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SHARP GREEN FUNCTION ESTIMATES FOR $\Delta + \Delta^{\alpha/2}$ IN $C^{1,1}$ OPEN SETS AND THEIR APPLICATIONS

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ABSTRACT. We consider a family of pseudo differential operators $\{\Delta + a^\alpha \Delta^{\alpha/2}; a \in [0,1]\}$ on $\mathbb{R}^d$ that evolves continuously from Δ to $\Delta + \Delta^{\alpha/2}$, where $d \geq 1$ and $\alpha \in (0,2)$. It gives rise to a family of Lévy processes $\{X^a, a \in [0,1]\}$, where X^a is the sum of a Brownian motion and an independent symmetric α-stable process with weight a. Using a recently obtained uniform boundary Harnack principle with explicit decay rate, we establish sharp bounds for the Green function of the process X^a killed upon exiting a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$. Our estimates are uniform in $a \in (0,1]$ and taking $a \to 0$ recovers the Green function estimates for Brownian motion in D. As a consequence of the Green function estimates for X^a in D, we identify both the Martin boundary and the minimal Martin boundary of D with respect to X^a with its Euclidean boundary. Finally, sharp Green function estimates are derived for certain Lévy processes which can be obtained as perturbations of X^a.

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1. Introduction

Discontinuous Markov processes have been intensively studied in recent years due to their importance both in theory and applications; see, for instance, [1], [2], [5], [13], [14], [16], [17], [19], [25], [26], [28]. In contrast to the diffusion case, the infinitesimal generator of a discontinuous Markov process in $\mathbb{R}^d$ is a nonlocal (or integro-differential) operator. Most of the recent studies have concentrated on discontinuous Markov processes (and corresponding integro-differential operators) that do not have a diffusion component. See [7], [11] and the references therein for a summary of some of these recent results from the probability literature. For recent progress in PDE literature, we refer the readers to [8], [9], [10].

However, Markov processes with both diffusion and jump components are needed in many situations, like in finance and control theory. See, for example, [24], [31], [32]. On the other hand, the fact that such a process X has both diffusion and jump components is also the source of many technical difficulties in investigating the potential theory of X. The main difficulty in studying X stems from the fact that it runs on two different scales: on the small scale the diffusion part dominates, while on the large scale the jumps take over. Another difficulty is encountered at the exit of X from an open set: for diffusions, the exit is through the boundary, while for the pure jump processes, typically the exit happens by jumping out from the open set. For the process X, both cases will occur which makes the process X much more difficult to study.

Despite these difficulties, significant progress has been made in the last few years in understanding the potential theory of discontinuous Markov processes with both diffusion and jump components. Green function estimates (for the whole space) and the Harnack inequality for some processes with both diffusion and jump components were established in [33], [35]. The parabolic Harnack inequality and heat kernel estimates were studied in [37] for the sum of a Brownian motion and an independent symmetric stable process, and in [15] for much more general diffusions with jumps (see also [20]). Moreover, an a priori Hölder estimate is established in [15] for bounded parabolic functions. Very recently, the boundary Harnack principle for some one-dimensional Lévy processes with both diffusion and jump components was studied in [29], where sharp estimates on the Green functions of bounded open sets of $\mathbb{R}$ were also established. Most recently, a boundary Harnack principle with explicit decay rate for nonnegative harmonic functions of the sum of a Brownian motion and an independent symmetric stable process in $C^{1,1}$ open sets in $\mathbb{R}^d$ was obtained in [12].

The main goal of this paper is to use the boundary Harnack principle obtained in [12] to establish sharp Green function estimates in $C^{1,1}$ open sets for the Lévy processes that are sums of Brownian motions and independent symmetric stable processes. These processes, although very specific, serve as a model case for more general Lévy processes and Markov processes that have both diffusion and jump components, just as Brownian motion does for diffusions. We hope the results of this paper and some of the techniques used in this paper will shed light on the fine potential theoretic properties of general Lévy processes and on general Markov processes with both diffusion and jump components.

Let us now fix the notation and state the main result of this paper. Throughout this paper, we assume that $d \geq 1$ is an integer and $\alpha \in (0,2)$. Let $X^0 = (X^0_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ with generator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, and let $Y = (Y_t, t \geq 0)$ be an independent (rotationally) symmetric α-stable process in $\mathbb{R}^d$. For $a > 0$, we define the process $X^a = (X^a_t, t \geq 0)$ by $X^a_t = X^0_t + aY_t$, called *the sum of a Brownian motion and an independent symmetric stable process with weight a.*

Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$, let $X^{a,D}$ be the process X^a killed upon exiting D and let $G^a_D(x,y)$ denote the Green function of $X^{a,D}$ (for precise definitions see Section 2). Our main goal is to establish sharp two-sided estimates for $G^a_D(x,y)$. Let $\delta_D(x)$ denote the Euclidean distance between the point $x \in D$ and the boundary ∂D. The main result of this paper is the following theorem. Here and in the sequel, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. Define for $d \geq 3$ and $a > 0$,

$$g^a_D(x,y) := \begin{cases} \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right), \\ \quad \text{when } x, y \text{ are in the same component of } D, \\ \frac{a^\alpha}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right), \\ \quad \text{when } x, y \text{ are in different components of } D; \end{cases}$$

for $d = 2$ and $a > 0$,

$$g^a_D(x,y) := \begin{cases} \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right), \\ \quad \text{when } x, y \text{ are in the same component of } D, \\ a^\alpha \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right), \\ \quad \text{when } x, y \text{ are in different components of } D; \end{cases} \tag{1.1}$$

and for $d = 1$ and $a > 0$,

$$g^a_D(x,y) := \begin{cases} (\delta_D(x)\delta_D(y))^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|}, \\ \quad \text{when } x, y \text{ are in the same component of } D, \\ a^\alpha\left((\delta_D(x)\delta_D(y))^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|}\right), \\ \quad \text{when } x, y \text{ are in different components of } D. \end{cases} \tag{1.2}$$

THEOREM 1.1. *Let $M > 0$. Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exists $C = C(D, M, \alpha) > 1$ such that for all $x, y \in D$ and all*

$a \in (0, M]$

$$C^{-1} g_D^a(x,y) \le G_D^a(x,y) \le C g_D^a(x,y). \tag{1.3}$$

Note that the above estimates are uniform in $a \in (0, M]$. In case $d = 1$, a (nonuniform) estimate is established by [29]. Letting $a \downarrow 0$ in (1.3) recovers the Green function estimates for Brownian motion killed upon exiting D; for the latter, see [18, p. 182] for $d = 2$ and [40] for $d \ge 3$, respectively. Note that when x and y are in the same component of D the estimates for $G_D^a(x,y)$ are the same as for the Brownian motion killed upon exiting D, but contrary to the latter case, $G_D^a(x,y)$ is nonzero when x and y are in different components. This, of course, is a consequence of X^a having jumps, and estimates in the case when x and y are in different components follow easily from the jump structure together with the estimates for a single component. Furthermore, our estimates on $G_D^a(x,y)$ give the rate at which $G_D^a(x,y)$ vanishes as $a \downarrow 0$ when x and y are in different components.

The rest of the paper is organized as follows. Section 2 gives preliminary and background materials. Theorem 1.1 is proved in Sections 3, 4 and 5. The proof of the theorem in the case $d \ge 3$ is by now quite standard. Once the interior estimates are established, the full estimates in connected $C^{1,1}$ open sets follow from the boundary Harnack principle by the method developed by Bogdan [6] and Hansen [23]. However, this method is not applicable when $d \le 2$ since Brownian motion is recurrent in this case. When $d = 2$, the above method ceases to work due to the nature of the logarithmic potential associated with the Laplacian. We use a capacitary argument to derive the interior upper bound estimate for the Green function; see Lemmas 4.5, 4.6 and Corollary 4.7. By a scaling consideration and applying the uniform boundary Harnack principle, we can then get sharp Green function upper bound estimates. For the lower bound estimates, we compare the process with the subordinate killed Brownian motion when D is connected, and then extend it to general bounded $C^{1,1}$ open set by using the jumping structure of the process. The proof of these estimates for $d = 2$ is presented in Section 4. The case $d = 1$ is dealt with in Section 5, where we follow the arguments of [29], making use of the reflected process at supremum and the ascending ladder processes. In Section 6, using the boundary Harnack principle and our Green function estimates, we show that both the Martin and the minimal Martin boundary of the process $X^{a,D}$ can be identified with the Euclidean boundary when D is a bounded $C^{1,1}$ open set. In the last section, we extend our results on X^a to symmetric Lévy processes that can be obtained from X^a through certain perturbations. In particular, for every $m > 0$, we obtain sharp Green function estimates of $\Delta + m - (m^{2/\alpha} - \Delta)^\alpha$ in any bounded $C^{1,1}$ open set with zero exterior condition. The process corresponding to $\Delta + m - (m^{2/\alpha} - \Delta)^\alpha$ is a Lévy process that is the sum of a Brownian motion and an independent relativistic α-stable process with mass m.

Throughout this paper, we use the capital letters $C_1, C_2, \dots$ to denote constants in the statement of results, and their labeling will be fixed. The lowercase constants $c_1, c_2, \dots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the constants $c_1, c_2, \dots$ starts anew in every proof. The dependence of the constant c on the dimension d and $\alpha \in (0,2)$ may not be mentioned explicitly. The constant $\alpha \in (0,2)$ will be fixed throughout this paper. We will use ":=" to denote a definition, which is read as "is defined to be." $B(x,r)$ denotes the open ball in $\mathbb{R}^d$ centered at x with radius $r>0$. Recall that for any $x \in D$, $\delta_D(x)$ denotes the distance between x and ∂D, and for $a,b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial)=0$. Lebesgue measure in $\mathbb{R}^d$ will be denoted by dx. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

2. Preliminaries

A (rotationally) symmetric α-stable process $Y=(Y_t, t\ge 0, \mathbb{P}_x, x\in\mathbb{R}^d)$ in $\mathbb{R}^d$ is a Lévy process with the characteristic exponent $|\xi|^\alpha$, that is,

$$\mathbb{E}_x\left[e^{i\xi\cdot(Y_t-Y_0)}\right]=e^{-t|\xi|^\alpha} \quad \text{for every } x\in\mathbb{R}^d \text{ and } \xi\in\mathbb{R}^d.$$

The infinitesimal generator of Y is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2}u(x)=\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}}\big(u(y)-u(x)\big)\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}}\,dy,$$

where $\mathcal{A}(d,-\alpha):=\alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Here, Γ is the Gamma function defined by $\Gamma(\lambda):=\int_0^\infty t^{\lambda-1}e^{-t}\,dt$ for every $\lambda>0$.

Suppose X^0 is a Brownian motion in $\mathbb{R}^d$ with generator $\Delta=\sum_{i=1}^d\frac{\partial^2}{\partial x_i^2}$, and Y is a symmetric α-stable process in $\mathbb{R}^d$. Assume that X^0 and Y are independent. For any $a>0$, we define the process $X^a=(X_t^a, t\ge 0)$ by $X_t^a := X_t^0 + aY_t$. As already mentioned, the process X^a is called the sum of the Brownian motion X^0 and the independent symmetric α-stable process Y with weight a. It is a Lévy process with the characteristic exponent $\Phi^a(\xi)=|\xi|^2+a^\alpha|\xi|^\alpha$, $\xi\in\mathbb{R}^d$, and its infinitesimal generator is $\Delta+a^\alpha\Delta^{\alpha/2}$. The process X^a has a jointly continuous transition density that will be denoted by $p^a(t,x,y)$. From the Chung–Fuchs criterion (see [3, Theorem I.17]), it easily follows that, when $a>0$, X^a is transient if and only if $\alpha<d$, while it is well known that X^0 is transient if and only if $d\ge 3$.

There is another representation of the process X^a which will be useful in Sections 3, 4 and 5. It can be obtained by subordinating X^0 with an independent subordinator $T_t^a := t + a^2 T_t$ where $T=(T_t, t\ge 0)$ is an $\alpha/2$-stable subordinator, that is, the processes $(X^a_t, t \geq 0)$ and $(X^0_{T^a_t}, t \geq 0)$ have the same distribution. Note that the Laplace exponent of T^a is $\phi^a(\lambda) = \lambda + a^\alpha \lambda^{\alpha/2}$. Let $\mathcal{M}_{\alpha/2}(t) := \sum_{n=0}^\infty (-1)^n t^{n\alpha/2}/\Gamma(1+n\alpha/2)$. It follows by a straightforward integration that

$$\int_0^\infty e^{-\lambda t} \mathcal{M}_{1-\alpha/2}\big(a^{2\alpha/(2-\alpha)} t\big)\, dt = \frac{1}{\phi^a(\lambda)},$$

which shows that the potential density u^a of the subordinator T^a is given by

$$u^a(t) = \mathcal{M}_{1-\alpha/2}\big(a^{2\alpha/(2-\alpha)} t\big). \tag{2.1}$$

Since, for any $a > 0$, ϕ^a is a complete Bernstein function, we know that (see, for instance, [33]) $u^a(\cdot)$ is a completely monotone function. In particular, $u^a(\cdot)$ is a decreasing function. Since $u^a(t) = u^1(a^{2\alpha/(2-\alpha)} t)$, we see that $a \mapsto u^a(t)$ is a decreasing function. Moreover, since the drift of T^a is equal to 1, we have that $u^a(0+) = 1$ and so

$$u^a(t) \leq 1 \quad \text{for } t > 0. \tag{2.2}$$

The Lévy measure of X^a has a density with respect to the Lebesgue measure given by

$$J^a(x, y) := j^a(y - x) := j^a(|y - x|) = a^\alpha \mathcal{A}(d, -\alpha)|x - y|^{-(d+\alpha)}, \tag{2.3}$$

which is called the Lévy intensity of X^a. It determines a Lévy system for X^a, which describes the jumps of the process X^a: For any nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, x, x) = 0$ for all $s > 0$ and $x \in \mathbb{R}^d$, and stopping time T (with respect to the filtration of X^a),

$$\mathbb{E}_x\left[\sum_{s \leq T} f(s, X^a_{s-}, X^a_s)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^a_s, y) J^a(X^a_s, y)\, dy\right) ds\right] \tag{2.4}$$

(see, for example, [13, Proof of Lemma 4.7] and [14, Appendix A]).

The quadratic form $(\mathcal{E}^a, \mathcal{F})$ associated with the generator $\Delta + a^\alpha \Delta^{\alpha/2}$ of X^a is given by

$$\mathcal{F} = W^{1,2}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d; dx) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^d; dx) \text{ for every } 1 \leq i \leq d \right\}$$

and for $u, v \in \mathcal{F}$,

$$\begin{aligned} \mathcal{E}^a(u, v) = &\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x)\, dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) \frac{\mathcal{A}(d, -\alpha) a^\alpha}{|x - y|^{d+\alpha}}\, dx\, dy. \end{aligned}$$

In probability theory, the quadratic form $(\mathcal{E}^a, W^{1,2}(\mathbb{R}^d))$ is called the Dirichlet form of X^a. Let $\mathcal{E}^a_1(u, u) := \mathcal{E}^a(u, u) + \int_{\mathbb{R}^d} u(x)^2\, dx$. Note that for every $a > 0$, there is a positive constant $c = c(a, d, \alpha) \geq 1$ so that

$$\int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 + u(x)^2\right) dx \\ \leq \mathcal{E}_1^a(u,u) \leq c \int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 + u(x)^2\right) dx \quad \text{for } u \in W^{1,2}(\mathbb{R}^d).$$

Thus the processes X^a, $a \geq 0$, share the same family of sets having zero capacity.

For any open set $D \subset \mathbb{R}^d$, $\tau_D^a := \inf\{t > 0 : X_t^a \notin D\}$ denotes the first exit time from D by X^a. We denote by $X^{a,D}$ the subprocess of X^a killed upon leaving D. The infinitesimal generator of $X^{a,D}$ is $(\Delta + a^\alpha \Delta^{\alpha/2})|_D$. It is known (see [15]) that $X^{a,D}$ has a continuous transition density $p_D^a(t,x,y)$ with respect to the Lebesgue measure.

Definition 2.1. A real-valued function u defined on $\mathbb{R}^d$ is said to be:

(1) harmonic in $D \subset \mathbb{R}^d$ with respect to X^a if for every open set B whose closure is a compact subset of D,

$$\mathbb{E}_x[|u(X^a_{\tau^a_B})|] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x[u(X^a_{\tau^a_B})] \quad \text{for every } x \in B; \tag{2.5}$$

(2) regular harmonic in $D \subset \mathbb{R}^d$ with respect to X^a if it is harmonic in D with respect to X^a and

$$u(x) = \mathbb{E}_x[u(X^a_{\tau^a_D})] \quad \text{for every } x \in D;$$

(3) harmonic for $X^{a,D}$ if it is harmonic for X^a in D and vanishes outside D;

(4) superharmonic in $D \subset \mathbb{R}^d$ with respect to X^a if for every open set B whose closure is a compact subset of D,

$$\mathbb{E}_x[|u(X^a_{\tau^a_B})|] < \infty \quad \text{and} \quad u(x) \geq \mathbb{E}_x[u(X^a_{\tau^a_B})] \quad \text{for every } x \in B. \tag{2.6}$$

It follows from [15] that every harmonic function u in D with respect to X^a is continuous in D and $\int_{\mathbb{R}^d} |u(y)|(1 \wedge |y|^{-(d+\alpha)})\, dy < \infty$.

Using the parabolic Harnack inequality from [15, Theorem 6.7] and a scaling argument, the following uniform Harnack principle was established in [12].

Proposition 2.2 (Uniform Harnack principle). *Suppose that $M > 0$. There exists a constant $C_1 = C_1(\alpha, M) > 0$ such that for any $r \in (0,1]$, $a \in [0, M]$, $x_0 \in \mathbb{R}^d$ and any function u which is nonnegative in $\mathbb{R}^d$ and harmonic in $B(x_0, r)$ with respect to X^a we have*

$$u(x) \leq C_1 u(y) \quad \textit{for all } x, y \in B(x_0, r/2).$$

We recall that an open set D in $\mathbb{R}^d$ with $d \geq 2$ is said to be $C^{1,1}$ if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$,

$\nabla\phi(0) = (0,\ldots,0)$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(y)| \le \Lambda|x-y|$, and an orthonormal coordinate system CS_Q: $y = (y_1,\ldots,y_{d-1},y_d) =: (\widetilde{y}, y_d)$ with its origin at Q such that

$$B(Q,R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS_Q : y_d > \phi(\widetilde{y})\}.$$

The pair (R,Λ) is called the characteristics of the $C^{1,1}$ open set D. Note that a $C^{1,1}$ open set may be disconnected. Observe that the distance between any two distinct connected open components of D is at least R. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set may be unbounded. It is well known that any $C^{1,1}$ open set D satisfies the uniform exterior ball condition: There exists $\widetilde{R} > 0$ such that for every $z \in \partial D$, there is a ball B^z of radius $\widetilde{R}$ such that $B^z \subset (\overline{D})^c$ and $\partial B^z \cap \partial D = \{z\}$. Without loss of generality, throughout this paper, we assume that the characteristics (R,Λ) of a $C^{1,1}$ open set satisfy $R = \widetilde{R}$.

Observe that for any $C^{1,1}$ open set with $C^{1,1}$ characteristics (R,Λ), there exists a constant $\kappa \in (0,1/2]$, which depends only on (R,Λ), such that for each $Q \in \partial D$ and $r \in (0,R)$, $D \cap B(Q,r)$ contains a ball $B(A_r(Q),\kappa r)$ of radius κr. In the rest of paper, whenever we deal with $C^{1,1}$ open sets, the constants Λ, R and κ will have the meaning described above.

Let $Q \in \partial D$. We will say that a function $u : \mathbb{R}^d \to \mathbb{R}$ vanishes continuously on $D^c \cap B(Q,r)$ if $u = 0$ on $D^c \cap B(Q,r)$ and u is continuous at every point of $\partial D \cap B(Q,r)$.

The following theorem is the main result of [12].

THEOREM 2.3 (Uniform boundary Harnack principle). *Suppose that $M > 0$. For any $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R,Λ), there exists a positive constant $C_2 = C_2(\alpha,d,\Lambda,R,M)$ such that for all $a \in [0,M]$, $r \in (0,R]$, $Q \in \partial D$ and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q,r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q,r)$, we have*

$$\frac{u(x)}{u(y)} \le C_2 \frac{\delta_D(x)}{\delta_D(y)} \qquad \text{for every } x,y \in D \cap B(Q,r/2). \tag{2.7}$$

A subset D of $\mathbb{R}^d$ is said to be Greenian for X^a if $X^{a,D}$ is transient. A Greenian set for X^0 will be simply called Greenian. As mentioned in the second paragraph of Section 2, when $d \ge 2$ and $a > 0$, any nonempty open set $D \subset \mathbb{R}^d$ is Greenian for X^a; and any nonempty open set in $\mathbb{R}^d$ is Greenian when $d \ge 3$. An open set $D \subset \mathbb{R}^2$ is Greenian if and only if D^c is nonpolar (or equivalently, has positive capacity). In particular, every bounded open set in $\mathbb{R}^2$ is Greenian.

For any $a > 0$ and any Greenian open subset D of $\mathbb{R}^d$ for X^a, we use $G^a_D(x, y)$ to denote the Green function of $X^{a,D}$, that is,

$$G^a_D(x, y) := \int_0^\infty p^a_D(t, x, y)\, dt, \tag{2.8}$$

where $p^a_D(t, x, y)$ is the continuous transition density of $X^{a,D}$ with respect to the Lebesgue measure. The function $G^a_D(\cdot, \cdot)$ is finite off the diagonal. It follows immediately from (2.8) that $G^a_D(x, y)$ is a positive continuous symmetric function off the diagonal of $D \times D$ such that for any Borel measurable function $f \geq 0$,

$$\mathbb{E}_x\left[\int_0^{\tau^a_D} f(X^a_s)\, ds\right] = \int_D G^a_D(x, y) f(y)\, dy.$$

We set G^a_D equal to zero outside $D \times D$. The function $G^a_D(x, y)$ is also called the Green function of X^a in D. For any $x \in D$, $G^a_D(\cdot, x)$ is superharmonic in D with respect to X^a, harmonic in $D \setminus \{x\}$ with respect to X^a and regular harmonic in $D \setminus \overline{B(x, \varepsilon)}$ with respect to X^a for every $\varepsilon > 0$.

Recall that a point z on the boundary ∂D of an open set D is said to be a regular boundary point for X^a if $\mathbb{P}_z(\tau^a_D = 0) = 1$. An open set D is said to be regular for X^a if every point in ∂D is a regular boundary point for X^a. It is easy to check that every $C^{1,1}$ open set D is regular for X^a for all $a > 0$ and using the argument in the last paragraph of the proof of [18, Theorem 2.4], we conclude that for any bounded $C^{1,1}$ open set D, $G^a_D(\cdot, z)$ vanishes continuously on ∂D for every $z \in D$.

Now, as a corollary of the uniform boundary Harnack principle and the fact that, for any bounded $C^{1,1}$ open set D, $G^a_D(x, \cdot)$ vanishes continuously on ∂D for every fixed $x \in D$, we have the following proposition.

PROPOSITION 2.4. *Suppose that $M > 0$. For any bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists a positive constant $C_3 = C_3(\alpha, d, \Lambda, R, M) > 1$ such that for all $Q \in \partial D$, $r \in (0, R)$ and $a \in (0, M]$ we have*

$$\frac{G^a_D(x, z_1)}{G^a_D(y, z_1)} \leq C_3 \frac{G^a_D(x, z_2)}{G^a_D(y, z_2)}, \tag{2.9}$$

when $x, y \in D \setminus \overline{B(Q, r)}$ and $z_1, z_2 \in D \cap B(Q, r/2)$.

The following scaling property will be used below: If $(X^{a,D}_t, t \geq 0)$ is the subprocess in D of the sum of a Brownian motion and an independent symmetric stable process in $\mathbb{R}^d$ with weight a, then $(\lambda X^{a,D}_{\lambda^{-2}t}, t \geq 0)$ is the subprocess in λD of the sum of a Brownian motion and an independent symmetric stable process in $\mathbb{R}^d$ with weight $a\lambda^{(\alpha-2)/\alpha}$. So for any $\lambda > 0$, we have

$$p^{a\lambda^{(\alpha-2)/\alpha}}_{\lambda D}(t, x, y) = \lambda^{-d} p^a_D(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \quad \text{for } t > 0 \text{ and } x, y \in \lambda D. \tag{2.10}$$

By integrating the above equation with respect to t, we get that when D is Greenian for X^a,

$$G_D^a(x,y) = \lambda^{d-2} G_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(\lambda x, \lambda y) \quad \text{for } x, y \in D. \tag{2.11}$$

In particular, for $d=2$, we have

$$G_D^a(x,y) = G_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(\lambda x, \lambda y) \quad \text{for } x, y \in D. \tag{2.12}$$

3. Higher dimensional case: $d \geq 3$

In this section, we assume that $d \geq 3$. We will use $G^a(x,y) = G^a(y-x) = G^a_{\mathbb{R}^d}(x,y)$ to denote the Green function of X^a in $\mathbb{R}^d$.

Recall that u^a is the potential density of the subordinator $T_t^a = t + a^2 T_t$ given in (2.1). The Green function G^a of X^a is also given by the following formula [33]

$$G^a(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} u^a(t)\, dt, \quad x \in \mathbb{R}^d. \tag{3.1}$$

Using this formula, we can easily see that G^a is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

Lemma 3.1. *Suppose that $M > 0$. For all $a \in [0, M]$, we have*

$$G^M(x) \leq G^a(x) \leq \frac{1}{|x|^{d-2}} \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. We have seen that for all $t > 0$, $u^a(t) \leq 1$, and the function $a \mapsto u^a(t)$ is decreasing on $\mathbb{R}_+$, cf. (2.2) and the text preceding it. The desired inequalities follow immediately from these properties and (3.1). □

Lemma 3.2. *Suppose that $M > 0$. There exist $R_1 > 0$ and $L_1 \geq 2$ such that for all $a \in [0, M]$*

$$G^a(x) \geq 2G^a(L_1 x) \quad \textit{for all } |x| < R_1. \tag{3.2}$$

Proof. By [33, Theorem 3.1], there exists $c_1 = c_1(\alpha, d, M) > 0$ such that

$$\lim_{|x| \to 0} G^M(x)|x|^{d-2} = c_1. \tag{3.3}$$

Let $L_1 = (2/c_1)^{2/(d-2)} \vee 2$ and take $0 < \delta_1 < c_1(1 - L^{-(d-2)/2})$. Using (3.3), we can choose a positive constant $R_1 > 0$ such that

$$(c_1 - \delta_1)\frac{1}{|x|^{d-2}} \leq G^M(x) \quad \text{when } |x| \leq R_1. \tag{3.4}$$

Thus, by Lemma 3.1, for every $|x| < R_1$

$$G^a(x) \geq G^M(x) \geq (c_1 - \delta_1)\frac{1}{|x|^{d-2}} \geq L_1^{\frac{d-2}{2}} \frac{c_1}{|L_1 x|^{d-2}} \geq 2G^a(L_1 x). \qquad \square$$

The next proposition gives the interior estimates for G_D^a.

PROPOSITION 3.3. *Suppose that* $M > 0$. *For any bounded and connected* $C^{1,1}$ *open set* D *in* $\mathbb{R}^d$ *there exists a positive constant* C_4 *such that for every* $a \in [0, M]$

$$G_D^a(x,y) \le C_4 \frac{1}{|x-y|^{d-2}} \quad \text{for all } x, y \in D \tag{3.5}$$

and

$$G_D^a(x,y) \ge C_4^{-1} \frac{1}{|x-y|^{d-2}} \quad \text{when } 2|x-y| \le \delta_D(x) \wedge \delta_D(y). \tag{3.6}$$

Proof. Since $G_D^a(x,y) \le G^a(x,y)$, (3.5) is an immediate consequence of Lemma 3.1. So we only need to show (3.6). Without loss of generality, we assume that $\delta_D(y) \le \delta_D(x)$.

Recall that $L_1 \ge 2$ and R_1 are the constants from Lemma 3.2. By Lemmas 3.1, 3.2 and (3.4), we have

$$\begin{aligned} G^a(x,y) - G^a(L_1 x, L_1 y) &\ge \frac{1}{2} G^a(x,y) \ge \frac{1}{2} G^M(x,y) \\ &\ge c_1 \frac{1}{|x-y|^{d-2}} \quad \text{when } |x-y| \le R_1 \end{aligned} \tag{3.7}$$

for some positive constant c_1.

Case 1: $L_1|x-y| \le \delta_D(y)$. We consider three subcases separately:

(a) $\delta_D(y) \le R_1$. Note that, since $L_1|x-y| \le \delta_D(y)$,

$$\left|X^a_{\tau^a_{B(y,\delta_D(y))}} - y\right| \ge \delta_D(y) \ge L_1|x-y|.$$

Thus by the fact that $G^a(\cdot)$ is radially decreasing and (3.7),

$$\begin{aligned} G_D^a(x,y) \ge G^a_{B(y,\delta_D(y))}(x,y) &= G^a(x,y) - \mathbb{E}_x\left[G^a(X^a_{\tau^a_{B(y,\delta_D(y))}}, y)\right] \\ &\ge G^a(x,y) - G^a(L_1 x, L_1 y) \ge c_1 \frac{1}{|x-y|^{d-2}}. \end{aligned}$$

(b) $\delta_D(y) > R_1$ and $L_1|x-y| \le R_1$. In this case, $|X^a_{\tau^a_{B(y,R_1)}} - y| \ge R_1 \ge L_1|x-y|$ and, again by the fact that $G^a(\cdot)$ is radially decreasing and (3.7),

$$\begin{aligned} G_D^a(x,y) \ge G^a_{B(y,R_1)}(x,y) &= G^a(x,y) - \mathbb{E}_x\left[G^a(X^a_{\tau^a_{B(y,R_1)}}, y)\right] \\ &\ge G^a(x,y) - G^a(L_1 x, L_1 y) \ge c_1 \frac{1}{|x-y|^{d-2}}. \end{aligned}$$

(c) $\delta_D(y) > R_1$ and $L_1|x-y| > R_1$. In this case, we have $\delta_D(x) \ge \delta_D(y) \ge L_1|x-y| \ge R_1$. Choose a point $w \in \partial B(y, R_1/(2L_1))$. Then from the argument in (b), we get

$$G_D^a(w,y) \ge c_1 \frac{1}{(R_1/(2L_1))^{d-2}}.$$

Since D is a bounded and connected $C^{1,1}$ open set and $|x-w| \le |x-y| + |y-w| \le \delta_D(y)/L_1 + R_1/(2L_1) \le 3\delta_D(y)/(2L_1)$, by Proposition 2.2 and a chain argument, we have

$$G_D^a(x,y) \ge c_2 G_D^a(w,y) \ge c_3 \frac{1}{(R_1/(2L_1))^{d-2}}$$
$$\ge c_3 2^{d-2}(2L_1/R_1)^{2(d-2)} \frac{1}{|x-y|^{d-2}}.$$

Case 2: $2|x-y| \le \delta_D(y) < L_1|x-y|$. Take $x_0 \in \partial B(y, \delta_D(y)/(L_1+1))$. Then

$$|x-y| \le \frac{1}{2}\delta_D(y) \le L_1|x_0 - y| = \frac{L_1}{L_1+1}\delta_D(y) \le \delta_D(y) \wedge \delta_D(x_0).$$

Since D is a bounded and connected $C^{1,1}$ open set and $|x_0 - x| \le |x_0 - y| + |y - x| \le (\frac{1}{L_1+1} + \frac{1}{2})\delta_D(y)$, by Proposition 2.2, a chain argument and the argument in the first case, there are constants $c_i = c_i(D, \alpha, L_1, M) > 0, i = 4, 5, 6$, such that

$$G_D^a(x,y) \ge c_4 G_D^a(x_0, y) \ge c_5 \frac{1}{|x_0 - y|^{d-2}} \ge c_6 \frac{1}{|x-y|^{d-2}}.$$

This completes the proof of the proposition. □

Suppose that D is a bounded and connected $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R, Λ) and corresponding κ. Fix $z_0 \in D$ with $\kappa R < \delta_D(z_0) < R$, and let $\varepsilon_1 := \kappa R/24$. For $x, y \in D$, define $r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and

$$\mathcal{B}(x,y) := \left\{ z \in D : \delta_D(z) > \frac{\kappa}{2} r(x,y), |x-z| \vee |y-z| < 5r(x,y) \right\}$$

if $r(x,y) < \varepsilon_1$, and $\mathcal{B}(x,y) := \{z_0\}$ otherwise.

Put $C_5 := C_4 2^{d-2} \delta_D(z_0)^{-d+2}$. Then by (3.5),

$$G_D^a(\cdot, z_0) \le C_5 \quad \text{on } D \setminus B\big(z_0, \delta_D(z_0)/2\big).$$

Now we define

$$g^a(x) := G_D^a(x, z_0) \wedge C_5.$$

Note that if $\delta_D(z) \le 6\varepsilon_1$, then $|z - z_0| \ge \delta_D(z_0) - 6\varepsilon_1 > \delta_D(z_0)/2$ since $6\varepsilon_1 < \delta_D(z_0)/4$, and therefore $g^a(z) = G_D^a(z, z_0)$.

Using the uniform Harnack principle (Proposition 2.2) and Proposition 2.4, the following form of Green function estimates follows from [23, Theorem 2.4].

THEOREM 3.4. *Suppose that $M > 0$. For any bounded and connected $C^{1,1}$ open set D in $\mathbb{R}^d$, there exists $C_6 = C_6(D, M, \alpha) > 0$ such that for all x, y in D and all $a \in (0, M]$*

$$C_6^{-1} \frac{g^a(x) g^a(y)}{g^a(A)^2} |x-y|^{-d+2} \le G_D^a(x,y) \le C_6 \frac{g^a(x) g^a(y)}{g^a(A)^2} |x-y|^{-d+2},$$

where $A \in \mathcal{B}(x,y)$.

Suppose D is a bounded and connected $C^{1,1}$ open set. For all $x, y \in D$, we let Q_x and Q_y be points on ∂D such that $\delta_D(x) = |x - Q_x|$ and $\delta_D(y) = |y - Q_y|$, respectively. It is easy to check that if $r(x,y) < \varepsilon_1$

$$A_{r(x,y)}(Q_x), A_{r(x,y)}(Q_y) \in \mathcal{B}(x,y). \tag{3.8}$$

(Recall that, for any $Q \in \partial D$, $A_r(Q)$ is a point such that $B(A_r(Q), \kappa r) \subset D \cap B(Q,r)$.) Indeed, by the definition of $A_{r(x,y)}(Q_x)$, $\delta_D(A_{r(x,y)}(Q_x)) \geq \kappa r(x,y) > \kappa r(x,y)/2$. Moreover,

$$\begin{aligned}\left|x - A_{r(x,y)}(Q_x)\right| &\leq |x - Q_x| + \left|Q_x - A_{r(x,y)}(Q_x)\right| \\ &\leq \delta_D(x) + r(x,y) \leq 2r(x,y)\end{aligned}$$

and $|y - A_{r(x,y)}(Q_x)| \leq |y - x| + |x - A_{r(x,y)}(Q_x)| \leq 3r(x,y)$. This verifies the claim (3.8).

Recall the fact that $g^a(z) = G_D^a(z, z_0)$ if $\delta_D(z) < 6\varepsilon_1$. By Theorem 2.3 and the fact that $\kappa r(x,y) \leq \delta_D(A_{r(x,y)}(Q_y)) \leq r(x,y)$, there exists $c_1 > 1$ such that for every $a \in (0, M]$ and all $x, y \in D$ with $\delta_D(x) < 6\varepsilon_1$ and $\delta_D(y) < 6\varepsilon_1$,

$$c_1^{-1}\frac{\delta_D(x)}{r(x,y)} \leq \frac{g^a(x)}{g^a(A_{r(x,y)}(Q_x))} = \frac{G_D^a(x,z_0)}{G_D^a(A_{r(x,y)}(Q_x), z_0)} \leq c_1\frac{\delta_D(x)}{r(x,y)} \tag{3.9}$$

and

$$c_1^{-1}\frac{\delta_D(y)}{r(x,y)} \leq \frac{g^a(y)}{g^a(A_{r(x,y)}(Q_y))} = \frac{G_D^a(y,z_0)}{G_D^a(A_{r(x,y)}(Q_y), z_0)} \leq c_1\frac{\delta_D(y)}{r(x,y)}. \tag{3.10}$$

Proof of Theorem 1.1 *when* $d \geq 3$. First, we assume that D is connected.

Combining inequalities (3.9) and (3.10) with Proposition 3.3, Theorem 3.4 and the fact that

$$\frac{\delta_D(x)\delta_D(y)}{(r(x,y))^2} \leq \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \leq \frac{9}{4}\frac{\delta_D(x)\delta_D(y)}{(r(x,y))^2} \tag{3.11}$$

(see [6]), we get the inequalities (1.3).

Next we assume that D is not connected. Let (R, Λ) be the $C^{1,1}$ characteristics of D. Note that D has only finitely many components and the distance between any two distinct components of D is at least $R > 0$. Assume first that x and y are in two distinct components of D. Let $D(x)$ be the component of D that contains x. Then by the strong Markov property and the Lévy system (2.4) of X^a, we have

$$\begin{aligned}G_D^a(x,y) &= \mathbb{E}_x[G_D^a(X^a_{\tau^a_{D(x)}}, y)] \\ &= \mathbb{E}_x\left[\int_0^{\tau^a_{D(x)}}\left(\int_{D\setminus D(x)} j^a(|X_s^a - z|)G_D^a(z,y)\,dz\right)ds\right].\end{aligned}$$

Consequently,

$$
\begin{aligned}
(3.12)\qquad & j^a(\operatorname{diam}(D))\mathbb{E}_x\left[\tau^a_{D(x)}\right]\int_{D\setminus D(x)} G^a_D(y,z)\,dz \\
&\quad \le G^a_D(x,y) \le j^a(R)\mathbb{E}_x\left[\tau^a_{D(x)}\right]\int_{D\setminus D(x)} G^a_D(y,z)\,dz.
\end{aligned}
$$

Applying the two-sided estimates (1.3) established in the first part of this proof to $D(x)$, we get

$$
(3.13)\qquad c_1^{-1}\delta_D(x) = c_1^{-1}\delta_{D(x)}(x) \le \mathbb{E}_x\left[\tau^a_{D(x)}\right] \le c_1\delta_{D(x)}(x) = c_1\delta_D(x)
$$

for some $c_1 = c_1(D,M,\alpha) > 1$. Clearly, using (3.13),

$$
\begin{aligned}
\int_{D\setminus D(x)} G^a_D(y,z)\,dz &\ge \int_{D(y)} G^a_{D(y)}(y,z)\,dz = \mathbb{E}_y\left[\tau^a_{D(y)}\right] \\
&\ge c_2\delta_D(y).
\end{aligned}
$$

On the other hand, it follows from (3.5) that $\sup_{z\in D, a\in(0,M]}\mathbb{E}_z[\tau^a_D] \le c_3 < \infty$. Moreover by (3.13) and the Lévy system (2.4) of X^a,

$$
\begin{aligned}
&\int_{D\setminus D(x)} G^a_D(y,z)\,dz \\
&\quad \le \mathbb{E}_y[\tau^a_D] = \mathbb{E}_y\left[\tau^a_{D(y)}\right] + \mathbb{E}_y\left[\mathbb{E}_{X_{\tau^a_{D(y)}}}[\tau^a_D]\right] \\
&\quad \le c_4\delta_D(y) + \mathbb{E}_y\left[\int_0^{\tau^a_{D(y)}}\left(\int_{D\setminus D(y)} j^a(|X^a_s - z|)\mathbb{E}_z[\tau^a_D]\,dz\right)ds\right] \\
&\quad \le c_4\delta_D(y) + c_5 j^M(R)\mathbb{E}_y\left[\tau^a_{D(y)}\right] \le c_6\delta_D(y).
\end{aligned}
$$

We conclude from the last three displays, (3.12) and the form of j^a given in (2.3) that there is a constant $c_7 = c_7(D,M,\alpha) \ge 1$ such that for every $a \in (0,M]$,

$$
(3.14)\qquad c_7^{-1}a^\alpha\delta_D(x)\delta_D(y) \le G_D(x,y) \le c_7 a^\alpha\delta_D(x)\delta_D(y).
$$

Since for x and y in different components of D, $R \le |x-y| \le \operatorname{diam}(D)$, we have established (1.3).

Now we assume that x, y are in the same component U of D. Applying (1.3) to U, we get

$$
\begin{aligned}
G^a_D(x,y) \ge G^a_U(x,y) &\ge c_8\frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|^2}\right) \\
&= c_8\frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right).
\end{aligned}
$$

For the upper bound, we use the strong Markov property, the Lévy system (2.4), and (3.13), (3.14) to get

$$
\begin{aligned}
G^a_D(x,y) &= G^a_U(x,y) + \mathbb{E}_x[G^a_D(X^a_{\tau^a_U}, y)] \\
&\le c_9 \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \\
&\quad + \mathbb{E}_x\left[\int_0^{\tau^a_U}\left(\int_{D\setminus U} j^a(|X^a_s - z|) G^a_D(z,y)\,dz\right) ds\right] \\
&\le c_9 \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \\
&\quad + j^M(R)\mathbb{E}_x[\tau^a_U]\int_{D\setminus U} G^a_D(y,z)\,dz \\
&\le c_9 \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \\
&\quad + c_{10}\delta_D(x)\delta_D(y)\int_{D\setminus U}\delta_D(z)\,dz.
\end{aligned}
\tag{3.15}
$$

Since the boundedness of D implies

$$
\delta_D(x)\delta_D(y) \le c_{11}\frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right),
$$

we have from (3.15)

$$
G^a_D(x,y) \le c_{12}\frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right). \qquad \square
$$

Define

$$
a(x,y,z,w) := \begin{cases} 1, & \text{when } x,y,z,w \text{ are in the same component of } D, \\ a^{-\alpha}, & \text{when } x,y \in D(x), z \notin D(x) \text{ and } w \in D(z), \\ a^{\alpha}, & \text{when } x,w \in D(x), \#\big(\{y,z\} \cap D(x)\big) = 1, \\ a^{\alpha}, & \text{when } x,y,z,w \text{ are all in different components of } D, \\ a^{2\alpha}, & \text{when } x,w \in D(x), \{y,z\} \cap D(x) = \emptyset. \end{cases}
\tag{3.16}
$$

The next theorem will be used in Section 7.

THEOREM 3.5 (Generalized 3G theorem). *Suppose that $M > 0$. For any bounded $C^{1,1}$ open set D in $\mathbb{R}^d$, there exists a constant $C_8 = C_8(D, \alpha, M)$ such* *that for all $x, y, z, w \in D$ and $a \in (0, M]$,*

$$\begin{aligned} \frac{G_D^a(x,y)G_D^a(z,w)}{G_D^a(x,w)} &\le C_8 a(x,y,z,w)\left(\frac{|x-w|\wedge|y-z|}{|x-y|}\vee 1\right) \\ &\quad\times\left(\frac{|x-w|\wedge|y-z|}{|z-w|}\vee 1\right)\frac{|x-w|^{d-2}}{|x-y|^{d-2}|z-w|^{d-2}}. \end{aligned} \tag{3.17}$$

Proof. Recall that $r(x,y) = \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and let

$$g_D(x,y) := \frac{1}{|x-y|^{d-2}} \frac{\delta_D(x)\delta_D(y)}{(r(x,y))^2}$$

and

$$H(x,y,z,w) := \frac{|x-w|^{d-2}}{|x-y|^{d-2}|z-w|^{d-2}}.$$

By Theorem 1.1 for the case $d \ge 3$ and (3.11),

$$\frac{G_D^a(x,y)G_D^a(z,w)}{G_D^a(x,w)} \le c_1 a(x,y,z,w)\frac{g_D(x,y)g_D(z,w)}{g_D(x,w)} \tag{3.18}$$

$$= c_1 a(x,y,z,w)\frac{\delta_D(y)\delta_D(z)r(x,w)^2}{r(x,y)^2 r(z,w)^2} H(x,y,z,w). \tag{3.19}$$

1. If $|x-w| \le \delta_D(x) \wedge \delta_D(w)$, $g_D(x,w) \ge |x-w|^{-d+2}$. Thus, by (3.18)
$$\frac{G_D^a(x,y)G_D^a(z,w)}{G_D^a(x,w)} \le c_1 a(x,y,z,w)H(x,y,z,w).$$
2. Note that if $y = z$, since $r(x,w) \le 2r(x,y) + 2r(y,w)$,
$$\frac{\delta_D(y)\delta_D(y)r(x,w)^2}{r(x,y)^2 r(y,w)^2} \le 8\left(\frac{\delta_D(y)^2}{r(y,w)^2} + \frac{\delta_D(y)^2}{r(x,y)^2}\right) \le 8.$$

Thus,

$$\frac{g_D(x,y)g_D(y,w)}{g_D(x,w)} \le 8H(x,y,y,w). \tag{3.20}$$

Now consider the case $|y-z| \le \delta_D(y) \wedge \delta_D(z)$. In this case $g_D(y,z) \ge |y-z|^{-d+\alpha}$. Thus, using (3.20), we obtain that

$$\frac{g_D(x,y)g_D(z,w)}{g_D(x,w)} = \frac{g_D(x,y)g_D(y,z)}{g_D(x,z)} \frac{g_D(x,z)g_D(z,w)}{g_D(x,w)} \frac{1}{g_D(y,z)} \tag{3.21}$$

$$\begin{aligned}&\le 64\frac{|x-z|^{d-2}}{|x-y|^{d-2}|y-z|^{d-2}}\frac{|x-w|^{d-2}}{|x-z|^{d-2}|z-w|^{d-2}}\frac{1}{g_D(y,z)}\\ &= 64\frac{|x-w|^{d-2}}{|x-y|^{d-2}|y-z|^{d-2}|z-w|^{d-2}}\frac{1}{g_D(y,z)}.\end{aligned}$$

Thus, by (3.18) and (3.21), we have

$$\frac{G_D^a(x,y)G_D^a(z,w)}{G_D^a(x,w)} \le c_2 a(x,y,z,w)H(x,y,z,w).$$

3. Now we assume that $|x-w| > \delta_D(x)\wedge\delta_D(w)$ and $|y-z| > \delta_D(y)\wedge\delta_D(z)$. Since $\delta_D(x)\vee\delta_D(w) \le \delta_D(x)\wedge\delta_D(w) + |x-w|$, using the assumption $\delta_D(x)\wedge\delta_D(w) < |x-w|$, we obtain $r(x,w) < 2|x-w|$. Similarly, $r(y,z) < 2|y-z|$. By (3.19), we only need to show that

(3.22)
$$\frac{\delta_D(y)\delta_D(z)r(x,w)^2}{r(x,y)^2 r(z,w)^2} \le c_3\left(\frac{|x-w|\wedge|y-z|}{|x-y|}\vee 1\right)\left(\frac{|x-w|\wedge|y-z|}{|z-w|}\vee 1\right).$$

Since $r(x,w) \le 2r(x,y) + 2r(y,w) \le 2r(x,y) + 4r(y,z) + 4r(z,w)$, we have

$$\begin{aligned}\frac{\delta_D(y)\delta_D(z)r(x,w)^2}{r(x,y)^2 r(z,w)^2} &\le c_4\left(\frac{\delta_D(y)\delta_D(z)}{r(z,w)^2} + \frac{\delta_D(y)\delta_D(z)}{r(x,y)^2} + \frac{\delta_D(y)\delta_D(z)r(y,z)^2}{r(x,y)^2 r(z,w)^2}\right)\\ &\le c_4\left(\frac{\delta_D(y)}{r(z,w)} + \frac{\delta_D(z)}{r(x,y)} + \frac{r(y,z)^2}{r(x,y)r(z,w)}\right)\\ &\le c_4\left(\frac{r(y,z)}{r(z,w)} + \frac{r(y,z)}{r(x,y)} + \frac{r(y,z)^2}{r(x,y)r(z,w)}\right),\end{aligned}$$

which is, by [25, Lemma 3.15], less than or equal to

$$2c_4\left(\frac{r(y,z)}{r(x,y)}\vee 1\right)\left(\frac{r(y,z)}{r(z,w)}\vee 1\right).$$

On the other hand, clearly

$$\begin{aligned}\frac{\delta_D(y)\delta_D(z)r(x,w)^2}{r(x,y)^2 r(z,w)^2} &= \frac{\delta_D(y)\delta_D(z)}{r(x,y)r(z,w)}\frac{r(x,w)^2}{r(x,y)^2 r(z,w)^2}\\ &\le \left(\frac{r(x,w)}{r(x,y)}\vee 1\right)\left(\frac{r(x,w)}{r(z,w)}\vee 1\right).\end{aligned}$$

Thus,

$$\frac{\delta_D(y)\delta_D(z)r(x,w)^2}{r(x,y)^2 r(z,w)^2} \le c_5\left(\frac{r(y,z)\wedge r(x,w)}{r(x,y)}\vee 1\right)\left(\frac{r(y,z)\wedge r(x,w)}{r(z,w)}\vee 1\right).$$

Now applying the fact that $r(x,w) < 2|x-w|$, $r(y,z) < 2|y-z|$, $r(x,y) \ge |x-y|$ and $r(z,w) \ge |z-w|$, we arrive at (3.22).

We have proved the theorem. $\square$

Note that, since we consider disconnected open sets too, we can not apply [25, Theorem 1.1] directly to get the generalized 3G theorem.

Taking $y=z$ in (3.17), we get the classical 3G estimates, that is,

$$\begin{aligned}\frac{G_D^a(x,z)G_D^a(z,w)}{G_D^a(x,w)} &\le C_8 a(x,z,z,w)\frac{|x-w|^{d-2}}{|x-z|^{d-2}|z-w|^{d-2}}\\ &\le C_8(M^{2\alpha}\vee 1)\frac{|x-w|^{d-2}}{|x-z|^{d-2}|z-w|^{d-2}}.\end{aligned}$$

4. Two dimensional case

In this section, we assume $d=2$ and prove Theorem 1.1 for this case. Unlike the case of $d\ge 3$, due to the recurrence of planar Brownian motions, the methods in [6], [23] are not applicable in dimension $d=2$ even though we have the Harnack and boundary Harnack principles. We use a capacitary approach and some recent results on the subordinate killed Brownian motions instead.

First, we derive the lower bound. The method we use relies on comparing the process $X^{a,D}$, which is the killed subordinate Brownian motion, with another process, the subordinate killed Brownian motion. This method also works for dimensions $d\ge 3$.

To be more precise, let D be a bounded open set in $\mathbb{R}^2$ and $X^{0,D}$ the killed Brownian motion in D. Let $(T_t^a : t\ge 0)$ be a subordinator independent of X^0 which can be written as $T_t^a = t + a^2 T_t$ where $(T_t : t\ge 0)$ is an $\alpha/2$-stable subordinator. The process $(Z_t^{a,D} : t\ge 0)$ defined by $Z_t^{a,D} = X^{0,D}_{T_t^a}$ is called a subordinate killed Brownian motion in D. Let u^a be the potential density of T^a (see (2.1)). It follows from [36] that the Green function $R_D^a(x,y)$ of $Z^{a,D}$ is given by

$$R_D^a(x,y) = \int_0^\infty p_D^0(t,x,y)u^a(t)\,dt, \tag{4.1}$$

where $p_D^0(t,x,y)$ is the transition density of the killed Brownian motion $X^{0,D}$. It is well known (see, for instance, [38, Proposition 3.1]) that

$$R_D^a(x,y)\le G_D^a(x,y),\quad (x,y)\in D\times D. \tag{4.2}$$

THEOREM 4.1. *Suppose that $M>0$. For any bounded $C^{1,1}$* connected *open set D in $\mathbb{R}^2$, there exists a positive constant $C_9=C_9(\alpha,M,D)$ such that for all $x,y\in D$ and all $a\in(0,M]$,*

$$G_D^a(x,y)\ge R_D^a(x,y)\ge C_9\log\left(1+\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right).$$

Proof. First, recall the following lower bound for the transition density of the killed Brownian motion $X^{0,D}$ obtained in [39] which states that for any $A>0$, there exist positive constants c_0 and c_1 such that for any $t\in(0,A]$ and any $x, y \in D$,

$$p_D^0(t,x,y) \geq c_0\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{t}\right)t^{-1}\exp\left(-\frac{c_1|x-y|^2}{t}\right). \tag{4.3}$$

It follows from (2.1) that

$$u^a(t) = u^1\left(a^{\frac{2\alpha}{2-\alpha}}t\right) \quad \text{for } t>0. \tag{4.4}$$

Let $T=(\mathrm{diam}(D))^2$. Since $u^1(t)$ is a completely monotone function with $u^1(0+)=1$, by (4.4), for any $a \in (0, M]$

$$u^a(t) \geq u^1\left(M^{\frac{2\alpha}{2-\alpha}}T\right), \quad t \in (0,T]. \tag{4.5}$$

By a change of variables $s=\frac{|x-y|^2}{t}$, we have

$$\begin{aligned} &\int_0^T \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{t}\right)t^{-1}e^{-c_1\frac{|x-y|^2}{t}}\,dt \\ &\quad= \int_{\frac{|x-y|^2}{T}}^\infty \left(1 \wedge \frac{\delta_D(x)\delta_D(y)s}{|x-y|^2}\right)s^{-1}e^{-c_1 s}\,ds. \end{aligned} \tag{4.6}$$

Define

$$f_D(x,y) = \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}. \tag{4.7}$$

Since $1/f_D(x,y) \geq |x-y|^2/\operatorname{diam}(D)^2 = |x-y|^2/T$, we split the last integral into two parts:

$$\begin{aligned} &\int_1^\infty \left(1 \wedge \frac{\delta_D(x)\delta_D(y)s}{|x-y|^2}\right)s^{-1}e^{-c_1 s}\,ds \\ &\quad\geq \left(\int_1^\infty s^{-1}e^{-c_1 s}\,ds\right)\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right), \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \int_{\frac{|x-y|^2}{T}}^1 s^{-1}\left(1 \wedge (f_D(x,y)s)\right)ds &\geq \int_{\frac{|x-y|^2}{T}}^1 s^{-1}\mathbf{1}_{\{s \geq 1/f_D(x,y)\}}\,ds \\ &= \log\left(f_D(x,y) \vee 1\right). \end{aligned} \tag{4.9}$$

Combining (4.3) and (4.6)–(4.9), we have

$$\begin{aligned} \int_0^T p_D^0(t,x,y)\,dt &\geq c_2\left(1 \wedge f_D(x,y)\right) + c_2\log\left(f_D(x,y) \vee 1\right) \\ &\geq c_3 \log\left(1+\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right). \end{aligned}$$

So it follows from (4.1), (4.2) and (4.5) that

$$\begin{aligned} G_D^a(x,y) \geq R_D^a(x,y) &\geq u^1\left(M^{\frac{2}{2-\alpha}}T\right)\int_0^T p_D^0(t,x,y)\,dt \\ &\geq c_4 \log\left(1+\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right). \end{aligned}$$

□

Integrating the estimate in Theorem 4.1 with respect to y yields the following corollary.

COROLLARY 4.2. *Suppose that $M>0$. For any bounded connected $C^{1,1}$ open set D in $\mathbb{R}^2$, there exists a positive constant $C_{10}=C_{10}(\alpha,M,D)$ such that for all $x\in D$ and all $a\in(0,M]$,*

$$\mathbb{E}_x[\tau_D^a]\geq C_{10}\delta_D(x).$$

The inequalities in the next lemma can be proved by elementary calculus and will be used several times without being mentioned explicitly.

LEMMA 4.3. *For any $L>0$, there exists a constant $C_{11}=C_{11}(L)>1$ such that*

$$C_{11}^{-1}b\leq\log(1+b)\leq b\quad\textit{for any } 0<b\leq L$$

and

$$C_{11}^{-1}\log(1+s)\leq\log(1+Ls)\leq C_{11}\log(1+s)\quad\textit{for any } 0<s<\infty.$$

Using Corollary 4.2, Theorem 4.1 can be extended to general (not necessarily connected) bounded $C^{1,1}$ open sets. Recall that g_D^a is defined by (1.1).

THEOREM 4.4. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^2$ with characteristics (R,Λ). There exists a positive constant $C_{12}=C_{12}(\alpha,M,D)$ such that for all $x,y\in D$ and all $a\in(0,M]$,*

$$G_D^a(x,y)\geq C_{12}g_D^a(x,y).$$

Proof. Recall that $f_D(x,y)$ is defined in (4.7). If x and y are in the same component, say $x,y\in U$, then by monotonicity,

$$G_D^a(x,y)\geq G_U^a(x,y)\geq c_1\log\bigl(1+f_U(x,y)\bigr)=c_1\log\bigl(1+f_D(x,y)\bigr). \tag{4.10}$$

If x,y are in the different components of D, using Corollary 4.2 and Lemma 4.3, and by following the second part of the proof of Theorem 1.1 in case $d\geq 3$ (that is, the paragraph containing (3.12) and (3.13)), we get

$$\begin{aligned} G_D^a(x,y)&\geq c_2a^\alpha\delta_D(x)\delta_D(y)\geq c_2R^2a^\alpha\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\\ &\geq c_3a^\alpha\log\biggl(1+\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\biggr). \end{aligned}$$

This completes the proof of the theorem. □

Recall that when $d\geq 2$ and $a>0$, any non-empty open set $D\subset\mathbb{R}^d$ is Greenian for X^a. For any Greenian open set D, any Borel subset A of D and

$a \geq 0$, we define

$$\text{Cap}^a_D(A) := \sup\left\{\eta(A) : \eta \text{ is a measure supported on } A \text{ with } \int_D G^a_D(x,y)\eta(dy) \leq 1\right\}. \tag{4.11}$$

It is known (cf. [22]) that for any open subset A of D,

$$\text{Cap}^a_D(A) = \inf\left\{\mathcal{E}^a(u,u) : u \in W^{1,2}(\mathbb{R}^d), u = 0 \text{ on } D^c, u \geq 1 \text{ a.e. on } A\right\}$$

and for any Borel subset A of D,

$$\text{Cap}^a_D(A) = \inf\{\text{Cap}^a_D(B) : A \subset B \text{ and } B \text{ is open}\}.$$

Since $\mathcal{E}^0 \leq \mathcal{E}^a$, for any Greenian open set $D \subset \mathbb{R}^d$ and every $a \in [0, M]$

$$\text{Cap}^0_D(A) \leq \text{Cap}^a_D(A) \quad \text{for every } A \subset D. \tag{4.12}$$

LEMMA 4.5. *There exists $C_{13} > 0$ such that*

$$\text{Cap}^0_{B(0,1)}(\overline{B(0,r)}) \geq \frac{C_{13}}{\log(1/r)} \quad \textit{for every } r \in (0, 3/4).$$

Proof. Recall that (see, e.g., [18, p. 178])

$$G^0_{B(0,1)}(x,y) = \frac{1}{2\pi}\log\left(1 + \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\right). \tag{4.13}$$

Let $\mathcal{P}$ denote the family of all probability measures on $\overline{B(0,r)}$. It follows from [21, p. 159] that

$$\text{Cap}^0_{B(0,1)}(\overline{B(0,r)}) = \left(\inf_{\mu \in \mathcal{P}} \sup_{x \in \text{supp}(\mu)} G^0_{B(0,1)}\mu(x)\right)^{-1}. \tag{4.14}$$

Let m_r be the normalized Lebesgue measure on $\overline{B(0,r)}$. By (4.14),

$$\text{Cap}^0_{B(0,1)}(\overline{B(0,r)}) \geq \frac{1}{\sup_{x \in \overline{B(0,r)}} G^0_{B(0,1)} m_r(x)}. \tag{4.15}$$

Further, by using symmetry in the first equality, and (4.13) in the second line, we have

$$\begin{aligned}\sup_{x \in \overline{B(0,r)}} G^0_{B(0,1)} m_r(x) = G^0_{B(0,1)} m_r(0) &= \int_{B(0,r)} G^0_{B(0,1)}(0,y) m_r(dy) \\ &= \frac{1}{\pi r^2}\int_{B(0,r)} \frac{1}{2\pi}\log\frac{1}{|y|^2}\,dy \\ &= \frac{1}{\pi r^2}\frac{r^2}{2}\left(1 + 2\log\frac{1}{r}\right) \leq c\log\frac{1}{r},\end{aligned}$$

for some constant $c > 0$. This together with (4.15) yields the desired capacity estimate. $\square$

For any Borel subset V, we use σ^a_V to denote the first hitting time of V by X^a: $\sigma^a_V = \inf\{t > 0 : X^a_t \in V\}$.

LEMMA 4.6. *Suppose that $M > 0$. There exists $C_{14} > 0$ such that for every $a \in (0, M]$, any Greenian open set D in $\mathbb{R}^2$ containing $B(0,1)$ and any $x \in \overline{B(0, \frac{3}{4})}$*

$$G^a_D(x,0) \le \frac{C_{14}}{\mathrm{Cap}^0_D(\overline{B(0,|x|/2)})} \mathbb{P}_x\big(\sigma^a_{\overline{B(0,|x|/2)}} < \tau_D\big).$$

Proof. Fix $x \in \overline{B(0,3/4)}$ and let $r := |x|/2$. Since $\overline{B(0,r)}$ is a compact subset of D, there exists a capacitary measure μ^a_r for $\overline{B(0,r)}$ with respect to $X^{a,D}$ such that

$$\mathrm{Cap}^a_D(\overline{B(0,r)}) = \mu^a_r(\overline{B(0,r)})$$

(see, for example, [4, Section VI.4] for details). Then by Proposition 2.2, we have

$$\begin{aligned}
(4.16)\qquad \int_{\overline{B(0,r)}} G^a_D(x,y)\mu^a_r(dy) &\ge \Big(\inf_{y\in\overline{B(0,r)}} G^a_D(x,y)\Big)\mu^a_r(\overline{B(0,r)}) \\
&\ge c_1 G^a_D(x,0)\mathrm{Cap}^a_D(\overline{B(0,r)}) \\
&\ge c_1 G^a_D(x,0)\mathrm{Cap}^0_D(\overline{B(0,r)})
\end{aligned}$$

for some constant $c_1 > 0$. In the last inequality above, we have used (4.12).

On the other hand,

$$\begin{aligned}
(4.17)\qquad &\int_{\overline{B(0,r)}} G^a_D(x,y)\mu^a_r(dy) \\
&= \int_{\overline{B(0,r)}} \mathbb{E}_x\big[G^a_D\big(X^{a,D}_{\sigma^a_{\overline{B(0,r)}}}, y\big)\big]\mu^a_r(dy) \\
&\le \Big(\sup_{w\in\overline{B(0,r)}} \int_{\overline{B(0,r)}} G^a_D(w,y)\mu^a_r(dy)\Big)\mathbb{P}_x\big(\sigma^a_{\overline{B(0,r)}} < \tau_D\big) \\
&\le \mathbb{P}_x\big(\sigma^a_{\overline{B(0,r)}} < \tau_D\big).
\end{aligned}$$

In the last inequality above, we have used (4.11).

Combining (4.16) and (4.17), we have

$$G^a_D(x,0) \le \frac{c_1^{-1}}{\mathrm{Cap}^0_D(\overline{B(0,r)})}\mathbb{P}_x\big(\sigma^a_{\overline{B(0,r)}} < \tau_D\big). \qquad \square$$

COROLLARY 4.7. *Suppose that $M > 0$. There exists $C_{15} > 0$ such that for every $a \in (0, M]$ and every $x \in \overline{B(0,3/4)}$*

$$G^a_{B(0,1)}(x,0) \le C_{15}\log(1/|x|).$$

Proof. It follows from Lemmas 4.5 and 4.6 that

$$\begin{aligned} G^a_{B(0,1)}(x,0) &\le \frac{C_{14}}{\mathrm{Cap}^0_{B(0,1)}(\overline{B(0,|x|/2)})}\mathbb{P}_x\big(\sigma^a_{\overline{B(0,|x|/2)}}<\tau_{B(0,1)}\big)\\ &\le C_{14}C_{13}^{-1}\log(2/|x|)\le c\log(1/|x|)\end{aligned}$$

for some constant $c>0$. $\square$

LEMMA 4.8. *Suppose that $M>0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^2$ with characteristics (R,Λ). There exists $C_{16}=C_{16}(D)>0$ such that for every $a\in(0,M]$ and all $x,y\in D$ with $|x-y|\le\frac{3}{4}\delta_D(x)<\frac{3}{4}R$,*

$$G^a_D(x,y)\le C_{16}\log\left(\frac{\delta_D(x)}{|x-y|}\right).$$

Proof. By our assumption, D satisfies the uniform exterior ball condition with radius $R>0$.

Fix $x,y\in D$ with $|x-y|\le\frac{3}{4}\delta_D(x)$ and let $r:=\delta_D(x)$. Since $r<R$, without loss of generality, we may assume $x=(0,0)$, $(0,-r)\in\partial D$ with $B((0,-2r),r)\in\mathbb{R}^2\setminus D$.

Let $\widehat{a}:=ar^{(2-\alpha)/\alpha}$, $\widehat{y}:=r^{-1}y$ and $\widehat{D}:=r^{-1}D$. Then by (2.12),

$$G^a_D(0,y)=G^{\widehat{a}}_{\widehat{D}}(0,\widehat{y}). \tag{4.18}$$

By the strong Markov property, we have

$$G^{\widehat{a}}_{\widehat{D}}(0,\widehat{y})=G^{\widehat{a}}_{B(0,1)}(0,\widehat{y})+\mathbb{E}_0\big[G^{\widehat{a}}_{\widehat{D}}\big(X^{\widehat{a}}_{\tau^{\widehat{a}}_{B(0,1)}},\widehat{y}\big)\big]. \tag{4.19}$$

Note that $\widehat{a}=ar^{(2-\alpha)/\alpha}\le MR^{(2-\alpha)/\alpha}$. Define

$$h(z,w):=\mathbb{E}_z\big[G^{\widehat{a}}_{\widehat{D}}\big(X^{\widehat{a}}_{\tau^{\widehat{a}}_{B(0,1)}},w\big)\big].$$

For each fixed $z\in B(0,1)$, the function $w\mapsto h(z,w)$ is harmonic in $B(0,1)$ with respect to $X^{\widehat{a}}$ and for each fixed $w\in B(0,1)$, $z\mapsto h(z,w)$ is harmonic in $B(0,1)$ with respect to process $X^{\widehat{a}}$. So it follows from Proposition 2.2,

$$h(0,\widehat{y})\le c_1\min_{z,w\in B(0,5/6)}h(z,w)\le c_1\min_{z,w\in B(0,5/6)}G^{\widehat{a}}_{\widehat{D}}(z,w)\le c_1G^{\widehat{a}}_{\widehat{D}}(0,x_1),$$

where $|x_1|=1/2$. In the second inequality we used that $G^{\widehat{a}}_{\widehat{D}}(\cdot,w)$ is superharmonic in $B(0,1)$ for $X^{\widehat{a}}$. Note that $\widehat{D}\subset E:=\mathbb{R}^2\setminus B((0,-2),1)$. Thus, by Lemma 4.6,

$$h(0,\widehat{y})\le c_1G^{\widehat{a}}_{\widehat{D}}(0,x_1)\le c_1G^{\widehat{a}}_{E}(0,x_1)\le\frac{c_2}{\mathrm{Cap}^0_E(\overline{B(0,1/4)})}<\infty. \tag{4.20}$$

On the other hand, by Corollary 4.7

$$G^{\widehat{a}}_{B(0,1)}(0,\widehat{y})\le c_3\log\left(\frac{1}{|\widehat{y}|}\right)=c_3\log\left(\frac{\delta_D(0)}{|y|}\right). \tag{4.21}$$

It follows from (4.18)–(4.21) that

$$G_D^a(x,y) = G_D^a(0,y) \le c_4 + c_3 \log\left(\frac{\delta_D(0)}{|y|}\right) \le c_5 \log\left(\frac{\delta_D(x)}{|x-y|}\right),$$

which proves the lemma. □

LEMMA 4.9. *Suppose that $M>0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^2$. If x and y are in the same component of D with*

$$\frac{1}{c}\big(\delta_D(x) \vee \delta_D(y)\big) \le |x-y| \le c\big(\delta_D(x) \wedge \delta_D(y)\big)$$

for some $c>1$, then there exists $C_{17} = C_{17}(c,D) > 0$ such that for every $a \in [0,M]$

$$G_D^a(x,y) \le C_{17}.$$

Proof. Without loss of generality, we assume $\delta_D(x) \le \delta_D(y)$. If $\frac{1}{c}\delta_D(y) \le |x-y| \le \frac{3}{4}\delta_D(x)$, then the lemma follows from Lemma 4.8. In the case $\frac{3}{4}\delta_D(x) \le |x-y| \le c\delta_D(x)$, since x,y are in the same component of D, we use Proposition 2.2 and a standard Harnack chain argument. □

THEOREM 4.10. *Suppose that $M>0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^2$. There exists $C_{18} = C_{18}(D) > 0$ such that for every $a \in (0,M]$ and all $x,y \in D$*

$$G_D^a(x,y) \le C_{18} \log\left(1 + \frac{1}{|x-y|^2}\right).$$

Proof. Let $L = \max\{2\mathrm{diam}(D), 2\}$.

(i) If $|x-y| < 1/4$, by Lemma 4.8 applied to $B(x,L)$,

$$G_D^a(x,y) \le G_{B(x,L)}^a(x,y) \le c_1 \log\left(\frac{1}{|x-y|}\right) \le c_2 \log\left(1 + \frac{1}{|x-y|^2}\right).$$

(ii) If $1/4 \le |x-y|$, by (2.12) and Corollary 4.7,

$$\begin{aligned} G_D^a(x,y) &\le G_{B(x,L)}^a(x,y) \le G_{B(L^{-1}x,1)}^{aL^{(2-\alpha)/\alpha}}(L^{-1}x, L^{-1}y) \le c_3 \log\left(\frac{L}{|x-y|}\right) \\ &\le c_3 \log(4L) \le c_4 \le c_5 \log\left(1 + \frac{1}{|x-y|^2}\right). \end{aligned}$$

□

Now we are ready to prove Theorem 1.1 for $d=2$. Recall that $f_D(x,y)$ is defined in (4.7).

Proof of Theorem 1.1 when $d=2$. By Theorem 4.4, we only need to consider the upper bound. We divide its proof into two steps.

Step 1. We first consider the case that x and y are in the same component of D. Without loss of generality, throughout this proof, we assume $\delta_D(x) \le \delta_D(y)$.

Fix $z_0 \in D$ with $\kappa R < \delta_D(z_0) < R$, and let $\varepsilon_1 := \kappa R/24$. Choose $Q_x, Q_y \in \partial D$ with $|Q_x - x| = \delta_D(x)$ and $|Q_y - y| = \delta_D(y)$. We consider the following five cases separately.

(a) If $\delta_D(x) \geq \varepsilon_1 \kappa^2/32$, by Theorem 4.10

$$G_D^a(x,y) \leq c_1 \log\left(1 + \frac{1}{|x-y|^2}\right) \leq c_2 \log\bigl(1 + f_D(x,y)\bigr).$$

(b) Suppose $\delta_D(x) < \varepsilon_1\kappa^2/32$ and $\delta_D(y) \geq \varepsilon_1\kappa/4$. Let $r := \varepsilon_1\kappa/16$ and put $x_1 = A_{r\kappa/2}(Q_x)$. One can easily check that $|z_0 - Q_x| \geq r$ and $|y - Q_x| \geq r$. So by (2.9), Theorem 2.3 and Theorem 4.10, we have

$$G_D^a(x,y) \leq c_3 G_D^a(x_1,y)\frac{G_D^a(x,z_0)}{G_D^a(x_1,z_0)} \leq c_4 \delta_D(x) \leq c_5 f_D(x,y)$$

for some $c_3, c_4, c_5 > 0$. Note that $f_D(x,y) < c_6$ in this case because D is bounded and $|x-y| \geq \delta_D(y) - \delta_D(x) \geq \varepsilon_1\kappa(1/4 - \kappa/32) > 0$. So it follows from the above display and Lemma 4.3 that

$$G_D^a(x,y) \leq c_7 \log\bigl(1 + f_D(x,y)\bigr).$$

(c) Suppose $\delta_D(x) < \varepsilon_1\kappa^2/32$, $\delta_D(y) \leq \varepsilon_1\kappa/4$ and $|x-y| < \delta_D(y)/2$. From $\delta_D(y) \leq |x-y| + \delta_D(x)$ we conclude that $\delta_D(y) < 2\delta_D(x)$ and so $|x-y| < \delta_D(x)$. This together with Lemma 4.8 gives that

$$G_D^a(x,y) \leq c_8 \log\left(\frac{\delta_D(y)}{|x-y|}\right) \leq c_9 \log\bigl(1 + f_D(x,y)\bigr).$$

(d) If $\frac{1}{2}\delta_D(y) \leq |x-y| \leq (24/\kappa^2)\delta_D(x)$, by Lemma 4.9,

$$G_D^a(x,y) \leq c_{10} \leq c_{11} \log\bigl(1 + f_D(x,y)\bigr).$$

(e) The remaining case is

$$\delta_D(x) \leq \frac{\varepsilon_1\kappa^2}{32}, \qquad \delta_D(x) \leq \delta_D(y) \leq \frac{\varepsilon_1\kappa}{4}$$

and

$$|x-y| > \max\left\{\frac{24}{\kappa^2}\delta_D(x), \frac{\delta_D(y)}{2}\right\}.$$

We claim that in this case

$$G_D^a(x,y) \leq c_{12} f_D(x,y). \tag{4.22}$$

By Lemma 4.3, the above implies that $G_D^a(x,y) \leq c_{13}\log(1 + f_D(x,y))$ since in this case

$$f_D(x,y) = \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \leq 4.$$

We now proceed to prove (4.22) by considering the following two subcases.

(i) $(24/\kappa^2)\delta_D(x) \le |x-y| \le (4/\kappa)\delta_D(y)$: Let $r := \delta_D(y)/3$. Put $x_1 = A_{r\kappa/2}(Q_x)$. One can easily check that $|z_0 - Q_x| \ge r$ and $|y - Q_x| \ge r$. So by (2.9) and Theorem 2.3, we have

$$G_D^a(x,y) \le c_{14} G_D^a(x_1,y) \frac{G_D^a(x,z_0)}{G_D^a(x_1,z_0)} \le c_{15} G_D^a(x_1,y) \frac{\delta_D(x)}{|x-y|}.$$

Moreover,

$$\frac{24}{\kappa^2}\delta_D(x) \le |x-y| \le \frac{3}{2}|x_1 - y| \le \frac{6}{\kappa}\delta_D(y) \le \frac{72}{\kappa^3}\delta_D(x_1),$$

implying that

$$\delta_D(y) \le |x-y| + \delta_D(x) \le \left(\frac{3}{2} + \frac{36}{\kappa^2}\right)|x_1 - y|.$$

It follows from Lemma 4.9 that $G_D^a(x_1,y) \le c_{16}$. Therefore,

$$G_D^a(x,y) \le c_{17} \frac{\delta_D(x)}{|x-y|} \le c_{18} f_D(x,y).$$

(ii) $\delta_D(x) \le \delta_D(y) \le (\kappa/4)|x-y|$: Let $r := \frac{1}{2}(|x-y| \wedge \varepsilon_1)$. Put $x_1 = A_{r\kappa/2}(Q_x)$ and $y_1 = A_{r\kappa/2}(Q_y)$. Then, since $|z_0 - Q_x| \ge r$ and $|y - Q_x| \ge r$, by (2.9), we have

$$c_{19}^{-1} \frac{G_D^a(x_1,y)}{G_D^a(x_1,z_0)} \le \frac{G_D^a(x,y)}{G_D^a(x,z_0)} \le c_{19} \frac{G_D^a(x_1,y)}{G_D^a(x_1,z_0)}$$

for some $c_{19} > 1$. On the other hand, since $|z_0 - Q_y| \ge r$ and $|x_1 - Q_y| \ge r$, applying (2.9),

$$c_{19}^{-1} \frac{G_D^a(x_1,y_1)}{G_D^a(x_1,y)} \le \frac{G_D^a(y_1,z_0)}{G_D^a(y,z_0)} \le c_{19} \frac{G_D^a(x_1,y_1)}{G_D^a(x_1,y)}.$$

Putting the four inequalities above together we get

$$c_{19}^{-2} \frac{G_D^a(x_1,y_1)}{G_D^a(x_1,z_0)G_D^a(y_1,z_0)} \le \frac{G_D^a(x,y)}{G_D^a(x,z_0)G_D^a(y,z_0)} \le c_{19}^2 \frac{G_D^a(x_1,y_1)}{G_D^a(x_1,z_0)G_D^a(y_1,z_0)}.$$

Moreover, $\frac{1}{3}|x-y| < |x_1 - y_1| < 2|x-y|$ and

$$\frac{4}{3\kappa}\big(\delta_D(x_1) \vee \delta_D(y_1)\big) \le \frac{1}{3}|x-y| \le |x_1 - y_1| \le \frac{64}{\kappa^3 \varepsilon_1}\big(\delta_D(x_1) \wedge \delta_D(y_1)\big).$$

Thus by Lemma 4.9 and Theorem 2.3, we have

$$G_D^a(x,y) \le c_{20} \frac{G_D^a(x,z_0)G_D^a(y,z_0)}{G_D^a(x_1,z_0)G_D^a(y_1,z_0)} \le c_{21} f_D(x,y)$$

for some $c_{20}, c_{21} > 0$. This completes the proof of the claim (4.22) and therefore of the theorem when x and y are are in the same component of D.

Step 2. Next, we consider the case that x and y are in two different components of D. This part of the proof is the same as the second part of the proof of Theorem 1.1 when $d \ge 3$ [that is, the paragraph containing

(3.12) and (3.13)]. The only place that needs modification is the proof of $\sup_{z\in D}\mathbb{E}_x[\tau^a_D]\le c_{22}<\infty$. When $d=2$, we can not use (3.5) to deduce it. However, since D is bounded, there is $K>0$ so that $D\subset B(0,K)$. It follows from *Step 1* that

$$\sup_{z\in D, a\in(0,M]}\mathbb{E}_z[\tau^a_D]\le \sup_{z\in B(0,K), a\in(0,M]}\mathbb{E}_z[\tau^a_{B(0,K)}]\le c_{23}<\infty.$$

This completes the proof of Theorem 1.1. □

THEOREM 4.11 (3G theorem for $d=2$). *Suppose that $M>0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^2$. Then there exist positive constants $C_{19}=C_{19}(D,\alpha,M)$ and $C_{20}=C_{20}(D,\alpha,M)$ such that for all $x,y,z\in D$ and $a\in(0,M]$*

$$\begin{aligned}\frac{G^a_D(x,y)G^a_D(y,z)}{G^a_D(x,z)} &\le C_{19}\big(\log\big(1+f_D(x,y)\big)+\log\big(1+f_D(y,z)\big)+1\big)\\ &\le C_{20}\left(\left(\log\frac{1}{|x-y|}\vee 1\right)+\left(\log\frac{1}{|y-z|}\vee 1\right)\right).\end{aligned}$$

Proof. Note that, if x,z are in different components of D, either x,y or y,z are in different components of D. Thus, by Theorem 1.1 for $d=2$ and the fact that $a\in[0,M]$, we have

$$\frac{G^a_D(x,y)G^a_D(y,z)}{G^a_D(x,z)}\le c_1\frac{\log(1+f_D(x,y))\log(1+f_D(y,z))}{\log(1+f_D(x,z))}$$

for some $c_1=c_1(M,D,\alpha)$. Now following the proof of [18, Theorem 6.24], we get the theorem. □

REMARK 4.12. By considering how many different components of D that x,y and z fall into, we could get more precise 3G estimates with the dependence on a explicitly spelled out. Theorem 4.11 will not be used in the remainder of this paper.

5. One dimensional case

In this section we assume $d=1$ and prove Theorem 1.1 for this case. We will follow the ideas in [29].

Let $\overline{X}^a$ be the supremum process of X^a defined by $\overline{X}^a_t=\sup\{0\vee\overline{X}^a_s:0\le s\le t\}$ and let $\overline{X}^a-X^a$ be the reflected process at the supremum. The local time at zero of $\overline{X}^a-X^a$ is denoted by $L^a=(L^a_t:t\ge 0)$ and the inverse local time by $\{\tau^a_t:t\ge 0\}$, where $\tau^a_t:=\inf\{s:L^a_s>t\}$. The inverse local time $\{\tau^a_t:t\ge 0\}$ is a subordinator. The (ascending) ladder height process of X^a is the process $H^a=(H^a_t:t\ge 0)$ defined by $H^a_t=X^a_{\tau^a_t}$. The ladder height process is again a subordinator. It follows from [29] that H^a is a special subordinator with Laplace exponent given by

$$\chi^a(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\theta^2\lambda^2 + a^\alpha\theta^\alpha\lambda^\alpha)}{1+\theta^2}\,d\theta\right) \tag{5.1}$$

and that the drift coefficient of H^a is 1. When $a=0$, we have $\chi^0(\lambda)=\lambda$. Thus, if V^a is the potential measure of H^a and $V^a(x)=V^a([0,x])$, then, for every $a\ge 0$, V^a has a continuous, decreasing and strictly positive potential density v^a such that $v^a(0+)=1$. When $a=0$, we have $v^a\equiv 1$. The following results is a uniform version of [29, Proposition 2.3] in our present special case.

LEMMA 5.1. *Let M and R_2 be positive constants. There exists a constant $C_{21}=C_{21}(M,R_2)\in(0,1)$ such that for all $a\in[0,M]$ and $x\in(0,R_2]$,*

$$C_{21}\le v^a(x)\le C_{21}^{-1} \quad \textit{and} \quad C_{21}x\le V^a(x)\le C_{21}^{-1}x.$$

Proof. Since H^a is special, the potential density v^a is a decreasing function. Hence, $\inf_{0<t\le R_2} v^a(t)=v^a(R_2)$. It follows from (5.1) that the Laplace exponent χ^a is continuous in a. Thus, the potential measures converge vaguely, and by continuity and monotonicity of v^a, we get that $v^a(t)\to v^b(t)$ as $a\to b$ for all $t>0$. In particular, $v^a(R_2)\to v^b(R_2)$. Therefore, $c_1 := \inf_{0<t\le R_2, 0\le a\le M} v^a(t)>0$. Since $v^a(t)\le 1$ for all $t>0$ and all $a\ge 0$, we get that $c_2=\sup_{0<t\le R_2,0\le a\le M} v^a(t)=1$. Choose $c_3=c_3(M,R_2)\in(0,1)$ such that $c_3\le c_1\le c_2\le c_3^{-1}$. Since $V(x)=\int_0^x v(t)\,dt$, the claim follows immediately. □

THEOREM 5.2. *Suppose that $M>0$. For any bounded open interval D in $\mathbb{R}$, there exists a constant $C_{22}=C_{22}(\alpha,M,D)>1$ such that for all $x,y\in D$ and all $a\in(0,M]$,*

$$\begin{aligned} C_{22}^{-1}&\left(\big(\delta_D(x)\delta_D(y)\big)^{1/2}\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|}\right)\\ &\le G_D^a(x,y)\le C_{22}\left((\delta_D(x)\delta_D(y))^{1/2}\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|}\right).\end{aligned}$$

Proof. The proof of the lower bound is similar to that of Theorem 4.1 and [29, Proposition 3.3]. Using our Lemma 5.1 instead of [29, Proposition 2.3], we can follow the proof of [29, Proposition 3.1] to get the upper bound. We omit the details. □

Integrating the estimate in Theorem 5.2 with respect to y yields the following corollary.

COROLLARY 5.3. *Suppose that $M>0$. For any bounded open interval D in $\mathbb{R}$, there exists a positive constant $C_{23}=C_{23}(\alpha,M,D)>1$ such that for all $x\in D$ and all $a\in(0,M]$,*

$$C_{23}^{-1}\delta_D(x)\le\mathbb{E}_x[\tau_D^a]\le C_{23}\delta_D(x).$$

Using Corollary 5.3, we can repeat the proof of Theorem 1.1 for $d \geq 3$ case (see also [29, Theorem 3.8]) to generalize Theorem 5.2 to general (not necessarily connected) bounded $C^{1,1}$ open sets. Recall that g^a_D is defined by (1.2).

THEOREM 5.4. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}$. There exists a positive constant $C_{24} = C_{24}(\alpha, M, D) > 1$ such that for all $x, y \in D$ and all $a \in (0, M]$,*

$$C_{24}^{-1} g^a_D(x,y) \leq G^a_D(x,y) \leq C_{24} g^a_D(x,y).$$

6. Martin boundary and Martin kernel estimates

Throughout this section, we assume that $d \geq 1$ and D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R, Λ) and the corresponding κ. We will show in this section that the Martin boundary and the minimal Martin boundary of D with respect to X^a can both be identified with the Euclidean boundary ∂D of D. With the boundary Harnack principle given in Theorem 2.3, the arguments of this section are modifications of the corresponding parts of [5], [16], [26], [28]. For this reason, most of the proofs in this section will be omitted.

The next lemma follows from Theorem 2.3.

LEMMA 6.1. *Suppose that $M > 0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exists a positive constant $C_{25} = C_{25}(D, \alpha, M)$ such that for all $a \in (0, M]$, $Q \in \partial D$, $r \in (0, R/2)$, and nonnegative function u in $\mathbb{R}^d$ which is harmonic with respect to X^a in $D \cap B(Q, r)$ we have*

$$u(A_r(Q)) \leq C_{25} \left(\frac{2}{\kappa}\right)^k u\big(A_{(\kappa/2)^k r}(Q)\big), \quad k = 0, 1, \ldots. \tag{6.1}$$

LEMMA 6.2. *Suppose that $M > 0$. For every $b \in (0, \infty)$, there exist $C_{26} = C_{26}(M, b) > 0$ and $C_{27} = C_{27}(M, b) > 0$ such that for all $x_0 \in \mathbb{R}^d$, $a \in (0, M]$ and $r \in (0, b]$,*

$$C_{26} r^2 \leq \mathbb{E}_{x_0}\left[\tau^a_{B(x_0,r)}\right] \leq C_{27} r^2 \tag{6.2}$$

and

$$\mathbb{E}_x\left[\tau^a_{B(x_0,r)}\right] \leq C_{27} r \delta_{B(x_0,r)}(x). \tag{6.3}$$

Proof. See [15, Lemmas 2.3 and 2.4] or [35, Lemmas 2.2 and 2.3] for a proof of (6.2). The inequality (6.3) follows easily from Theorem 1.1. In fact, by (2.11) and Theorem 1.1 (with $Mb^{(2-\alpha)/\alpha}$ instead of M)

$$\begin{aligned} \mathbb{E}_x\left[\tau^a_{B(0,r)}\right] &= r^2 \int_{B(0,1)} G^{ar^{(2-\alpha)/\alpha}}_{B(0,1)}(r^{-1}x, z)\, dz \leq c r^2 \delta_{B(0,1)}(r^{-1}x) \\ &= c r \delta_{B(0,r)}(x). \end{aligned}$$

□

For an open set $U \subset \mathbb{R}^d$, let

$$K_U^a(x,z) := \int_U G_U^a(x,y) J^a(y,z)\,dy, \quad (x,z) \in U \times \overline{U}^c. \tag{6.4}$$

Then by (2.4), for any nonnegative measurable function f on $\mathbb{R}^d$,

$$\mathbb{E}_x[f(X^a_{\tau^a_U}); X^a_{\tau^a_U -} \neq X^a_{\tau^a_U}] = \int_{\overline{U}^c} K_U^a(x,z) f(z)\,dz.$$

From (6.4), Theorem 1.1 and Lemma 6.2, we immediately get the following proposition.

PROPOSITION 6.3. *Suppose that $M > 0$. There exist $C_{28} > 0$ and $C_{29} > 0$ such that for all $a \in (0, M]$ and $r \in (0, R)$ and $x_0 \in \mathbb{R}^d$,*

$$\begin{aligned} K^a_{B(x_0,r)}(x,y) &\le C_{28} r(r - |x - x_0|)(|y - x_0| - r)^{-d-\alpha} \\ &\quad \text{for } (x,y) \in B(x_0,r) \times \overline{B(x_0,r)}^c \end{aligned} \tag{6.5}$$

and

$$K^a_{B(x_0,r)}(x_0,y) \ge C_{29} r^2 |y - x_0|^{-d-\alpha} \quad \text{for } y \in \overline{B(x_0,r)}^c. \tag{6.6}$$

Using (6.6), the proof of the next lemma is similar to that of [26, Lemma 4.3] or [28, Lemma 5.3]. Thus, we skip the proof.

LEMMA 6.4. *Suppose that $a > 0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exists a positive constant $C_{30} = C_{30}(D, \alpha, a)$ such that for all $Q \in \partial D$, $r \in (0, R/2)$ and $w \in D \setminus B(Q,r)$,*

$$G_D^a(A_r(Q), w) \ge C_{30} r^2 \int_{B(Q,r)^c} j^a(|z - Q|/2) G_D^a(z,w)\,dz.$$

Using (6.5), Lemmas 6.1 and 6.4, the proof of the next lemma is similar to that of [26, Lemma 4.4] or [28, Lemma 5.4]. Thus, we skip the proof.

LEMMA 6.5. *Suppose that $a > 0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exist positive constants $C_{31} = C_{31}(D, \alpha, a)$ and $C_{32} = C_{32}(D, \alpha, a) < 1$ such that for any $Q \in \partial D$, $r \in (0, R/4)$ and $w \in D \setminus B(Q, 2r/\kappa)$, we have*

$$\mathbb{E}_x[G_D^a(X^a_{\tau^a_{D\cap B_k}}, w) : X^a_{\tau^a_{D\cap B_k}} \in B(Q,r)^c] \le C_{31} C_{32}^k G_D^a(x,w), \quad x \in D \cap B_k,$$

where $B_k := B(Q, (\kappa/2)^k r)$, $k = 0, 1, \ldots$.

Let $x_0 \in D$ be fixed and set

$$M_D^a(x,y) := \frac{G_D^a(x,y)}{G_D^a(x_0,y)}, \quad x, y \in D,\ y \neq x_0.$$

Now the next theorem follows from Theorem 2.3 and Lemma 6.5 (instead of [5, Lemma 13] and [5, Lemma 14], respectively) in very much the same way as in the case of symmetric stable processes in [5, Lemma 16] (with Green functions instead of harmonic functions). We omit the details.

THEOREM 6.6. *Suppose that $a>0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exist positive constants R_1, M_1, C_{33} and β depending on D, α and a such that for any $Q\in\partial D$, $r<R_1$ and $z\in D\setminus B(Q,M_1r)$, we have*

$$|M_D^a(z,x)-M_D^a(z,y)|\le C_{33}\left(\frac{|x-y|}{r}\right)^{\beta},\quad x,y\in D\cap B(Q,r).$$

In particular, the limit $\lim_{D\ni y\to w}M_D^a(x,y)$ exists for every $w\in\partial D$.

As the process $X^{a,D}$ satisfies Hypothesis (B) in Kunita and Watanabe [30], the process $X^{a,D}$ has a Martin boundary: For every $a\in(0,M]$, there is a compactification D_a^M of D, unique up to a homeomorphism, such that $M_D^a(x,y)$ has a continuous extension to $D\times(D_a^M\setminus\{x_0\})$ and $M_D^a(\cdot,z_1)=M_D^a(\cdot,z_2)$ if and only if $z_1=z_2$ (see, for instance, [30]). The set $\partial_a^M D=D_a^M\setminus D$ is called the Martin boundary of D for $X^{a,D}$. For $z\in\partial_a^M D$, set $M_D^a(\cdot,z)$ to be zero in D^c.

For each fixed $z\in\partial D$ and $x\in D$, let

$$M_D^a(x,z):=\lim_{D\ni y\to z}M_D^a(x,y),$$

which exists by Theorem 6.6. M_D^a is called the Martin kernel of D with respect to X^a. For each $z\in\partial D$, set $M_D^a(x,z)$ to be zero for $x\in D^c$. By Theorem 6.6, $M_D^a(z,x)$ is jointly continuous on $\{x\in D:\delta_D(x)>2\varepsilon\}\times\partial D$, and hence on $D\times\partial D$ after letting $\varepsilon\downarrow 0$.

The following Martin kernel estimate is an immediate consequence of the Green function estimates in Theorem 1.1. Recall that $D(x)$ denotes the component of D that contains x. Define

$$h_D^a(x,z):=\begin{cases}\frac{\delta_D(x)}{|x-z|^d}, & \text{if } x\in D(x_0), z\in\partial D(x_0)\text{ or}\\ & x\in D\setminus D(x_0), z\in\partial D\setminus\big(\partial D(x_0)\cup\partial D(x)\big),\\ a^{\alpha}\frac{\delta_D(x)}{|x-z|^d}, & \text{if } x\in D(x_0), z\in\partial D\setminus\partial D(x_0),\\ a^{-\alpha}\frac{\delta_D(x)}{|x-z|^d}, & \text{if } x\in D\setminus D(x_0), z\in\partial D(x_0).\end{cases}\tag{6.7}$$

THEOREM 6.7. *Suppose that $M>0$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exists $C_{34}:=C_{34}(x_0,D,\alpha,M)>1$ such that for all $a\in(0,M]$,*

$$C_{34}^{-1}h_D^a(x,z)\le M_D^a(x,z)\le C_{34}h_D^a(x,z)\quad\text{for } x\in D, z\in\partial D.$$

Theorem 6.7 in particular implies that $M_D^a(\cdot,z_1)$ differs from $M_D^a(\cdot,z_2)$ if z_1 and z_2 are two different points on ∂D.

Now using our Green function estimates, (6.5) and Lemma 6.1, one can follow the arguments in the proofs of [26, Lemmas 4.6 and 4.7] and [28, Lemmas 5.6 and 5.7] and get the next two lemmas.

LEMMA 6.8. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. For every $z\in\partial D$ and $B\subset\overline{B}\subset D$, $M_D^a(X_{\tau_B^a}^a,z)$ is $\mathbb{P}_x$-integrable.*

LEMMA 6.9. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. For every $z \in \partial D$ and $x \in D$,*

$$M_D^a(x,z) = \mathbb{E}_x\left[M_D^a\left(X^D_{\tau^a_{B(x,r)}}, z\right)\right] \quad \text{for every } 0 < r \le \frac{1}{2}\left(R \wedge \delta_D(x)\right). \tag{6.8}$$

Unlike the case in the proofs of [16, Theorem 2.2] and [26, Theorem 4.8], $\mathbb{P}_x(X^a_{\tau^a_U} \in \partial U) \neq 0$ for every smooth open set U. Thus, we give the details of the proof of the next result.

THEOREM 6.10. *For every $z \in \partial D$, the function $x \mapsto M_D^a(x,z)$ is harmonic in D with respect to X^a.*

Proof. Fix $z \in \partial D$ and let $h(x) := M_D^a(x,z)$. Consider an open set $D_1 \subset \overline{D_1} \subset D$ and $x \in D_1$ and put

$$r(x) = \frac{1}{2}\left(R \wedge \delta_D(x)\right) \quad \text{and} \quad B(x) = B(x, r(x)).$$

Define a sequence of stopping times $\{T_m, m \ge 1\}$ as follows:

$$T_1 = \inf\{t > 0 : X^a_t \notin B(X^a_0)\},$$

and for $m \ge 2$,

$$T_m = \begin{cases} T_{m-1} + \tau_{B(X^a_{T_{m-1}})} \circ \theta_{T_{m-1}}, & \text{if } X^a_{T_{m-1}} \in D_1, \\ \tau^a_{D_1}, & \text{otherwise.} \end{cases}$$

Note that $X^a_{\tau^a_{D_1}} \in \partial D_1$ on $\bigcap_{n=1}^\infty \{T_n < \tau^a_{D_1}\}$. Thus, since $\lim_{m\to\infty} T_m = \tau^a_{D_1}$ $\mathbb{P}_x$-a.s. and h is continuous in D,

$$\lim_{m\to\infty} h(X^a_{T_m}) = h(X^a_{\tau^a_{D_1}}), \quad \text{on } \bigcap_{n=1}^\infty \{T_n < \tau^a_{D_1}\}$$

and, since h is bounded on $\overline{D_1}$, by the dominated convergence theorem

$$\lim_{m\to\infty} \mathbb{E}_x\left[h(X^a_{T_m}); \bigcap_{n=0}^\infty \{T_n < \tau^a_{D_1}\}\right] = \mathbb{E}_x\left[h(X^a_{\tau^a_{D_1}}); \bigcap_{n=0}^\infty \{T_n < \tau^a_{D_1}\}\right].$$

Therefore, using Lemma 6.9

$$\begin{aligned} h(x) &= \lim_{m\to\infty} \mathbb{E}_x[h(X^a_{T_m})] \\ &= \lim_{m\to\infty} \mathbb{E}_x\left[h(X^a_{T_m}); \bigcup_{n=0}^\infty \{T_n = \tau^a_{D_1}\}\right] \\ &\quad + \lim_{m\to\infty} \mathbb{E}_x\left[h(X^a_{T_m}); \bigcap_{n=0}^\infty \{T_n < \tau^a_{D_1}\}\right] \\ &= \mathbb{E}_x\left[h(X^a_{\tau^a_{D_1}}); \bigcup_{n=0}^\infty \{T_n = \tau^a_{D_1}\}\right] \end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{E}_x\left[h(X^a_{\tau^a_{D_1}}); \bigcap_{n=0}^{\infty}\{T_n < \tau^a_{D_1}\}\right]\\
&= \mathbb{E}_x[h(X^a_{\tau^a_{D_1}})]. \qquad \square
\end{aligned}
$$

A consequence of Theorems 6.6, 6.7 and 6.10 is that, when D is a bounded $C^{1,1}$ open set, the Martin boundary of $X^{a,D}$ can be identified with the Euclidean boundary ∂D of D.

A positive harmonic function u for $X^{a,D}$ is minimal if, whenever v is a positive harmonic function for $X^{a,D}$ with $v \le u$ on D, one must have $v = cu$ for some constant c. The set of points $z \in \partial_a^M D$ such that $M_D^a(\cdot, z)$ is minimal harmonic for $X^{a,D}$ is called the minimal Martin boundary of D for $X^{a,D}$.

With the explicit estimates from Theorem 6.7, by the same argument as that for [16, Theorem 3.7], we have the following.

THEOREM 6.11. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and $a > 0$. For every $z \in \partial D$, $M_D^a(\cdot, z)$ is a minimal harmonic function for $X^{a,D}$. Thus the minimal Martin boundary of D can be identified with the Euclidean boundary.*

We know from the general theory in Kunita and Watanabe [30] that nonnegative superharmonic functions with respect to $X^{a,D}$ (or equivalently, superharmonic functions with respect to X^a that vanish on D^c) admit a Martin representation. Thus, by Theorem 6.11 we conclude that, for every superharmonic function $u \ge 0$ with respect to $X^{a,D}$, there is a unique Radon measure μ in D and a finite measure ν on ∂D such that

$$
u(x) = \int_D G_D^a(x,y)\mu(dy) + \int_{\partial D} M_D^a(x,z)\nu(dz). \tag{6.9}
$$

Furthermore, u is harmonic for $X^{a,D}$ if and only if the measure $\mu = 0$.

7. Perturbation results

In this section, we assume $d \ge 1$ and fix $a > 0$. We consider a symmetric Lévy process Z which can be thought of as a perturbation of X^a, and show that under certain conditions, the Green function of Z^D, the process Z killed upon exiting a bounded $C^{1,1}$ open set D, is comparable to the Green function of $X^{a,D}$, see Theorem 7.13. Together with Theorem 1.1, this gives sharp bounds for the Green function G_D^Z of Z^D.

The approach of this section is motivated by [19], where perturbations of pure jump Lévy processes are discussed. Even though they consider pure jump Lévy processes, some results work for our case as well.

Throughout this section, Z is a symmetric Lévy process in $\mathbb{R}^d$ such that its Lévy measure has a density $J^Z(x,y) = j^Z(y-x)$ with respect to the Lebesgue measure. We assume that

$$
j_1^a(x) := j^a(x) - j^Z(x)
$$

is nonnegative and integrable in $\mathbb{R}^d$ and put $\mathcal{J}^a := \int_{\mathbb{R}^d} j_1^a(y)\,dy$. We also assume that the transition density of the Lévy process Z exists and we denote it by $p^Z(t,x,y) = p^Z(t, y-x)$.

Recall that $p^a(t,x,y) = p^a(t,y-x)$ is the transition density function of X^a. It is well known that (see [15], [37])

$$p^a(t,x,y) \le c_1(t^{-d/\alpha} \wedge t^{-d/2}) \wedge \left(t^{-d/2} e^{-c_2|x-y|^2/t} + \frac{t}{|x-y|^{d+\alpha}}\right), \tag{7.1}$$
$$(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$$

for some $c_i = c_i(a,d,\alpha) > 0$, $i = 1,2$. Thus, by following the proof of [19, Lemma 2.6], we have the following.

LEMMA 7.1. *$p^Z(t,x)$ is bounded on the set $\{(t,x) : t > 0, |x| > \varepsilon\}$ for $\varepsilon > 0$.*

Recall that for every bounded open set D, G_D^a is the Green function of $X^{a,D}$. We know from [27, Corollary 3.12] that there is a constant $c = c(D,a) > 0$ such that

$$c\mathbb{E}_x[\tau_D^a]\mathbb{E}_y[\tau_D^a] \le G_D^a(x,y), \quad x, y \in D. \tag{7.2}$$

The proofs of the following three results are the same as those of [19, Lemmas 2.2, 2.4, 2.5]. So we omit their proofs here. In the remainder of this section, the dependence of the constants on Z will not be mentioned explicitly.

LEMMA 7.2. *For every bounded open set D, there exists $C_{35} = C_{35}(D,a) > 0$ such that for all $x \in D$ and $t \ge 1$ we have*

$$p_D^a(t,x,y) \le C_{35} t^{-2} \mathbb{E}_x[\tau_D^a]\mathbb{E}_y[\tau_D^a].$$

For any open set $U \subset \mathbb{R}^d$, $\tau_U^Z := \inf\{t > 0 : Z_t \notin U\}$ denotes the first exit time from U by Z. We denote by Z^D the subprocess of Z killed upon leaving D and $p_D^Z(t,x,y)$ the transition density for Z^D.

LEMMA 7.3. *For every bounded open set D, there exist a constant $C_{36} = C_{36}(D,a) > 1$ such that for every $x \in D$,*

$$C_{36}^{-1}\mathbb{E}_x[\tau_D^Z] \le \mathbb{E}_x[\tau_D^a] \le C_{36}\mathbb{E}_x[\tau_D^Z].$$

LEMMA 7.4. *For every bounded open set D and any $x \in D$ and $t > 0$, we have*

$$p_D^Z(t,x,\cdot) \le e^{\mathcal{J}_a t} p_D^a(t,x,\cdot) \quad a.e.$$

If, in addition, we assume that $p^Z(t,\cdot)$ is continuous then we have for $x, y \in D$,

$$\mathbb{E}_x[p^Z(t-\tau_D^Z, Z_{\tau_D^Z}, y) : t \ge \tau_D^Z] \le e^{2\mathcal{J}_a t}\mathbb{E}_x[p^a(t-\tau_D^a, X^a_{\tau_D^a}, y) : t \ge \tau_D^a]. \tag{7.3}$$

Using the above lemmas (for Lemma 7.4, only its first part is needed), and following the proof of [19, Theorem 3.1], we have

THEOREM 7.5. *For every bounded open set D, there exists $C_{37} = C_{37}(D, a) > 0$ such that for every $x \in D$,*

$$G_D^Z(x,y) \le C_{37} G_D^a(x,y) \quad a.e.\ y \in D. \tag{7.4}$$

Here are some assumptions that we might put on the process Z.

(A1) The transition density $p_D^Z(t,x,y)$ of Z^D is continuous and strictly positive in $(0,\infty) \times D \times D$.

(A2) There exist positive constants c and ρ such that $j_1^a(x) \le c|x|^{\rho-d}$ on $B(0,1)$.

(A3) There exists $R_0 > 0$ such that $\inf_{x \in B(0,R_0)} j^Z(x) > 0$.

Without loss of generality, we may and do assume that the constant ρ in (A2) is less than 1. Clearly assumption (A2) implies (A3).

PROPOSITION 7.6 ([27, Corollary 3.11]). *Suppose that* (A1) *and* (A3) *hold. Then for every bounded open set D, there exists constant $C_{38} = C_{38}(D, \alpha) > 0$ such that*

$$C_{38} \mathbb{E}_x[\tau_D^Z] \mathbb{E}_y[\tau_D^Z] \le G_D^Z(x,y) \quad \text{for all } (x,y) \in D \times D. \tag{7.5}$$

For the remainder part of this section, we assume D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$.

LEMMA 7.7. *Suppose that* (A1) *and* (A3) *hold. Then for every $\varepsilon > 0$, there exists $C_{39} = C_{39}(\varepsilon, D, a) > 0$ such that for all $x, y \in D$ satisfying $|x - y| \ge \varepsilon$,*

$$G_D^a(x,y) \le C_{39} G_D^Z(x,y).$$

Proof. It follows from Theorem 1.1 that there exists $c_1 = c_1(D, a) > 0$ such that

$$c_1^{-1} \mathbb{E}_x[\tau_D^a] \le \delta_D(x) \le c_1 \mathbb{E}_x[\tau_D^a], \quad x \in D. \tag{7.6}$$

Combining (7.6) with Theorem 1.1 yields that there exist $c_2, c_3 > 0$ so that for all x and $y \in D$ with $|x - y| \ge \varepsilon$,

$$G_D^a(x,y) \le c_2 \delta_D(x) \delta_D(y) \le c_3 \mathbb{E}_x[\tau_D^a] \mathbb{E}_y[\tau_D^a].$$

Therefore, by Lemma 7.3 and Proposition 7.6 we get

$$G_D^a(x,y) \le c_4 \mathbb{E}_x[\tau_D^Z] \mathbb{E}_y[\tau_D^Z] \le c_5 G_D^Z(x,y)$$

for some positive constants $c_4, c_5 > 0$. □

The next lemma can be proved by following the arguments in the proofs of [34, Lemmas 7, 9]. So we skip the proof of the next lemma.

LEMMA 7.8. *Suppose that* (A1) *holds. For all $x, w \in D$, we have*

$$G_D^a(x,w) \le G_D^Z(x,w) + \int_D \int_D G_D^a(x,y) j_1^a(y-z) G_D^a(z,w)\, dy\, dz.$$

PROPOSITION 7.9. *Suppose that $d \geq 3$ and that* (A2) *holds. There exists a positive constant $C_{40} = C_{40}(D, a)$ such that for every $(x, w) \in D \times D$*

$$\int_D \int_D G_D^a(x,y) j_1^a(y-z) G_D^a(z,w)\, dy\, dz \leq C_{40} G_D^a(x,w) |x-w|^{d-2}.$$

Proof. Using the generalized 3G inequality (Theorems 3.5) for the Green function of X^a, one can easily get the following

$$\begin{aligned}
&\int_D \int_D G_D^a(x,y) j_1^a(y-z) G_D^a(z,w)\, dy\, dz \\
&\quad \leq c G_D^a(x,w) \biggl(|x-w|^{d-2} \int_D \int_D \frac{|y-z|^{\rho-d}\, dy\, dz}{|x-y|^{d-2}|z-w|^{d-2}} \\
&\qquad + |x-w|^{d-1} \int_D \int_D \frac{|y-z|^{\rho-d}\, dy\, dz}{|x-y|^{d-1}|z-w|^{d-2}} \\
&\qquad + |x-w|^{d-1} \int_D \int_D \frac{|y-z|^{\rho-d}\, dy\, dz}{|x-y|^{d-2}|z-w|^{d-1}} \\
&\qquad + |x-w|^{d} \int_D \int_D \frac{|y-z|^{\rho-d}\, dy\, dz}{|x-y|^{d-1}|z-w|^{d-1}} \biggr)
\end{aligned}$$

for some constant $c = c(D, a) > 0$. Now combining the above with [19, Lemma 3.12],we easily get the conclusion of the proposition. □

PROPOSITION 7.10. *Suppose that $d \geq 1$ and that* (A2) *holds. There exists a positive constant $C_{41} = C_{41}(D, a)$ such that for every $(x, w) \in D \times D$*

$$\int_D \int_D G_D^a(x,y) j_1^a(y-z) G_D^a(z,w)\, dy\, dz \leq C_{41} \frac{\delta_D(x)\delta_D(w)}{|x-w|^{d-\rho}}.$$

Proof. Recall that for every $d \geq 1$, $G_D^a(x,y) \leq c_1 \frac{\delta_D(x)\delta_D(y)}{|x-y|^d}$ by Theorem 1.1. Therefore, by following the arguments in the proof of [34, Lemma 8] we have

$$\int_D G_D^a(x,y) \frac{1}{|y-z|^{d-\rho}}\, dy \leq c_2 \frac{\delta_D(x)}{|x-z|^{d-\rho}}. \tag{7.7}$$

Thus

$$\int_D G_D^a(x,y) j_1^a(y-z)\, dy \leq c_3 \frac{\delta_D(x)}{|x-z|^{d-\rho}}$$

and so, since $G_D^a(z,w) = G_D^a(w,z)$, by (7.7)

$$\begin{aligned}
\int_D \int_D G_D^a(x,y) j_1^a(y-z) G_D^a(z,w)\, dy\, dz &\leq c_3 \delta_D(x) \int_D G_D^a(z,w) \frac{1}{|x-z|^{d-\rho}}\, dz \\
&\leq c_4 \frac{\delta_D(x)\delta_D(w)}{|x-w|^{d-\rho}}.
\end{aligned}$$

□

LEMMA 7.11. *Suppose that* $d \in \{1,2\}$, $T > 0$ *and that* (A1) *holds. Then there exists a constant* $C_{42} = C_{42}(a,T) > 0$ *such that*

$$\sup_{0<t\le T} \big(p^a(t,x,y) - e^{-2\mathcal{J}_a t} p^Z(t,x,y)\big) \le C_{42} t^{1-d/2}.$$

Proof. Recall that $p^Z(t, y-x) := p^Z(t,x,y)$ and $p^a(t,y-x) = p^a(t,x,y)$. Let $\widehat{Z}$ be a pure jump Lévy process with Lévy density j_1^a in $\mathbb{R}^d$ independent of Z. Then $\widehat{Z}$ is a compound Poisson process with transition probability given by

$$P^{\widehat{Z}}(t,\cdot) = e^{-\mathcal{J}_a t}\delta_0(\cdot) + e^{-\mathcal{J}_a t}\sum_{n=1}^{\infty} \frac{t^n (j_1^a)^{*n}(\cdot)}{n!}.$$

The process $Z + \widehat{Z}$ has the same distribution as X^a. Thus, the distribution of X_t^a is equal to the convolution of $p^Z(t,\cdot)$ and $P^{\widehat{Z}}(t,\cdot)$. Consequently, we have

$$p^a(t,x) = p^Z(t,x)e^{-\mathcal{J}_a t} + e^{-\mathcal{J}_a t}\sum_{n=1}^{\infty}\frac{t^n p^Z(t,\cdot) * (j_1^a)^{*n}(x)}{n!}.$$

It follows from Lemma 7.4 and (7.1) that for $0 < t \le T$

$$\begin{aligned} p^Z(t,\cdot) * (j_1^a)^{*n}(x) &\le e^{\mathcal{J}_a t} p^a(t,\cdot) * (j_1^a)^{*n}(x) \\ &\le c_1 e^{\mathcal{J}_a t}(\mathcal{J}_a)^n (t^{-d/\alpha} \wedge t^{-d/2}) \le c_2 t^{-d/2} e^{\mathcal{J}_a t}(\mathcal{J}_a)^n \end{aligned}$$

for some positive constants c_1, c_2. Thus, it follows from Lemma 7.4 and the above two displays that for $0 < t \le T$

$$\begin{aligned} p^a(t,x) - e^{-2\mathcal{J}_a t}p^Z(t,x) &= p^a(t,x) - e^{-\mathcal{J}_a t}p^Z(t,x) + e^{-\mathcal{J}_a t}(1 - e^{-\mathcal{J}_a t})p^Z(t,x) \\ &\le c_2 \sum_{n=1}^{\infty} \frac{t^{n-d/2}(\mathcal{J}_a)^n}{n!} + (1 - e^{-\mathcal{J}_a t})p^a(t,x). \end{aligned}$$

Since by (7.1) $p^a(t,x) \le c_3 t^{-d/2}$ for $0 < t \le T$, we reach the conclusion of the lemma in view of the above display. $\square$

We also need the following lemma.

LEMMA 7.12. *Let* D *be a* $C^{1,1}$ *open set with* $C^{1,1}$ *characteristics* (R,Λ). *Then there is a constant* $C_{43} > 0$ *such that for all* $x \in D$ *and* $t > 0$,

$$\mathbb{P}_x(\tau_D^a > t) \le C_{43}\frac{\delta_D(x)}{t + \sqrt{t}}.$$

Proof. When $t \ge 1$, the above inequality follows immediately from Markov's inequality and (7.6). To establish the inequality for the case of $0 < t < 1$, we will use a result from [12].

We will only give the proof for the case $d \ge 2$. The proof in the case $d = 1$ is similar but simpler. Without loss of generality, we can always assume that $R \le 1$ and $\Lambda \ge 1$. By definition, for every $Q \in \partial D$, there is a $C^{1,1}$-function $\phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi_Q(0) = 0$, $\nabla\phi_Q(0) = (0,\dots,0)$,

$\|\nabla\phi_Q\|_\infty \le \Lambda$, $|\nabla\phi_Q(x) - \nabla\phi_Q(z)| \le \Lambda|x-z|$, and an orthonormal coordinate system $CS_Q: y = (\widetilde{y}, y_d)$ such that $B(Q,R)\cap U = \{y = (\widetilde{y}, y_d) \in B(0,R)$ in $CS_Q : y_d > \phi(\widetilde{y})\}$. Define

$$\rho_Q(x) := x_d - \phi_Q(\widetilde{x}),$$

where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_Q. Note that for every $Q \in \partial U$ and $x \in B(Q,R)\cap U$, we have $(1+\Lambda^2)^{-1/2}\rho_Q(x) \le \delta_U(x) \le \rho_Q(x)$. We define for $r_1, r_2 > 0$

$$D_Q(r_1, r_2) := \{y \in U : r_1 > \rho_Q(y) > 0, |\widetilde{y}| < r_2\}.$$

Note that for $b > 0$,

$$\begin{aligned}\mathbb{P}_x(\tau_D^b > 1) &\le \mathbb{P}_x\big(\tau^b_{D_Q(\delta_0, r_0)} > 1\big) + \mathbb{P}_x\big(X^b_{\tau^b_{D_Q(\delta_0,r_0)}} \in D \text{ and } \tau^b_{D_Q(\delta_0,r_0)} \le 1\big) \\ &\le E_x\big[\tau^b_{D_Q(\delta_0,r_0)}\big] + \mathbb{P}_x\big(X^b_{\tau^b_{D_Q(\delta_0,r_0)}} \in D\big).\end{aligned}$$

Thus by [12, Lemma 3.5], there is a constant $c_1 = c_1(R, \Lambda, a)$ so that for every $b \in (0, a]$

$$\mathbb{P}_x(\tau_D^b > 1) \le c_1\delta_D(x) \quad \text{for every } x \in D. \tag{7.8}$$

Note that for $0 < \lambda \le 1$, $\lambda^{-1}D$ is a $C^{1,1}$ open set with $C^{1,1}$ characteristics (R, Λ). Hence by the scaling property of X^a in (2.10), we have from (7.8) that for $t \in (0,1]$,

$$\mathbb{P}_x(\tau_D^a > t) = \mathbb{P}_{t^{-1/2}x}\big(\tau^{at^{(2-\alpha)/(2\alpha)}}_{t^{-1/2}D} > 1\big) \le c_1\delta_{t^{-1/2}D}(t^{-1/2}x) = c_1\frac{\delta_D(x)}{\sqrt{t}}.$$

This completes the proof of the lemma. □

THEOREM 7.13. *Suppose that conditions* (A1) *and* (A2) *hold. There exists* $C_{44} = C_{44}(D, a) > 0$ *such that*

$$C_{44}^{-1}G_D^Z(x,w) \le G_D^a(x,w) \le C_{44}G_D^Z(x,w), \quad (x,w) \in D\times D. \tag{7.9}$$

Proof. By (7.4) and Lemma 7.7, we only need to show the second inequality in (7.9) for $|x-y|^2 < \varepsilon$, where $\varepsilon \in (0,1)$ is a constant to be chosen later. We consider the cases $d \ge 3$ and $d \le 2$ separately.

(a) $d \ge 3$: Applying Lemma 7.8 and then Proposition 7.9, we get

$$G_D^a(x,y) \le G_D^Z(x,y) + c_1 G_D^a(x,y)|x-y|^{d-2}.$$

Choose $\varepsilon > 0$ small so that

$$c_1 G_D^a(x,y)|x-y|^{d-2} \le \frac{1}{2}G_D^a(x,y) \quad \text{if } |x-y| < \varepsilon^{1/2}.$$

Thus,

$$G_D^a(x,y) \le 2G_D^Z(x,y) \quad \text{if } |x-y| < \varepsilon^{1/2}.$$

(b) $d\le 2$: We first note that, since $p^a_D(t,x,y)\le c_2(t^{-d/\alpha}\wedge t^{-d/2})$ by (7.1), using the semigroup property,

$$\begin{aligned}p^a_D(t,x,y)&=\int_D p^a_D(t/3,x,z)\int_D p^a_D(t/3,z,w)p^a_D(t/3,w,y)\,dw\,dz\\&\le c_2(t^{-d/\alpha}\wedge t^{-d/2})\int_D p^a_D(t/3,x,z)\,dz\int_D p^a_D(t/3,w,y)\,dw\\&=c_2(t^{-d/\alpha}\wedge t^{-d/2})\mathbb{P}_x(\tau^a_D>t/3)\mathbb{P}_y(\tau^a_D>t/3).\end{aligned}$$

By Lemma 7.12, we get

$$\begin{aligned}p^a_D(t,x,y)&\le c_3(t^{-d/\alpha}\wedge t^{-d/2})\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_D(y)}{\sqrt{t}}\wedge 1\right)\\&\le c_4(t^{-d/\alpha-1}\wedge t^{-d/2-1})\delta_D(x)\delta_D(y).\end{aligned}$$

Consequently,

$$\int_{t_0}^\infty p^a_D(t,x,y)\,dt\le c_5t_0^{-d/2}\delta_D(x)\delta_D(y)\quad\text{for every } t_0>0 \text{ and } x,y\in D. \tag{7.10}$$

Since

$$p^a_D(t,x,y)=p^a(t,x,y)-\mathbb{E}_x[p^a(t-\tau^a_D,X^a_{\tau^a_D},y):t\ge\tau^a_D]$$

and

$$p^Z_D(t,x,y)=p^Z(t,x,y)-\mathbb{E}_x[p^Z(t-\tau^Z_D,Z_{\tau^Z_D},y):t\ge\tau^Z_D],$$

it follows from (7.3) that

$$\begin{aligned}p^a_D(t,x,y)&=p^a(t,x,y)-\mathbb{E}_x[p^a(t-\tau^a_D,X^a_{\tau^a_D},y):t\ge\tau^a_D]\\&\le p^a(t,x,y)-e^{-2\mathcal{J}_at}\mathbb{E}_x[p^Z(t-\tau^Z_D,Z_{\tau^Z_D},y):t\ge\tau^Z_D]\\&=p^a(t,x,y)-e^{-2\mathcal{J}_at}\big(p^Z(t,x,y)-p^Z_D(t,x,y)\big)\\&\le p^a(t,x,y)+p^Z_D(t,x,y)-e^{-2\mathcal{J}_at}p^Z(t,x,y).\end{aligned}$$

So integrating over $[0,t_0]$ with $t_0=(\delta_D(x)\delta_D(y))^{1/2}$, which is bounded by $\mathrm{diam}(D)$, we have by Lemma 7.11 and (7.10) that

$$\begin{aligned}G^a_D(x,y)&=\int_0^{t_0}p^a_D(t,x,y)\,dt+\int_{t_0}^\infty p^a_D(t,x,y)\,dt\\&\le G^Z_D(x,y)+\int_0^{t_0}\big(p^a(t,x,y)-e^{-2\mathcal{J}_at}p^Z(t,x,y)\big)\,dt\\&\quad+c_5t_0^{-d/2}\delta_D(x)\delta_D(y)\\&\le G^Z_D(x,y)+c_6t_0^{2-d/2}+c_5t_0^{-d/2}\delta_D(x)\delta_D(y)\\&\le G^Z_D(x,y)+c_7(\delta_D(x)\delta_D(y))^{1-d/4}.\end{aligned}\tag{7.11}$$

Since $G^a_D(x,y) \ge c_8(\delta_D(x)\delta_D(y))^{1-d/2}$ for $|x-y|^2 \le \delta_D(x)\delta_D(y)$ by Theorem 1.1, we have from (7.11) that

$$\text{(7.12)}\qquad \begin{aligned} &G^a_D(x,y) \le G^Z_D(x,y) + c_9(\delta_D(x)\delta_D(y))^{d/4} G^a_D(x,y) \\ &\quad \text{if } |x-y|^2 \le \delta_D(x)\delta_D(y). \end{aligned}$$

On the other hand, applying Lemma 7.8 and then Proposition 7.10, we get

$$G^a_D(x,y) \le G^Z_D(x,y) + c_{10}\frac{\delta_D(x)\delta_D(y)}{|x-y|^{d-\rho}}.$$

Since $c_{11}\frac{\delta_D(x)\delta_D(y)}{|x-y|^d} \le G^a_D(x,y)$ for $|x-y|^2 \ge \delta_D(x)\delta_D(y)$ by Theorem 1.1, we have

$$\text{(7.13)}\qquad G^a_D(x,y) \le G^Z_D(x,y) + c_{12}|x-y|^\rho G^a_D(x,y).$$

Now using (7.12) and (7.13), we can choose $\varepsilon \in (0,1)$ small so that

$$c_{12}G^a_D(x,y)|x-y|^\rho \le \frac{1}{2}G^a_D(x,y) \quad \text{if } \delta_D(x)\delta_D(y) \le |x-y|^2 < \varepsilon$$

and

$$c_9(\delta_D(x)\delta_D(y))^{d/4}G^a_D(x,y) \le \frac{1}{2}G^a_D(x,y) \quad \text{if } |x-y|^2 \le \delta_D(x)\delta_D(y) < \varepsilon.$$

Thus in these cases, $G^a_D(x,w) \le 2G^Z_D(x,w)$.

For the remaining case $\delta_D(x)\delta_D(y) \ge \varepsilon$, we use (7.11), Lemma 7.3 and (7.5) to get that

$$G^a_D(x,w) \le \big(1 + c_{13}(\delta_D(x)\delta_D(y))^{-d/4}\big)G^Z_D(x,y) \le (1 + c_{13}\varepsilon^{-d/4})G^Z_D(x,y).$$

The proof of the theorem is now complete. □

We now show that the theorem above covers the case of the sum of a Brownian motion and an independent relativistic stable process, and the case of the sum of a Brownian motion and an independent truncated stable process.

For any $m \ge 0$, a relativistic α-stable process Y^m in $\mathbb{R}^d$ with mass m is a Lévy process with characteristic function given by

$$\mathbb{E}_x\big[e^{i\xi\cdot(Y^m_t - Y^m_0)}\big] = \exp\big(-t\big((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m\big)\big), \quad \xi \in \mathbb{R}^d.$$

Suppose that Y^m is independent of the Brownian motion X^0. We define Z^m by $Z^m_t := X^0_t + Y^m_t$. We will call the process Z^m the independent sum of the Brownian motion X^0 and the relativistic α-stable process Y^m with mass m. The Lévy measure of Z^m has a density

$$J^{Z^m}(x) = \mathcal{A}(d,-\alpha)|x|^{-d-\alpha}\psi(m^{1/\alpha}|x|)$$

where

$$\psi(r) := 2^{-(d+\alpha)}\Gamma\left(\frac{d+\alpha}{2}\right)^{-1}\int_0^\infty s^{\frac{d+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}\,ds,$$

which is a decreasing smooth function of r^2. (see [17, pp. 276–277] for details). Thus,

$$0 \le j^1(x) - j^{Z^m}(x) \le c|x|^{2-\alpha-d}.$$

Moreover, the conditions (A1) and (A2) can be checked easily. Therefore as a corollary of Theorem 7.13, we have the following.

COROLLARY 7.14. *There exists a constant $C_{45} = C_{45}(D,\alpha) > 0$ such that*

$$C_{45}^{-1} G_D^1(x,y) \le G_D^{Z^m}(x,y) \le C_{45} G_D^1(x,y), \quad x,y \in D,$$

where $G_D^{Z^m}(x,y)$ is the Green function of Z^m in D.

By a λ-truncated symmetric α-stable process in $\mathbb{R}^d$ we mean a pure jump symmetric Lévy process $\widehat{Y}^\lambda = (\widehat{Y}_t^\lambda, t \ge 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ in $\mathbb{R}^d$ with the Lévy density $\mathcal{A}(d,-\alpha)|x|^{-d-\alpha}\mathbf{1}_{\{|x|<\lambda\}}$. Note that the Lévy exponent ψ^λ of $\widehat{Y}^\lambda$, defined by

$$\mathbb{E}_x\left[e^{i\xi\cdot(\widehat{Y}_t^\lambda - \widehat{Y}_0^\lambda)}\right] = e^{-t\psi^\lambda(\xi)} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

is given by

$$\psi^\lambda(\xi) = \mathcal{A}(d,-\alpha)\int_{\{|y|<\lambda\}} \frac{1-\cos(\xi\cdot y)}{|y|^{d+\alpha}}\,dy. \tag{7.14}$$

Suppose that $\widehat{Y}^\lambda$ is a λ-truncated symmetric α-stable process in $\mathbb{R}^d$ which is independent of the Brownian motion X^0. We define $\widehat{X}_t^\lambda := X_t^0 + \widehat{Y}_t^\lambda$ for $t \ge 0$. Then $\widehat{X}^\lambda$ has the same distribution as the Lévy process obtained from X^1 by removing jumps of size larger than λ.

Unlike the symmetric stable process Y, the process $\widehat{Y}^\lambda$ can only make jumps of size less than λ. In order to guarantee the strict positivity of the transition density $p_D^{\widehat{X}^\lambda}(t,x,y)$ for $\widehat{X}^{\lambda,D}$, we need to impose the following assumption on D.

DEFINITION 7.15. We say that an open set D in $\mathbb{R}^d$ is λ-roughly connected if for every $x,y \in D$, there exist finitely many distinct connected components $U_1,\dots,U_m$ of D such that $x \in U_1$, $y \in U_m$ and $dist(U_k,U_{k+1}) < \lambda$ for $1 \le k \le m-1$.

The following result is proved in [27].

PROPOSITION 7.16 ([27, Proposition 4.4]). *For any bounded λ-roughly connected open set D in $\mathbb{R}^d$, the transition density $p_D^{\widehat{X}^\lambda}(t,x,y)$ for $\widehat{X}^{\lambda,D}$ is strictly positive in $(0,\infty)\times D \times D$.*

The other conditions (A1) and (A2) can be checked easily. Therefore as a corollary of Theorem 7.13, we have the following.

COROLLARY 7.17. *Suppose D is a bounded λ-roughly connected $C^{1,1}$ open set in $\mathbb{R}^d$, $d \geq 1$. There exists $C_{46} = C_{46}(D,\alpha) > 0$ such that*

$$C_{46}^{-1} G_D^1(x,y) \leq G_D^{\widehat{X}^\lambda}(x,y) \leq C_{46} G_D^1(x,y), \quad x,y \in D,$$

where $G_D^{\widehat{X}^\lambda}(x,y)$ is the Green function of $\widehat{X}^\lambda$ in D.

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