\lambda^{-1}\left([0, s_\lambda/3]\right) \subseteq G_\lambda^{-1}\left([0, s_\lambda]\right).$$

Let Y_1 be the connected component of $f_\lambda^{-1}(X_1^\lambda)$ not containing 0 and let $f_{(1,1)}^{-1} : X_1 \to \mathbb{C}$ be the corresponding holomorphic inverse branch of f_λ. Thus,

$$(5.15) \qquad f_{\lambda,(1,1)}^{-1}\left(X_1^\lambda\right) \subseteq X_1^\lambda.$$

Let $f_{\lambda,(1,b)}^{-1}$ be another holomorphic inverse branch of f_λ defined on X_1^λ. We then have

$$(5.16) \qquad f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \subseteq Z^\lambda.$$

Looking at (5.10), (5.3), and (5.11), we see that

$$(5.17) \qquad f_\lambda(F_\lambda) \subseteq F_\lambda.$$

Since, in addition, $X_1^\lambda \cap Z^\lambda = \emptyset$ and $F_\lambda \subseteq Z^\lambda$, we conclude that $f_{\lambda,(1,b)}^{-1}(X_1^\lambda) \cap F_\lambda = \emptyset$. Along with (5.16) this gives that

$$f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \subseteq Z^\lambda \setminus F_\lambda.$$

Since $f_{\lambda,(1,b)}^{-1}(X_1^\lambda)$ is a connected set it must be contained in one of the two connected components of $Z^\lambda \setminus F_\lambda$, the more in the closure of one of these two

components. Set $b = 2$ if this closure is X_2^λ and set $b = 3$ if it is X_3^λ. We thus have

(5.18) $$f_{\lambda,(1,2)}^{-1}(X_1^\lambda) \subseteq X_2^\lambda \quad \text{and} \quad f_{\lambda,(1,3)}^{-1}(X_1^\lambda) \subseteq X_3^\lambda.$$

Now consider the case when $a \in \{2,3\}$. Without loss of generality, we may assume that $a = 2$. Since the map f_λ restricted to $A_\lambda(1)$ is of degree 2, there are two branches of f_λ^{-1} defined on X_2^λ whose images intersect $A_\lambda(1)$. Fix one of them and label it by $f_{\lambda,(2,b)}^{-1}$, $b \in \{2,3\}$. Since

$$f_{\lambda,(2,b)}^{-1}(X_2^\lambda) \subseteq f_\lambda^{-1}(Z^\lambda) \subseteq G_\lambda^{-1}([0, s_\lambda/3]) \subseteq G_\lambda^{-1}([0, s_\lambda]),$$

since $f_{\lambda,(2,b)}^{-1}$ is a connected set, and since $f_{\lambda,(2,b)}^{-1}(X_2^\lambda) \cap A_\lambda(1) \neq \emptyset$, we conclude that

(5.19) $$f_{\lambda,(2,b)}^{-1}(X_2^\lambda) \subseteq Z^\lambda.$$

Since $\text{Int}(X_2^\lambda) \cap F_\lambda = \emptyset$, we see from (5.17) that $f_{\lambda,(2,b)}^{-1}(\text{Int}(X_2^\lambda)) \cap F_\lambda = \emptyset$. Together with (5.19) this yields

$$f_{\lambda,(2,b)}^{-1}(\text{Int}(X_2^\lambda)) \subseteq Z^\lambda \setminus F_\lambda.$$

The same argument as above then gives that

$$f_{\lambda,(2,b)}^{-1}(\text{Int}(X_2^\lambda)) \subseteq X_2^\lambda \quad \text{or} \quad f_{\lambda,(2,b)}^{-1}(\text{Int}(X_2^\lambda)) \subseteq X_3^\lambda.$$

Thus,

(5.20) $$f_{\lambda,(2,b)}^{-1}(X_2^\lambda) \subseteq X_2^\lambda \quad \text{or} \quad f_{\lambda,(2,b)}^{-1}(X_2^\lambda) \subseteq X_3^\lambda.$$

Put b equal 2 or 3 according to whether the first or the second part of the above alternative holds. Note that

(5.21) $$f_{\lambda,(2,2)}^{-1}(0) = 0 \quad \text{and} \quad f_{\lambda,(2,3)}^{-1}(0) = \alpha_\lambda.$$

It is left to consider the branch $f_{\lambda,(2,1)}^{-1} : X_2^\lambda \to \mathbb{C}$ characterized by the property that

$$f_{\lambda,(2,1)}^{-1}(X_2^\lambda) \cap A_\lambda(1) = \emptyset.$$

By (5.19), $f_{\lambda,(2,2)}^{-1}(X_2^\lambda) \cup f_{\lambda,(2,3)}^{-1}(X_2^\lambda) \subseteq Z^\lambda$. Therefore, since f_λ is of degree 2 on Z^λ, we have that

(5.22) $$f_{\lambda,(2,1)}^{-1}(X_2^\lambda) \cap Z^\lambda = \emptyset.$$

But $f_{\lambda,(2,1)}^{-1}(X_2^\lambda) \subseteq G_\lambda^{-1}([0, s_\lambda])$, and since $f_{\lambda,(2,3)}^{-1}(X_2^\lambda)$ is connected, we conclude from (5.22) and from the definition of Z^λ and X_2^λ, that,

(5.23) $$f_{\lambda,(2,1)}^{-1}(X_2^\lambda) \subseteq X_1^\lambda.$$

Now, we shall define the open simply connected sets W_1^λ, W_2^λ and W_3^λ. Fix $\xi \in (s,t)$ and define W_1^λ to be the connected component of $G_\lambda^{-1}([0,\xi])$ not

containing 0. Clearly W_1^λ is an open topological disk with smooth boundary and

(5.24) $X_1^\lambda \subseteq W_1^\lambda$ and $\overline{W}_1^\lambda \cap \overline{\bigcup_{n=0}^{\infty} \{f_\lambda^n(c_\lambda^{(1)}), f_\lambda^n(c_\lambda^{(2)})\}} = \emptyset.$

Since, by (5.20), $f_{\lambda,(2,3)}^{-1}(X_2^\lambda) \subseteq X_3^\lambda$, and since, by (5.21), $0 \notin f_{\lambda,(2,3)}^{-1}(X_2^\lambda)$, it follows from (5.14) that there exists an open topological disk $U_2^\lambda \supseteq f_{\lambda,(2,3)}^{-1}(X_2^\lambda)$ whose closure is disjoint from $\overline{\bigcup_{n=0}^{\infty} \{f_\lambda^n(c_\lambda^{(1)}), f_\lambda^n(c_\lambda^{(2)})\}}$. In virtue of (5.14), there exists an open topological disk $W_2^\lambda \subseteq \mathbb{C}$ with the following properties.

(a) $X_2^\lambda \subseteq W_2^\lambda \subseteq G_\lambda^{-1}([0,\xi))$,

(b) $f_\lambda(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}) \cap \overline{W}_2^\lambda = \emptyset$, and if $f_{\lambda,(2,3)}^{-1}$ is a holomorphic extension of $f_{\lambda,(2,3)}^{-1}$ onto W_2^λ (which exists because of (b) and for which we keep the same symbol $f_\lambda, (2,3)^{-1}$), then

(c) $\overline{f_{\lambda,(2,3)}^{-1}(W_2^\lambda)} \subset U_2^\lambda.$

And from (c),

(5.25) $\overline{f_{\lambda,(2,3)}^{-1}(W_2^\lambda)} \cap \overline{\bigcup_{n=0}^{\infty} \{f_\lambda^n(c_\lambda^{(1)}), f_\lambda^n(c_\lambda^{(2)})\}} = \emptyset.$

The sets U_3^λ and W_3^λ are defined verbatim with 2 and 3 mutually interchanged. In virtue of (5.24) and (a), there exists $\delta > 0$ such that

(5.26) $B(X_i^\lambda, \delta) \subseteq W_i^\lambda$

for all $i = 1, 2, 3$. By (5.24), the family $\mathcal{F}_1^\lambda$ of all holomorphic inverse branches of all iterates of f_λ is well-defined on an open set containing $\overline{W}_1^\lambda$. Since $J(f_\lambda) \cap W_1^\lambda \neq \emptyset$ this family is normal and all its limit functions are constant. Likewise, the family $\mathcal{F}_2^{\lambda\prime}$ of all holomorphic inverse branches of all iterates of f_λ is well defined on U_2^λ. Let $\mathcal{F}_2^\lambda = \{\phi \circ f_{(2,3)}^{-1} : \phi \in \mathcal{F}_2^\lambda\}$ and let $\mathcal{F}_3^\lambda$ be defined analogously. Again, since $J(f_\lambda) \cap W_2^\lambda \neq \emptyset$ and $J(f) \cap W_3^\lambda \neq \emptyset$, both families $\mathcal{F}_2^\lambda$ and $\mathcal{F}_3^\lambda$ are normal and all their limit functions are constant. Using (c), we therefore conclude that there exists $q_\lambda = q \geq 1$ such that if $\phi \in \mathcal{F}^\lambda := \mathcal{F}_1^\lambda \cup \mathcal{F}_2^\lambda \cup \mathcal{F}_3^\lambda$ is a holomorphic inverse branch of f_λ^n with $n \geq q$ (we say $\phi \in \mathcal{F}_n^\lambda$), then

(5.27) $\mathrm{diam}(\phi(W_i^\lambda)) < \delta$ and $\sup\{|\phi'(z)| : z \in W_i^\lambda\} < \frac{1}{2},$

where $i = 1, 2$ or 3 according to whether $\phi \in \mathcal{F}_i^\lambda$. Note that each such element $\phi \in \mathcal{F}_i^\lambda$ forms a unique holomorphic extension of some unique element $f_{\lambda,\omega_1}^{-1} \circ f_{\lambda,\omega_2}^{-1} \circ \cdots \circ f_{\lambda,\omega_n}^{-1}$, where all $\omega_j \in \{1,2,3\}^2$. Let S_λ be the system determined by the set of vertices $V = \{1,2,3\}$, the set of edges $E_q = (\{1,2,3\})^q$, the spaces X_v^λ and W_ω^λ, $v \in V$, described above, the maps $t(a_1, b_1, a_2, b_2, \ldots, a_q, b_q) =$

a_q, $i(a_1, b_1, a_2, b_2, \ldots, a_q, b_q) = b_1$, the generators $f_{\lambda,\tau}^{-q} : X_{t(\tau)}^\lambda \to X_{i(\tau)}^\lambda$, $\tau = (a_1, b_1, a_2, b_2, \ldots, a_q, b_q) \in E$ and $f_{\lambda,\tau}^{-q} = f_{\lambda,(a_1,b_1)}^{-1} \circ f_{\lambda,(a_2,b_2)}^{-1} \circ \cdots \circ f_{\lambda,(a_q,b_q)}^{-1}$, $\Omega = \{(2, 2, \ldots, 2), (3, 3, \ldots, 3)\}$, and the incidence matrix $A : E \times E \to \{0, 1\}$ consisting of all entries equal to 1. After all these definitions and preparations, it is rather easy to prove the following proposition.

PROPOSITION 5.1. *For every $\lambda \in D_0$, S_λ is a pf-irreducible PGDMS.*

Proof. Conditions (2a) and (2d) follow directly from the definition of the sets X_v^λ, $v \in V$. Condition (2b) is fulfilled by (5.15), (5.18), (5.20) (taken also with 2 replaced by 3) and the definition of $f_{\lambda,\tau}^{-q}$, $\tau \in E_q$, given above. Condition (2c) follows from the fact that the interiors $\{\mathrm{Int}(X_v^\lambda)\}_{v \in V}$ are mutually disjoint and that the generators of the system S are formed by continuous inverse branches of a single map, namely f_λ^q. Let us deal with condition (2e). If $\omega \in E^*$ is a hyperbolic word, say $|\omega| = n$, then $\phi_\omega \in \mathcal{F}_n^\lambda$, and it follows from (5.27) that $\mathrm{diam}(\phi_\omega(W_{t(\omega)}^\lambda)) < \delta$. But $\phi_\omega(X_{t(\omega)}^\lambda) \subseteq X_{i(\omega)}^\lambda$, and using (5.26), we conclude that

$$\phi_\omega\big(W_{t(\omega)}^\lambda\big) \subseteq B\big(X_{i(\omega)}^\lambda, \delta\big) \subseteq W_{i(\omega)}^\lambda.$$

Condition (2e) is established. Conditions (2f) and (2g) follow directly from Koebe Distortion Theorem. Condition (2h) is established by (5.27). To see that conditions (2i) and (2j) hold, consider without loss of generality the parabolic map $\phi_{(2,2)^q}$. It is enough to note that the sets $\phi_{(2,2)^q}^n(2B \cap X_2^\lambda)$ converge to the parabolic point 0, by the local behavior of parabolic points, where B is a sufficiently small ball centered at 0, and that the family of maps $\phi_{(2,2)^q}^n$, restricted to some sufficiently small neighborhood of $X_2^\lambda \setminus B$, is well-defined and normal. In conclusion, S_λ is a parabolic graph directed Markov system. It is obvious that S_λ is a pf-system since the incidence matrix A consists of 1s only and since $E_q \setminus \Omega$ is not empty. We are done. $\square$

Now, we shall prove the following.

LEMMA 5.2. *For every $\lambda_0 \in D_0$ there exists $R_0 > 0$ such that with suitably chosen sets X_v^λ, $\lambda \in B(\lambda_0, R_0)$, the family $\{S_\lambda\}_{\lambda \in B(\lambda_0, R_0)}$ of holomorphic PGDMSs is holomorphic.*

Proof. It follows from the construction of systems S_λ and local behavior of maps f_λ around zero, that the only non-trivial task to be done is to verify that the family $\{S_\lambda\}_{\lambda \in D_0}$ satisfies conditions (c), (d), and (b) of the definition of holomorphic families of holomorphic PGDMSs. In order to do it, fix $\lambda_0 \in D_0$. Put $O_{\lambda_0} = G_{\lambda_0}^{-1}((3\rho_{\lambda_0}, +\infty])$. Then $f_{\lambda_0}(O_{\lambda_0}) \subseteq G_{\lambda_0}^{-1}((9\rho_{\lambda_0}, +\infty])$ and taking $R_1 > 0$ sufficiently small, we will have $f_\lambda(O_{\lambda_0}) \subseteq G_{\lambda_0}^{-1}((8\rho_{\lambda_0}, +\infty]) \subseteq O_{\lambda_0}$ for all $\lambda \in B(\lambda_0, R_1)$. Consequently,

(5.28) $$f_\lambda^{-1}\big(G_{\lambda_0}^{-1}([0, 3\rho_{\lambda_0}])\big) \subseteq G_{\lambda_0}^{-1}\big([0, 3\rho_{\lambda_0}]\big).$$

We have by our construction,

$$(5.29) \quad W_1^{\lambda_0} \cup W_2^{\lambda_0} \cup W_3^{\lambda_0} \subseteq G_{\lambda_0}^{-1}\big([0, t_{\lambda_0}]\big) \subseteq G_{\lambda_0}^{-1}\big([0, \rho_{\lambda_0}]\big) \subseteq G_{\lambda_0}^{-1}\big([0, 3\rho_{\lambda_0}]\big).$$

There also exists $R_2 \in (0, R_1]$ so small that

$$\big(U_2^{\lambda_0} \cup U_3^{\lambda_0}\big) \cap \bigcup_{\lambda \in B(\lambda_0, R_1)} \bigcup_{n=0}^{\infty} \big\{f_\lambda^n\big(c_\lambda^{(1)}\big), f_\lambda^n\big(c_\lambda^{(2)}\big)\big\} = \emptyset.$$

From continuity of the functions $D_0 \ni \lambda \mapsto c_\lambda^{(1)}, c_\lambda^{(2)}$, and consequently, of the function, $\lambda \mapsto \rho_{\lambda_1}$, we can choose the numbers $s_\lambda < \zeta_\lambda$ such that

$$(5.30) \qquad\qquad \lim_{\lambda \to \lambda_0} s_\lambda = s_{\lambda_0} < \zeta_{\lambda_0} = \lim_{\lambda \to \lambda_0} \zeta_\lambda.$$

We can also choose ω_λ so that $\lim_{\lambda \to \lambda_0} \omega_\lambda = \omega_{\lambda_0}$, $\lim_{\lambda \to \lambda_0} T_\lambda^c = T_{\lambda_0}^c$, $\lim_{\lambda \to \lambda_0} \overline{B}_\lambda = \overline{B}_{\lambda_0}$ (the two latter in the sense of Hausdorff metric on compact subsets of the complex plane $\mathbb{C}$). Consequently, also $\lim_{\lambda \to \lambda_0} \beta_\lambda^\infty = \beta_{\lambda_0}^\infty$, $\lim_{\lambda \to \lambda_0} \tilde{\beta}_\lambda^\infty = \tilde{\beta}_{\lambda_0}^\infty$, and $\lim_{\lambda \to \lambda_0} \overline{H}_\lambda = \overline{H}_{\lambda_0}$. Therefore (see (5.12))

$$(5.31) \qquad\qquad \lim_{\lambda \to \lambda_0} F_\lambda = F_{\lambda_0}.$$

It follows immediately from (5.30) that

$$\lim_{\lambda \to \lambda_0} X_1^\lambda = X_1^{\lambda_0} \quad \text{and} \quad \lim_{\lambda \to \lambda_0} Z^\lambda = Z^{\lambda_0}.$$

Along with (5.31), this implies that

$$\lim_{\lambda \to \lambda_0} X_i^\lambda = X_i^{\lambda_0} \quad \text{for } i = 1, 2, 3.$$

Because of this and (5.30), we can find for every $i = 1, 2, 3$ one open set $U_i^{\lambda_0}$ and one open set $W_i^{\lambda_0} \subseteq \tilde{W}_i^{\lambda_0}$ that all satisfy all the requirements for the sets U_i^λ and W_i^λ from the construction leading to Proposition 5.1 up to formula (5.26) if λ is sufficiently close to λ_0, say $\lambda \in B(\lambda_0, R_3)$, $R_3 \in (0, R_2]$. That is, we can from now on either set

$$W_i^\lambda := W_i^{\lambda_0} \quad \text{or} \quad W_i^\lambda := \tilde{W}_i^{\lambda_0}$$

for all $i = 1, 2, 3$ and for all $\lambda \in B(\lambda_0, R_3)$. We can even find compact sets Y_1, Y_2 and Y_3 such that $X_i^\lambda \subseteq Y_i \subseteq W_i^{\lambda_0}$ for all $i = 1, 2, 3$ and all $\lambda \in B(\lambda_0, R_3)$. Hence, the condition (d) of the definition of holomorphic families of holomorphic PGDMS is satisfied. Recall that for every $\lambda \in B(\lambda_0, R_3)$ the family $\mathcal{F}^\lambda$ is bijectively parametrized by the set $\tilde{E}^*$, where $E = \{1, 2, 3\}^2$, $\Omega = \{(2, 2), (2, 3)\}$, and $\hat{E}$ is defined accordingly. In fact, in view of our construction of the sets $W_i^{\lambda_0}$, it follows from the Implicit Function theorem and Monodromy theorem, that for every $\omega \in \hat{E}^*$, there exists a holomorphic function $g_\omega : B(\lambda_0, R_3) \times \tilde{W}_{t(\omega)}^{\lambda_0} \to \mathbb{C}$ such that, abusing slightly notation, we have

$$\big\{g_\omega|_{\{\lambda\} \times \tilde{W}_i^{\lambda_0}} : \omega \in \hat{E}^* \text{ and } t(\omega) = i\big\} = \mathcal{F}_i^\lambda$$

for all $i = 1, 2, 3$. In virtue of (5.28) and (5.29) we have,

$$g_\omega\big(B(\lambda_0, R_3) \times \tilde{W}^{\lambda_0}_{t(\omega)}\big) \subseteq G^{-1}_{\lambda_0}\big([0, 3\rho_{\lambda_0}]\big)$$

for all $\omega \in \hat{E}^*$. Since the set $G^{-1}_{\lambda_0}([0, 3\rho_{\lambda_0}])$ is bounded, we thus conclude that for each $i = 1, 2, 3$, the family

$$\Gamma_i = \big\{ g_\omega : B(\lambda_0, R_3) \times \tilde{W}^{\lambda_0}_{t(\omega)} : \omega \in \hat{E}^* \text{ and } t(\omega) = i \big\}$$

is normal. Since for each $\lambda \in B(\lambda_0, R_3)$ and each $i \in \{1, 2, 3\}$, $X^\lambda_i \subseteq W^{\lambda_0}_i$ and $J_\lambda \cap X^\lambda_i \neq \emptyset$, all the limit functions of the normal family $\mathcal{F}^\lambda_i$ are constant. But this means that all the limit functions of the family Γ_i depend only on the first coordinate λ. Therefore (remember that $W^{\lambda_0}_i \subset \tilde{W}^{\lambda_0}_i$), there exists $R_4 \in (0, R_3]$ and $q \geq 1$ such that

$$\mathrm{diam}\big(g_\omega\big(B(\lambda_0, R_4) \times \tilde{W}^{\lambda_0}_i\big)\big) < \delta$$

and

$$\sup\left\{ \left| \frac{\partial g_\omega}{\partial z}(\lambda, z) \right| : (\lambda, z) \in B(\lambda_0, R) \times W^{\lambda_0}_i \right\} < \frac{1}{2}$$

for all $i \in \{1, 2, 3\}$ and all $\omega \in \hat{E}^*$ with $|\omega| \geq q$. Now, as in the previous section, we conclude from this, (5.26), the inclusion $\phi^\lambda_\omega(X^\lambda_{t(\omega)}) \subseteq X^\lambda_{t(\omega)}$, and equality $\phi^\lambda_\omega = g_{\omega}|_{\{\lambda\} \times B(\lambda_0, R_4)}$ (with obvious abuse of notation) that

$$\phi^\lambda_\omega\big(W^{\lambda_0}_{t(\omega)}\big) \subseteq B\big(X^{\lambda_0}_{t(\omega)}, \delta\big) \subseteq W^{\lambda_0}_{t(\omega)}$$

for all $\lambda \in B(\lambda_0, R_4)$ and all $\omega \in \hat{E}^*$ with $|\omega| \geq q$. Now, define the systems S_λ, $\lambda \in B(\lambda_0, R_4)$, as appearing in Proposition 5.1 with the help of this same $q \geq 1$. Since obviously all the maps $B(\lambda_0, R) \times W^{\lambda_0}_i \ni (\lambda, z) \mapsto \phi^\lambda_\omega(z)$, $\omega \in \hat{E}^*$ are holomorphic, the family $\{\hat{S}_\lambda\}_{\lambda \in B(\lambda_0, R_4)}$ is analytic, meaning that condition (b) of the definition of holomorphic families of holomorphic PGDMSs is satisfied. Since clearly, all the maps $B(\lambda_0, R) \times B(0, R_0) \ni (\lambda, z) \mapsto \phi^\lambda_e(z)$, with e being $(2, 2)^q$ or $(3, 3)^q$ and $R_0 \in [0, R_4]$ sufficiently small, are holomorphic, we see that condition (c) of the definition of holomorphic families of holomorphic PGDMSs is satisfied, and we may therefore conclude that the family $\{S_\lambda\}_{\lambda \in B(\lambda_0, R_0)}$ is holomorphic. We are done. □

As an immediate consequence of this lemma and Corollary 4.10, we get the following main result of our paper.

THEOREM 5.3. *The Hausdorff dimension function* $D_0 \ni \lambda \mapsto \mathrm{HD}(J(f_\lambda))$ *is real-analytic.*

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