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ZERO SETS OF REAL POLYNOMIALS CONTAINING COMPLEX VARIETIES

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ABSTRACT. We give necessary conditions for real algebraic hypersurfaces (possibly with singularities) to contain nontrivial germs of complex hypersurfaces. Moreover, if a real hypersurface S in $\mathbf{C}^2$ is defined by a real polynomial of a sufficiently general form and if S contains a nontrivial analytic disk, then, using the above result, we show that S must contain certain complex lines.

1. Introduction

Given a real polynomial p in $\mathbf{C}^n, n \geq 2$, we are concerned with the problem of finding *explicit* necessary and sufficient conditions so that $S := \{z : p(z) = 0\}$ contains a germ of a complex variety i.e., there exists a complex variety V in some open subset U of $\mathbf{C}^n$ such that $V \subset S$. This problem is connected with a well known theorem of Trépreau in [Tr] stating that if a is a *smooth* point of a real hypersurface S such that there is no germ of a complex hypersurface passing through a, then there exists a one side neighbourhood of a in S for which we have the extension property i.e., there exists a neighbourhoods basis $\{U_j\}_{j\geq 1}$ of a and a connected component Ω of $\mathbf{C}^n \backslash S$ such that holomorphic functions on $\Omega \cap U_j$ extends holomorphically to U_j for every $j \geq 1$. The main result of this note (Theorem 2.2) asserts that if a real algebraic hypersurface S contains a a germ of complex variety then S must include an algebraic variety. Moreover, if S contains an algebraic hypersurface then we obtain an algebraic decomposition formula for the defining function of S. It follows easily from this result that if S is defined by the vanishing locus of a real valued, real homogeneous polynomial in $\mathbf{C}^2$ then either S does not contain a germ of a complex hypersurface or S includes a complex line passing through the origin (cf. Proposition 2.3). Moreover, we also give in Corollary 2.4,

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sufficient conditions for a wide class of real algebraic hypersurface in $\mathbf{C}^2$ to contain a nontrivial analytic disk. Here by a nontrivial analytic disk we mean the image in $\mathbf{C}^n$ of the unit disk in $\mathbf{C}$ under some nonconstant holomorphic map. The final result of the note is a complete description of complex curves lying in a certain class of real algebraic hypersurfaces in $\mathbf{C}^2$.

We should say that a different approach to our problem, using an interesting variant of the classical Frobenius theorem, has been discussed in [HT].

2. Main results

We adopt the following terminology. By a complex (resp. real) polynomial in $\mathbf{C}^n$, we mean a polynomial in $z_1, \ldots, z_n$ (resp. in $z_1, \overline{z_1}, \ldots, z_n, \overline{z_n}$), where $(z_1, \ldots, z_n)$ are coordinates of $\mathbf{C}^n$. For a point $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, we denote $\overline{z} = (\overline{z_1}, \ldots, \overline{z_n})$. Given a real analytic function f on an open subset U of $\mathbf{C}^n$, we denote by $\hat{f}$ the complexification of f i.e., $\hat{f}$ is holomorphic in $U \times \overline{U}$, where $\overline{U} = \{\overline{z} : z \in U\}$ and satisfies $\hat{f}(z, \overline{z}) = f(z)$ for every $z \in U$. If p is a complex polynomial in $\mathbf{C}^n$, then the *conjugate* $\overline{p}$ is defined as $\overline{p}(z) := \overline{p(\overline{z})}$. Finally, if p is a real or complex polynomial in $\mathbf{C}^n$ then we denote by $\tilde{p}$ the leading homogeneous component of p.

The first result of the note is the following elementary fact.

PROPOSITION 2.1. *Let f be a real valued, $\mathcal{C}^s(1 \leq s \leq \infty)$ smooth function in a neighbourhood of $0 \in \mathbf{C}^n$. Assume that there is a smooth complex submanifold H of codimension k defined in a neighbourhood of the origin satisfying $0 \in H \subset \{f = 0\}$. Let $g_1, \ldots, g_k$ be holomorphic defining functions for H. Then there exist complex valued $\mathcal{C}^{s-1}$ smooth functions $h_1, \ldots, h_k$ defined near 0 such that*

$$f = \sum_{j=1}^{k} (h_j g_j + \overline{h_j}\overline{g_j}).$$

If f is real analytic, then h_j can be chosen to be real analytic.

Proof. After a biholomorphic change of coordinates, we may assume that $g_j = z_j$ for all $1 \leq j \leq k$. For every $z = (z_1, \ldots, z_k, z')$, where $z' = (z_{k+1}, \ldots, z_n)$ near the origin we set $F(t, z) = f(tz_1, \ldots, tz_k, z')$. Since $f(z) = 0$ whenever $z_1 = \cdots = z_k = 0$, by the fundamental theorem of calculus we get

$$f(z) = F(1, z) = \int_0^1 \frac{\partial}{\partial t} F(t, z)\, dt = \sum_{j=1}^{k} \left(h_j\left(z_1, z'\right) z_j + \overline{h_j\left(z_1, z'\right)}\, \overline{z_j}\right),$$

where $h_j(z_1, z') := \int_0^1 \frac{\partial}{\partial z_1} f(tz_1, z')\, dt$. The desired conclusion now follows. □

As happens frequently, more precise information can be derived for real algebraic hypersurfaces.

THEOREM 2.2. *Let f be a non constant, real valued, real polynomial in $\mathbf{C}^n, f \not\equiv 0$. Assume that the hypersurface $S := \{z \in \mathbf{C}^n : f(z) = 0\}$ contains a germ of a complex variety of pure codimension $k, 1 \le k \le n-1$. Then the following assertions hold.*

(i) *S contains an irreducible algebraic variety of pure codimension $k' \le k$.*

(ii) *If $k = 1$, then there exist an irreducible complex polynomial p in $\mathbf{C}^n$, a real polynomial q in $\mathbf{C}^n$ such that for every $\xi \in H := \{z \in \mathbf{C}^n : p(z) = 0\} \cap S$ we have*

(a) $H \subset \{z \in \mathbf{C}^n : \hat{f}(z, \overline{\xi}) = 0\} \subset S$.

(b) $\deg f = \deg p + \deg q$ *and*

$$f = pq + \overline{pq} \quad \text{on } \mathbf{C}^n. \tag{1}$$

In particular, $H \subset S$. Moreover, given p with $2\deg p > \deg f$ then there exists at most a real polynomial q satisfying (1).

Recall that by an algebraic variety we mean the common zero set of a finitely many complex polynomials in $\mathbf{C}^n$.

Proof of Theorem 2.2. (i) We will use some ideas from [DF]. By assumption, there exists an open set U in $\mathbf{C}^n$ and holomorphic functions $g_1, \dots, g_k$ on U such that

$$V := \{z : z \in U, g_1(z) = \cdots = g_k(z) = 0\} \subset S$$

and $\partial g_1 \wedge \cdots \wedge \partial g_k \not\equiv 0$ on V. Pick a point $a \in V$ such that $\partial g_1(a) \wedge \cdots \wedge \partial g_k(a) \neq 0$. Then, by Proposition 2.1, there is a neighbourhood W around a and real analytic functions $h_1, \dots, h_k$ on W such that $V \cap W$ is a (connected) smooth complex submanifold in W and that

$$f(z) = \sum_{j=1}^{k} \big(h_j(z) g_j(z) + \overline{h_j(z) g_j(z)}\big) \quad \forall z \in W.$$

This implies that

$$\hat{f}(z, w) = \sum_{j=1}^{k} \big(h_j(z, w) g_j(z) + \hat{h}_j(z, w) \overline{g_j}(w)\big) \quad \forall (z, w) \in W \times \overline{W}. \tag{2}$$

It follows from (2) that

$$\hat{f}(z, \overline{w}) = 0 \quad \forall z, w \in V \cap W. \tag{3}$$

Set

$$W' = \bigcap_{w \in V \cap W} \{z \in W : \hat{f}(z, \overline{w}) = 0\}, \qquad W'' = \bigcap_{w \in W'} \{z \in W : \hat{f}(z, \overline{w}) = 0\}.$$

Then W', W'' are complex subvarieties of W and $V \subset W'$ in view of (3). It follows that $W'' \subset W'$. Notice also that $\hat{f}(z,\overline{w}) \equiv 0$ on $W' \times (V \cap W)$. By reality of f, we obtain

$$\hat{f}(z,\overline{w}) = \overline{\hat{f}(w,\overline{z})} \quad \forall z,w \in \mathbf{C}^n.$$

It follows that $\hat{f}(z,\overline{w}) = 0$ on $(V \cap W) \times W'$. By the definition of W'', we get that $V \cap W \subset W''$. Since, obviously, $\hat{f}(z,\overline{w}) = 0$ on $W'' \times W'$, we have $\hat{f}(z,\overline{z}) = 0$ for every $z \in W''$. Thus, $W'' \subset S$. Now we set

$$W^* = \bigcap_{w \in W'} \{z \in \mathbf{C}^n : \hat{f}(z,\overline{w}) = 0\}.$$

By the above reasoning, we can choose an irreducible branch $\tilde{W}$ of W^* such that $V \cap W \subset \tilde{W}$. It is easy to see that $\tilde{W}$ is an algebraic variety of pure codimension $k' \le k$ in $\mathbf{C}^n$. Since $f \equiv 0$ on $\tilde{W} \cap W$, we infer that $f \equiv 0$ on $\tilde{W}$ as well. Thus, $\tilde{W} \subset S$.

(ii) If $k = 1$, then by (i) and Noetherian property of the rings of complex polynomials we can find an irreducible complex polynomial p in $\mathbf{C}^n$ and a point $\xi \in S$ such that (a) holds. Now we pick a regular point a on the algebraic hypersurface $\{p = 0\}$. Then there is a small neighbourhood W of a such that $S \cap W$ is a smooth connected complex hypersurface. To get the conclusion (b), we use a reasoning inspired by the proof of Lemma 3.8 in [BG]. After a linear change of coordinates, we may arrange that f and p are monic polynomial in z_1 i.e.,

$$\hat{f}(z,w) = z_1^s + \sum_{j=0}^{s-1} z_1^j f_j(z',w), \qquad \hat{p}(z) = z_1^d + \sum_{j=0}^{d-1} p_j(z') z_1^j,$$

where $s = \deg f, z = (z_1, z'), d = \deg p$ and f_j (resp. p_j) are polynomials in z', w (resp. in z'). Then we can find an algebraic subvariety A of $\mathbf{C}^{n-1}$ such that the equation $p(z_1, z') = 0$ has exactly d distinct roots for every fixed $z' \notin A$. Consider the algebraic variety

$$V := \{(z,w) \in \mathbf{C}^n \times \mathbf{C}^n : p(z) = \overline{p}(w) = 0\}.$$

Clearly, V is irreducible and of codimension 2 in $\mathbf{C}^{2n}$ and $V' := V \cap (W \times \overline{W})$ is smooth and connected. Since the polynomial $f(z,w)$ vanishes on the totally real manifold $V' \cap \{(z,w) : z = \overline{w}\}$ of codimension 2, we deduce $f(z,w)$ vanishes on V' and therefore also on V. Now we can write

$$f(z,w) = a_d(z,w)p(z) + \sum_{j=0}^{d-1} a_j(z',w) z_1^j, \tag{4}$$

and

$$a_j(z',w) = b_{j,d}(z',w)\overline{p}(w) + \sum_{k=0}^{d-1} b_{j,k}(z',w')w_1^k \quad \forall 0 \le j \le d-1, \tag{5}$$

where $a_j, b_k, 0 \le j \le d, 0 \le k \le d-1$ are complex polynomials. Notice that $\deg a_d = s - d$. Inserting (5) into (4), on V we have

$$\sum_{\overrightarrow{0\le j,k\le d-1}} b_{j,k}(z',w')z_1^j w_1^k = 0. \tag{6}$$

Fix $(z',w') \notin (A \times \mathbf{C}^{n-1}) \cup (\mathbf{C}^{n-1} \times \overline{A})$ where $\overline{B} = \{\overline{z'} : z' \in A\}$. Then we get d distinct roots $z_{1,1}, \dots, z_{1,d}$ of the equation $p(z_1, z') = 0$ and d distinct roots $w_{1,1}, \dots, w_{1,d}$ of the equation $p(w_1, w') = 0$. Now in (6) we replace z_1 by $z_{1,i}$ with fixed i and w_1 by $w_{1,1}, \dots, w_{1,d}$. This yields a d linear system in terms of $\sum_{1\le j\le d} b_{j,k}(z',w')z_{1,i}^j$ for $0 \le k \le d-1$. Since the determinant of the Vandermonde matrix $(w_{1,\alpha}^{\beta})_{1\le\alpha,\beta\le d}$ is nonzero, we get

$$\sum_{1\le j\le d} b_{j,k}(z',w')z_{1,i}^j = 0 \quad \forall 0 \le k \le d-1.$$

Next we fix k and vary i. By the same argument as above, we obtain $b_{j,k}(z',w') = 0$ for $0 \le j,k \le d-1$. Combining this and (4) and (5) one gets a polynomial b_0 in z,w such that for every $(z,w) \in \mathbf{C}^n \times \mathbf{C}^n$ such that $(z',w') \notin (A \times \mathbf{C}^{n-1}) \cup (\mathbf{C}^{n-1} \times \overline{A})$ we have

$$\hat{f}(z,w) = a_d(z,w)p(z) + b_d(z,w)\overline{p}(w). \tag{7}$$

Since A is nowhere dense in $\mathbf{C}^{n-1}$ we infer that (6) holds on $\mathbf{C}^n \times \mathbf{C}^n$ entirely. It also follows from (7) that $\deg b_d \le s - d$. Now from the reality of f, by setting $q = 1/2(a_d + \overline{b_d})$ we obtain (1) as well as the conclusion on degree of q. Finally, assume that $2\deg p > \deg f$. Suppose that there are two distinct real polynomial q_1 and q_2 satisfying (1). By complexification, we get for every $(z,w) \in \mathbf{C}^n \times \mathbf{C}^n$

$$f(z,w) = p(z)q_i(z,w) + \overline{p}(w)\overline{q_i}(w,z), \quad i = 1,2.$$

This implies

$$\big(q_1(z,w) - q_2(z,w)\big)p(z) = \big(\overline{q_2}(z,w) - \overline{q_1}(z,w)\big)\overline{p}(w) \quad \forall (z,w) \in \mathbf{C}^n \times \mathbf{C}^n.$$

After a linear change of coordinates, we may write $\overline{p}(w)$ as a monic polynomial of degree d in w_1 where $w = (w_1, w')$. Thus, we have

$$q_1(z,w) - q_2(z,w) = r(z,w)\overline{p}(w) + \sum_{j=0}^{d-1} w_1^j r_j(z,w'), \tag{8}$$

where r_j are polynomials in z, w'. Since $\overline{p}$ is irreducible, there exists a complex subvariety B of $\mathbf{C}^{n-1}$ such that for all $w' \notin B$, the equation $q(w_1, w') = 0$ has exactly d distinct roots in w_1. Fix $w' \notin B$ and $z \in \mathbf{C}^n$ such that $p(z) = 0$.

We also deduce from (8) that the equation $\sum_{j=0}^{d-1} w_1^j r_j(z,w') = 0$ has d distinct roots in w_1. Thus $r_j(z,w') = 0$ for all $0 \le j \le d-1$. This implies that $r_j(z,w') = 0$ for all $z, w \in \mathbf{C}^n$. Therefore, $\overline{p}$ divides $q_1 - q_2$, and hence $\deg p \le \deg(q_1 - q_2) \le \deg f - \deg p$. This contradicts the assumption on degree of p. □

REMARKS. (a) It is not true that for every polynomial p satisfying $\{z \in \mathbf{C}^n : p(z) = 0\} \subset \{z \in \mathbf{C}^n : f(z) = 0\}$ we can find a real polynomial q satisfying the identity (2). For a simple example, we may take $p(z,w) = zw$ where $(z,w) \in \mathbf{C}^2$ and $f(z,w) = \mathrm{Re}(z\overline{w})$.

(b) We do not know if the assertion (ii) still holds if $k \ge 2$.

(c) Given a complex polynomial p with $2\deg p \le \deg f$, the representation (1) is no longer unique. Indeed, we can always write

$$f = pq + \overline{pq} = p(q - i\overline{p}) + \overline{p}(\overline{q} + ip).$$

(d) For every $\xi \in S$, the complex hypersurface $\{z \in \mathbf{C}^n : \hat{f}(z, \overline{\xi})\} = 0$ is called the Segre variety associated to ξ. This concept has proved to be quite fruitful in the study of real analytic hypersurfaces. For more profound applications of Segre varieties, the reader may consult [We], [BG], [DF] and the references given therein.

As an illustration of the strength of Theorem 2.2 we have the following proposition.

PROPOSITION 2.3. *Let f be a real valued, real polynomial of degree s in $\mathbf{C}^n$. Denote by $\tilde{f}$ the homogeneous component of f of degree s. Assume that $S := \{z : f(z) = 0\}$ contains a germ of a complex hypersurface. Then there exist an irreducible algebraic hypersurface H in $\mathbf{C}^n$ lying in S and a complex homogeneous polynomial of degree less than s whose zero set is included in*

$$\tilde{S}_\xi := \{z : \tilde{f}(z) = 0\} \cap \{z : \tilde{\hat{f}}_\xi(z) = 0\} \quad \forall \xi \in H,$$

where $\hat{f}_\xi(z) := \hat{f}(z, \overline{\xi})$. In particular, if $n = 2$ then the zero set of $\tilde{f}$ contains a complex line passing through the origin.

Proof. By Theorem 2.2(ii), we can find a real polynomial q and an irreducible complex polynomial p in $\mathbf{C}^n$ such that

(a) $H := \{z : p(z) = 0\} \subset \{z : \hat{f}(z, \overline{\xi}) = 0\} \subset S \, \forall \xi \in H$;

(b) $\deg p + \deg q = \deg f$ and $f \equiv pq + \overline{pq}$ on $\mathbf{C}^n$.

It follows from (a) and irreducibility of p that p divides $\hat{f}_\xi(z)$ for all $\xi \in H$. This implies that the zero set of $\tilde{p}$ is included in that of $\tilde{\hat{f}}_\xi$ for all $\xi \in H$. On the other hand, from (b) we obtain

$$\tilde{f} \equiv \tilde{p}\tilde{q} + \overline{\tilde{p}\tilde{q}} \quad \text{on } \mathbf{C}^n.$$

Putting all this together, we see that $\tilde{p}$ is the desired homogeneous polynomial. □

We have the following simple application of the above result to the case where the leading homogeneous component of the defining function for S contains no complex monomial.

COROLLARY 2.4. *Let S be the real algebraic hypersurface in $\mathbf{C}^2$ defined by a real valued, real polynomial f of degree $k \geq 2$. Suppose that $f = p + q + r$, where p (resp. q) are real valued, real homogeneous polynomials of degree k (resp. $k-1$) and r is a real valued, real polynomials of degree $k-2$. Assume that p, q satisfy the following conditions*:

(i) $p(z) = \Re(\overline{z_1}p_1(z) + \overline{z_2}p_2(z) + \overline{z_1}^2 p_3(z) + \overline{z_2}^2 p_4(z))$, *where p_1, p_2 are complex homogeneous polynomials of degree $k-1$ and p_3, p_4 are homogeneous polynomials of degree $k-2$.*

(ii) $q(z) = \Re(q_1(z) + q_2(z))$, *where q_1 is a complex homogeneous polynomial of degree $k-1$ and q_2 contains no complex monomial of degree $k-1$.*

(iii) $p_1(0, z_2) = \alpha_1 z_2^{k-1}$, $p_2(0, z_2) = \alpha_2 z_2^{k-1}$, $q(0, z_2) = \alpha_3 z_2^{k-1}$ *with* $\alpha_1 \alpha_2 \alpha_3 \neq 0$.

Assume that S includes a nontrivial analytic disk. Then S must contain one of the following complex lines.

(a) $l_1 = \{(w_1, w_2) : \overline{\alpha_1} w_1 + \overline{\alpha_2} w_2 + \overline{\alpha_3} = 0\}$.

(b) $l_2 = \{(w_1, w_2) : w_1 \overline{p_1(1,\lambda)} + w_2 \overline{p_2(1,\lambda)} + \overline{q_1(1,\lambda)} = 0\}$, *where λ is a constant satisfying*

$$p_1(1, \lambda) + \overline{\lambda} p_2(1, \lambda) = 0.$$

Proof. It follows from (i) and (ii) that for every $w = (w_1, w_2) \in \mathbf{C}^2$,

$$\tilde{\hat{f}}(z, \overline{w}) = \frac{1}{2}\big(\overline{w_1} p_1(z) + \overline{w_2} p_2(z) + q_1(z)\big).$$

Using Proposition 2.3, we can find an irreducible polynomial p, a complex line d passing through the origin such that $H := \{(w_1, w_2) \in \mathbf{C}^2 : p(w_1, w_2) = 0\} \subset S$ and for every $(z_1, z_2) \in d, (w_1, w_2) \in H$ we have

$$p(z_1, z_2) = 0, \qquad w_1 \overline{p_1(z_1, z_2)} + w_2 \overline{p_2(z_1, z_2)} + \overline{q_1(z_1, z_2)} = 0. \tag{9}$$

Consider two cases.

Case 1. $d = \{z_1 = 0\}$. It follows from (9) that S contains the complex line l_1.

Case 2. $d \neq \{z_1 = 0\}$. Then we may parameterize $d = \{(z_1, \lambda z_1) : z_1 \in \mathbf{C}\}$ where λ is a constant. Inserting into (9) and using homogenieties of p_1, p_2 and q we obtain

$$p_1(1, \lambda) + \overline{\lambda} p_2(1, \lambda) = 0, \qquad w_1 \overline{p_1(1, \lambda)} + w_2 \overline{p_2(1, \lambda)} + \overline{q_1(1, \lambda)} = 0.$$

The proof is complete. □

Given a real algebraic hypersurface S, it is rather hard to find *all* germs of complex varieties lying in S. However, it is possible in some special cases.

PROPOSITION 2.5. *Let $f = \Re(f_1\overline{f_2})$, where f_1 and f_2 are complex polynomials in $\mathbf{C}^2$. Assume that $f_1(0) = f_2(0) = 0$ and that the map $\Phi := (f_1, f_2)$ is proper from $\mathbf{C}^2$ onto $\mathbf{C}^2$. Let H be a nontrivial analytic disk lying in $S := \{z \in \mathbf{C}^2 : f(z) = 0\}$. Then H is contained in $\{af_1 + bf_2 = 0\}$ where a and ib are real numbers satisfying $a^2 - b^2 > 0$.*

Proof. We consider the particular case where $f(z_1, z_2) = z_1, g(z_1, z_2) = z_2$. By the proof of Proposition 2.3(b), we can find a point $(\alpha, \beta) \in S \backslash \{(0, 0)\}$ such that

$$H \subset \{\alpha z_1 + \beta z_2 = 0\} \subset S.$$

Assume that $\alpha \neq 0$, then we can choose $a = |\alpha|^2, b = \overline{\alpha}\beta$. For the general case, it suffices to notice that, since Φ is proper, the image $\Phi(H)$ of H under Φ, is a complex curve lying in $S' := \{\Re(z'_1\overline{z'_2}) = 0\}$. By the special case considered above, we conclude the proof. □

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