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ON HOEFFDING DECOMPOSITION IN L_p

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ABSTRACT. We give a new proof of a result by J. Bourgain which says that if $(\Omega, \mathcal{F}, P)$ is a product of probability spaces then V_d—the orthonormal in $L_2(\Omega, \mathcal{F}, P)$ projection on the space spanned by those $X \in L_2(\Omega, \mathcal{F}, P)$ which depend on most of d-variables is a bounded operator in $L_p(\Omega, \mathcal{F}, P)$ for $1 < p < \infty$. We prove that for $X \in L_p(\Omega, \mathcal{F}, P)$ $E|V_d(X)|^p \le C_{p,d} E|X|^p$ with $C_{p,d} = (c\frac{\hat{p}}{\ln \hat{p}})^{dp}$, where $\hat{p} = \max\{p, \frac{p}{p-1}\}$ and c is an universal constant.

1. Introduction and definitions

The Hoeffding decomposition of U-statistics of order d into sums of canonical statistics of order $k \le d$ plays a crucial role in proving limit theorems for U-statistics of order d. The reason is that usually in limit theorems each canonical statistic of order k requires a renormalization which depends on order k. The decomposition formula and its basic properties, for random variables with finite variances, were given by W. Hoeffdinig in 1948, [6]. The purpose of the present paper is to extend the Hoeffding decomposition to the class of random variables with finite pth moment and to prove basic estimates. The definition of Hoeffding projections, which we give below is somewhat unorthodox. It is put in more abstract and general form which is more convenient for our purposes. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and $(\mathcal{F}_i)_{i \in I}$ a family of independent σ-subfields of $\mathcal{F}$ such that $\mathcal{F} = \sigma\{\mathcal{F}_i, i \in I\}$, that is, $\mathcal{F}$ is the σ-field generated by $\bigcup_{i \in I} \mathcal{F}_i$. Equivalently, we can assume that $(\Omega, \mathcal{F}, P) = \bigotimes_{i \in I}(\Omega_i, \mathcal{F}_i, P_i)$ is a probability product space, and $\mathcal{F}_i$ is identified with events which depend on the ith coordinate. For each $J \subset I$, we put $\mathcal{F}_J = \sigma\{\mathcal{F}_i, i \in J\}$. For a Hilbert

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space H, we will denote by $L_p(J,H) = L_p(\Omega, \mathcal{F}_J, P; H)$ the space of all random variables X with values in H, with a separable range, $\mathcal{F}_J$-measurable and such that $\|X\|_p = (E|X|^p)^{\frac{1}{p}} < \infty$, where $|\cdot| = \|\cdot\|_H$ is the norm in H. $L_p(J,H)$ is a closed subspace of $L_p(I,H)$. The space $L_p(I,H)$ will be denoted simply by $L_p(H)$ or L_p when $H = R$. $|J|$ denotes the cardinality of J, $\mathcal{J} = \{J \in I : |J| < \infty\}$ and $\mathcal{J}_d = \{J \subset I : |J| = d\}$. By $E_{|J}(X)$, we will abbreviate the conditional expectation $E(X|\mathcal{F}_J)$ and by $E_J(X)$ the integration with respect to the variables in J, that is, $E_J(X) = E(X|\mathcal{F}_{I\setminus J})$. For $1 \le p \le \infty$ the operator $E_{|J} : L_p(H) \to L_p(J,H)$ is a projection of norm 1 and "onto". For $1 \le p < \infty$, its adjoint operator is again the operator $E_{|J}$. Moreover $E_{|J}E_{|K} = E_{|J\cap K}$ for each $J, K \subset I$. Therefore, there is a unique family $(Q_J)_{J\in\mathcal{J}}$ of projections in $L_p(H)$ such that $E_{|J} = \sum_{K\subset J} Q_K$ for each $J \in \mathcal{J}$ and $Q_J Q_K = 0$ for each different $J, K \in \mathcal{J}$. An easily verifiable formula for $Q_J, J \in \mathcal{J}$ is

$$Q_J = \sum_{K\subset J} (-1)^{|J|+|K|} E_{|K}.$$

By the martingale convergence theorem for each $1 \le p < \infty$, $X \in L_p(H)$

$$X = \lim_{J\in\mathcal{J}} E_{|J}(X) = \lim_{J\in\mathcal{J}} \sum_{K\subset J} Q_K(X),$$

where $\lim_{J\in\mathcal{J}}$ is the generalized limit in $L_p(H)$ over the directed, by inclusion, set—$\mathcal{J}$. The formula on Q_J proves that the projections $Q_J, J \in \mathcal{J}$ are selfadjoint operators in $L_2(H)$. Hence, it follows that they are projections of norm 1 and that for each $X \in L_2(H)$ the family $Q_K(X), K \in \mathcal{J}$ is orthogonal in Hilbert space $L_2(H)$. Thus, for each $X \in L_2(H)$ we have $X = \sum_{K\in\mathcal{J}} Q_K(X)$ and the convergence in $L_2(H)$ is unconditional, that is, it does not depend on order of the summation. In particular for each $X \in L_2(H)$, we obtain

$$X = \sum_{d=0}^{\infty} U_d(X), \quad \text{where } U_d(X) = \sum_{J\in\mathcal{J}_d} Q_J(X)$$

and the convergence of the both sums is in $L_2(H)$ unconditional. This is what is called the *Hoeffding decompostion* of random variable X into canonical U-statistics. $U_d(X)$ is called *canonical U-statistic of order d associated to X*.

We easily see that $U_0(X) = EX$ and for $d \ge 1$ the recurrence formula holds

$$U_d(X) = \sum_{J\in\mathcal{J}_d} E_{|J}\big(X - U_{d-1}(X)\big).$$

A more explicit formula for U_d can be written in the case of the following example:

Example. Let $(\varepsilon_i)_{i\in I}$ be a family of independent random variables each of them having two values and $E\varepsilon_i = 0$, $E\varepsilon_i^2 = 1$. Let $\mathcal{F}_i = \sigma\{\varepsilon_i\}$ for $i \in I$

and for $J \in \mathcal{J}$ let $w_J = \prod_{i\in J} \varepsilon_i$. In this case, we check easily that for $X \in L_2$

$$U_d(X) = \sum_{J\in\mathcal{J}_d} w_J E(w_J X).$$

In this paper, we investigate which of the above properties of the Hoeffding decomposition can be saved if we take L_p instead of L_2. The main result of the paper is the following theorem.

THEOREM. *For each nonnegative integer d and $1 < p < \infty$, $X \in L_p$ the sum*

$$U_d(X) = \sum_{J\in\mathcal{J}_d} Q_J(X),$$

is well defined and the convergence in L_p is unconditional. Moreover, $\|U_d(X)\|_p \le C_{p,d}\|X\|_p$, where $(c\frac{\hat{p}}{\ln \hat{p}})^d$, $\hat{p} = \max\{p, \frac{p}{p-1}\}$ and c is an universal constant.

Also we will show that in general theorem is false for $p = 1$ and for $p \neq 2$ it is not true that $X = \sum_{d=0}^{\infty} U_d(X)$ for all $X \in L_p$. Except for the constant theorem was proved by a different method by J. Bourgain, [1] which yields the constant $C_{p,d} = (c\frac{\hat{p}^{\frac{5}{2}}}{\ln^2 \hat{p}})^d$. For $d = 1$, the theorem was proved in [2] and we follow the method from that paper.

2. Rosenthal type inequalities for canonical U-statistics

A random variable X in $L_p(H)$ is said to be a finite canonical U-statistics of order d if X can be written in the form $X = \sum_{J\in\mathcal{A}} X_J$ for some finite set $\mathcal{A} \subset \mathcal{J}_d$ and the family $(X_J)_{J\in\mathcal{A}}$ fulfills $X_J \in L_p(J;H)$ and $E_{\{j\}}X_J = 0$ for each $j \in J \in \mathcal{J}_d$. The last two properties of X_J are the same as $X_J = Q_J(X)$. Thus, the above representation of X is unique and we can equivalently say that X is a finite canonical U-statistic of order d if $Q_J(X)$ is nonzero only for a finite collection of $J \in \mathcal{J}_d$ and $X = U_d(X) = \sum_{J\in\mathcal{J}_d} Q_J(X)$. The proof of the theorem is based on the Proposition, given below, which in the case of $d = 1$ coincides with the well known Rosenthal inequality for sums of independent random variables. Also, it is strongly related to Theorem 2.3 from [4] and Theorem 2 from [5], where only symmetric U-statistics are considered and the estimate is expressed in terms which complicate its usefulness for a proof of our main theorem. To specify constants in the proposition as well for its proof, we need to recall the Burkholder inequality, cf. [3], which is a martingale version of the Rosenthal Inequality. It states that for $p \ge 2$ there are constants $a(p), b(p)$ such that for each martingal $(M_j)_{j\in N}$ with values in a Hilbert space, with respect to a filtration $(\mathcal{G}_j)_{j\in N}$, $\Delta_j = M_j - M_{j-1}$ and $M = \lim_j M_j$ we have

$$E|M|^p \le a(p)E\left(\sum_{j=1}^{\infty} E(|\Delta_j|^2|\mathcal{G}_{j-1})\right)^{\frac{p}{2}} + b(p)\sum_{j=1}^{\infty} E|\Delta_j|^p.$$

For different possible choices of the constants $a(p), b(p)$, we refer to [8]. In particular, we can have $a(p) = b(p) = (c\frac{p}{\ln p})^p$ and c is an universal constant. For a fixed choice of the constants, let us put $c(p,d,k) = \binom{d}{k} a(p)^{d-k} b(p)^k$.

PROPOSITION. *Let $2 \le p < \infty$ and let d be a fixed nonnegative integer number. If $X \in L_p(H)$ and $X_J = Q_J(X)$ for $J \in \mathcal{J}_d$, then*

$$\sum_{k=0}^{d} c(p,d,k) \sum_{K \in \mathcal{J}_k} E\biggl(\sum_{K \subset J \in \mathcal{J}_d} E_{|K} |X_J|^2 \biggr)^{\frac{p}{2}} \le 2^d \bigl(a(p) + b(p)\bigr)^d E|X|^p.$$

If in addition X is a finite canonical U-statistics of order d, then

$$E|X|^p \le \sum_{k=0}^{d} c(p,d,k) \sum_{K \in \mathcal{J}_k} E\biggl(\sum_{K \subset J \in \mathcal{J}_d} E_{|K} |X_J|^2 \biggr)^{\frac{p}{2}}.$$

Proof. We will prove the second inequality, while the proof of the first will be postponed until the next section where it will be given together with a proof of theorem. Without lost of generality, we can assume that $I = \{1, 2, \ldots\}$. For $J \in \mathcal{J}$, $j \in I$ let $\max J = \max\{j : j \in J\}$ and $M_j = \sum_{J \in \mathcal{J}_d, \max J \le j} X_J$. Then $(M_j)_{j \in I}$ is a martingale with values in H with respect to the filtration $\mathcal{G}_j = \sigma(\mathcal{F}_i, i \le j)$. Its martingale differences are equal to $\Delta_j = M_j - M_{j-1} = \sum_{J \in \mathcal{J}_d, \max J = j} X_J$. Since X is a finite U-statistic $X = M_j$ for large enough j. Therefore to complete the proof of the first inequality it is enough to estimate the right side of of the above martingale inequality by the right side of the second inequality stated in the proposition. We will prove it by induction. For $d = 0$, it is trivial. For $d = 1$, we have that $\Delta_j = X_j$ and the two right sides of the inequalities coincide. Assume that proposition holds true for d. For the purpose of the proof, it is more convenient to view the probability space $(\Omega, \mathcal{F}, P)$ as a product of probability spaces as explained in the Introduction. For each $j \in I$ when we fix the jth coordinate the variable Δ_j is a canonical U-statistics of order $d-1$ and then by the inductive assumption $E_{|\{j\}} |\Delta_j|^p \le \sum_{\max K < j, |K| < d} c(p, d-1, |K|) E_{|\{j\}} (\sum_{K \cup \{j\} \subset J \in \mathcal{J}_d} E_{|K \cup \{j\}} |X_J|^2)^{\frac{p}{2}}$. Hence, summing the inequalities over $j \in I$ and taking $K \cup \{j\}$ as K we get

$$\sum_{j \in I} E|\Delta_j|^p \le \sum_{k=1}^{d} c(p, d-1, k-1) \sum_{K \in \mathcal{J}_k} \biggl(\sum_{K \subset J \in \mathcal{J}_d} E_{|K} |X_J|^2 \biggr)^{\frac{p}{2}}.$$

To estimate the first sum on the right side of the martingale inequality, let us notice that $E(|\Delta_j|^2 | \mathcal{G}_{j-1}) = E_{\{j\}} |\Delta_j|^2$ and thus $E(\sum_{j=1}^{\infty} E(|\Delta_j|^2 | \mathcal{G}_{j-1}))^{\frac{p}{2}} = E\|\tilde{\Delta}\|_{\tilde{H}}^p$ where: $\tilde{H} = l_2(L_2(H))$, that is, it is the space of all families of random variables $(Y_j)_{j \in I}$ in $L_2(H)$ such that $\|(Y_j)_{j \in I}\|_{\tilde{H}} = (\sum_{j \in I} E\|Y_j\|_H^2)^{\frac{1}{2}} < \infty$, $\tilde{\Delta} = (\tilde{\Delta}_j)_{j \in I}$ and where for a random variable $Y \in L_2(H)$ and $j \in I$ the random variable $\tilde{Y}_j \in L_2(L_2(H))$ is defined by $\tilde{Y}_j((\omega_i)_{i \in I}))((\omega'_i)_{i \in I})) = (Y -$

$E_{\{j\}}Y)((\omega''_i)_{i\in I})$ where $\omega''_j=\omega'_j$ and $\omega''_i=\omega_i$ for $i\neq j$. We easily check that $\tilde\Delta$ is a finite canonical U-statistics of order $d-1$ with values in $\tilde H$ and for each $J\in\mathcal{J}_{d-1}$ it is $\tilde\Delta_J=(W_j)_{j\in I}$ where $W_j=(\tilde X_{J\cup\{j\}})_j$ if $\max J<j$ and $W_j=0$ otherwise. Again by the inductive assumption we get

$$\begin{aligned}
E\|\tilde\Delta\|^p_{\tilde H} &\le \sum_{K\in\mathcal{J},|K|<d} c(p,d-1,|K|)E\Big(\sum_{K\subset J\in\mathcal{J}_{d-1}} E_{|K}\|\tilde\Delta_J\|^2_{\tilde H}\Big)^{\frac{p}{2}}\\
&= \sum_{K\in\mathcal{J},|K|<d} c(p,d-1,|K|)\\
&\quad \times E\Bigg(\sum_{K\subset J\in\mathcal{J}_{d-1}} E_{|K}\Big(\sum_{j>\max J} E_{\{j\}}|X_{J\cup\{j\}}|^2\Big)^{\frac{p}{2}}\Bigg)\\
&= \sum_{K\in\mathcal{J},|K|<d} c(p,d-1,|K|)E\Bigg(\sum_{K\subset J\in\mathcal{J}_{d-1}} E_{|K}\Big(\sum_{j>\max J} |X_{J\cup\{j\}}|^2\Big)^{\frac{p}{2}}\Bigg)\\
&\le \sum_{K\in\mathcal{J},|K|<d} c(p,d-1,|K|)E\Big(\sum_{K\subset J\in\mathcal{J}_{d}} E_{|K}|X_J|^2\Big)^{\frac{p}{2}}.
\end{aligned}$$

Thus, we obtain

$$E\Big(\sum_{j=1}^{\infty}E(|\Delta_j|^2|\mathcal{G}_{j-1})\Big)^{\frac{p}{2}} \le \sum_{k=0}^{d-1}c(p,d-1,k)\sum_{K\in\mathcal{J}_k}E\Big(\sum_{K\subset J\in\mathcal{J}_d}E_{|K}|X_J|^2\Big)^{\frac{p}{2}}.$$

The above inequality and the one proved earlier complete the induction step since $c(p,0)c(p,d-1,k)+c(p,1)c(p,d-1,k-1)=c(p,d,k)$ for $d=2,\dots$ and $0\le k\le d$ (where we put $c(p,d,-1)=0=c(p,d,d+1)$). Thus, the second inequality of the proposition is proved. □

3. Proof of the Theorem and of the first inequality of the Proposition

For $K\subset I$ and a Hilbert space H let $H'=L_2(L_2(H))$ and $D_K: L_2(H)\to L_2(H')$ be defined by $(D_K(Y)(\omega))(\omega')=Y(\omega'')$ where for $\omega=(\omega_i)_{i\in I},\ \omega'=(\omega'_i)_{i\in I},\ \omega''=(\omega_i)_{i\in I}$ is defined by $\omega''_i=\omega'_i$ for $i\in K$ and $\omega''_i=\omega_i$ for $i\in I\setminus K$. For a fixed $k\in N$ let $(\mathcal{J}_k,\mathcal{A}_k,\lambda_k)$ be a measure space where $\mathcal{A}_k$ is the family of all subsets of $\mathcal{J}_k$ and λ_k is the counting measure on $\mathcal{J}_k$. On the product of measure spaces $(\Omega\times\mathcal{J}_k,\mathcal{F}\otimes\mathcal{A}_k,P\otimes\lambda_k)$ let us consider the space $\mathcal{L}_p=L_p(\Omega\times\mathcal{J}_k,\mathcal{F}\otimes\mathcal{A}_k,P\otimes\lambda_k;H')$ of Bochner p-integrable functions on $\Omega\times\mathcal{J}_k$ with values in H'. For $1\le p\le\infty$ $\mathcal{L}_p$ is a Banach space. Next, for $0\le k\le d$ we define an operator $T_k: L_2(H)\to\mathcal{L}_2$ to be given by the formula $T_k(X)(\omega,K)=D_K(\sum_{K\subset J\in\mathcal{J}_d}Q_J(X))$ for $\omega\in\Omega,\ K\in\mathcal{J}_k$. To compute the norm of T_k let us observe that for any $X \in L_2(H)$ it is

$$\|T_k(X)\|_{\mathcal{L}_2}^2 = \sum_{K \in \mathcal{J}_k} E\left|\sum_{K \subset J \in \mathcal{J}_d} Q_J(X)\right|^2 = \sum_{K \in \mathcal{J}_k} \sum_{K \subset J \in \mathcal{J}_d} E|Q_J(X)|^2$$
$$\leq \binom{d}{k} \sum_{J \in \mathcal{J}_d} E|Q_J(X)|^2 \leq \binom{d}{k} E|X|^2.$$

The last inequality was explained in Introduction. Thus, $\|T\|_{L_2,\mathcal{L}_2}^2 \leq \binom{d}{k}$. For $X \in L_\infty(H)$, we have that

$$\|T_k(X)\|_{\mathcal{L}_\infty}^2 = \sup_{K \in \mathcal{J}_k} \operatorname{ess\,sup} E_{|K}\left|\sum_{K \subset J \in \mathcal{J}_d} Q_J(X)\right|^2$$
$$\leq \sup_{K \in \mathcal{J}_k} \operatorname{ess\,sup} E_{|K}|X|^2 \leq \|X\|_{L_\infty}^2.$$

Thus, $\|T_k\|_{L_\infty,\mathcal{L}_\infty}^2 \leq 1$. Hence, by the complex interpolation theorem we obtain that $\|T_k\|_{L_p,\mathcal{L}_p}^p \leq \binom{d}{k}$ for $2 \leq p \leq \infty$. Hence, it follows that for each $X \in L_p$ we have $\sum_{K \in \mathcal{J}_k} E(\sum_{K \subset J \in \mathcal{J}_d} E_{|K}|Q_J(X)|^2)^{\frac{p}{2}} = \|T_k^p(X)\|_{\mathcal{L}_\infty}^p \leq \binom{d}{k} E|X|^p$ and therefore $\sum_{k=0}^d c(p,d,k) \sum_{K \in \mathcal{J}_k} E(\sum_{K \subset J \in \mathcal{J}_d} E_{|K}|Q_J(X)|^2)^{\frac{p}{2}} \leq \sum_{k=0}^d c(p,d,k)\binom{d}{k} E|X|^p \leq 2^d(a(p)+b(p))^d$. Thus, the left side inequality of the proposition is proved. Now the proof of the theorem, for $2 \leq p < \infty$, follows quickly. Because $T_k(X) = T_k(U_d(X))$ for each $0 \leq k \leq d$ and since $U_d(X)$ is a canonical U-statistic of order d combining the inequalities of both sides from the proposition, we obtain that $E|U_d(X)|^p \leq (2(a(p)+b(p)))^d E|X|^p$. Since the adjoint operator of U_d is again U_d, we obtain that it has the same norm and $E|U_d(X)|^p \leq (2(a(q)+b(q)))^{d(p-1)} E|X|^p$ for $1 < p \leq 2$, where $q = \frac{p}{p-1}$. Thus, the inequality from the theorem is proved.

Moreover, for each family $\varepsilon = (\varepsilon_J)_{J \in \mathcal{J}_d}$ of signs the same arguments show that for $U_d^\varepsilon(X) = \sum_{J \in \mathcal{J}_d} \varepsilon_J Q_J(X)$ there holds $E|U_d^\varepsilon(X)|^p \leq c(d,p)E|X|^p$ with the constant $c(d,p)$ as before. This proves the unconditional convergence of the series $\sum_{J \in \mathcal{J}_d} Q_J(X)$. And the proof of the theorem is completed.

4. Concluding remarks

The fact that the theorem is not true for $p=1$ and any infinite product of nontrivial probability spaces was shown in [2]. It was proved there that in this case we can find $X \in L_1$ such that $U_1(X) = \sum_{i \in I} Q_{\{i\}}(X)$ is not convergent in L_1. In the case of independent σ-fields considered in the example in which we additionally assume that the random variables $\varepsilon_i, i \in I$ are identically distributed this is an easy observation. Indeed, if we put c to be the maximal value of $|\varepsilon|$ then for any $J \in \mathcal{J}_n$ for the random variable $X = \prod_{i \in J}(1 + c^{-1}\varepsilon_i)$ we have $E|X| = 1$ while $U_1(X) = c^{-1}(\sum_{i \in J} \varepsilon_i)$ and hence $E|U_1(X)| \geq c^{-1}\sqrt{\frac{n}{2}}$. The same example shows that the series $\sum_{d=0}^\infty U_d(X)$ need not converge in L_p

when $p \neq 2$. Assume to the contrary that it is so, then by Banach–Steinhaus theorem $E|U_d(X)|^p \leq CE|X|^p$ for some constant C and all $X \in L_p,\ d \in N$. We will show that this leads to a contradiction. For $J \in \mathcal{J}_n$ and $k \leq n$, let $W_{k,J} = \frac{1}{\sqrt{\binom{n}{k}}} \sum_{K \in \mathcal{J}_k, K \subset J} w_K$. It is known that the joint distributions of $W_{1,J}, \dots, W_{d,J}$ converge weakly to the joint distribution of $h_1(G), \dots, h_d(G)$ when $n = |J|$ converge to infinity and where G is a distributed by $N(0,1)$ and $h_d, d \in N$ are the Hermite orthonormal polynomials (normalized by $Eh_d^2(G) = 1$). If $f : R \to R$ is a continuous bounded function let $X = f(W_{1,J})$ then $U_d(X) = W_{d,J} EW_{d,J} f(W_{1,J})$. The above inequality and the weak convergence prove that $E|h_d(G)|^p |Eh_d(G) f(G)|^p \leq CE|f(G)|^p$. Since f is arbitrary this yields $(E|h_d(G)|^p)^{\frac{1}{p}} (E|h_d(G)|^q)^{\frac{1}{q}} \leq C^{\frac{1}{p}}$, where $q = \frac{p}{p-1}$. However, this contradicts the fact that for $1 < p,\ q = \frac{p}{p-1},\ p* = \max\{p-1, q-1\}$ there holds $(E|h_d(G)|^p)^{\frac{1}{p}} (E|h_d(G)|^q)^{\frac{1}{q}} \geq c\sqrt{\frac{p^{*d}}{d}}$ for some $c > 0$ and large enough d which follows form Theorem 2.1 in [7].

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