Illinois Journal of Mathematics
Volume 54, Number 3, Fall 2010, Pages 869–893
S 0019-2082

RECURRENCE AND TRANSIENCE PRESERVATION FOR VERTEX REINFORCED JUMP PROCESSES IN ONE DIMENSION

BURGESS DAVIS AND NOAH DEAN

Dedicated to Donald Burkholder

ABSTRACT. We show that the application of linear vertex reinforcement to one dimensional nearest neighbor Markov processes, yielding associated vertex reinforced jump processes, preserves both recurrence and transience. The analog for discrete time linear bond reinforcement is due to Takeshima. This together with another result we prove adds to the numerous known parallels between these two reinforcements. Martingales are the primary tool used to study vertex reinforced jump processes.

1. Introduction

Three of the self organizing processes called reinforced random walks involve linear reinforcement. Two of these processes are similar in many respects.

We study perturbations of nearest neighbor random walks on $\mathbb{Z}$, that is, of Markov processes on the integers with transition probabilities $p_{i,j}$ which for every integer i satisfy $p_{i,i+1} > 0$, $p_{i,i-1} > 0$, and $p_{i,i+1} + p_{i,i-1} = 1$. Often bond weights (conductances) or vertex weights are used to represent these transition probabilities. In bond weighting, each bond $(i, i+1)$ is assigned a positive number ω_i, and the probabilities of jumps from i to $i+1$ and $i-1$ are proportional to ω_i and ω_{i-1}, so that $p_{i,i+1} = \frac{\omega_i}{\omega_i+\omega_{i-1}}$. In vertex weighting, positive numbers w_i are assigned to each integer i and probabilities of jumps from i are proportional to w_{i+1} and w_{i-1} so that $p_{i,i+1} = \frac{w_{i+1}}{w_{i+1}+w_{i-1}}$.

Discrete time linear bond reinforced random walk was introduced by Coppersmith and Diaconis [CD87], and is here for brevity called Diaconis walk.

Received October 15, 2009; received in final form November 3, 2010.
2010 *Mathematics Subject Classification.* 60K37, 60J10.

A positive number (the initial weight) is assigned to each bond $(i, i+1)$, and the first jump is made with probabilities determined by these weights. The weight of a bond is increased by the positive constant δ each time it is crossed, and jumps at time n are made with probabilities proportional to the weights at time n. The analogous vertex reinforcement scheme, where the weight of an integer is increased by δ each time it is visited, behaves very differently. For example, if each integer has initial weight one, the bond reinforced walk visits every integer infinitely often (see [Pem92]) while the vertex reinforced walk visits exactly five states infinitely often (see [PV99] and [Tar04]).

Wendelin Werner (personal communication) has observed that in continuous time there is a third linear reinforcement, a vertex reinforcement. Again an initial positive weight w_i is assigned to each integer i; the weight w_i^t of i at time t equals w_i plus δ times the total time before t that the process has spent at i:

$$w_i^t = w_i + \delta \int_0^t \mathbb{I}(X_s = i)\, ds,$$

where X_t, $t \geq 0$, is the integer-valued process. Given the past up to time t and that $X_t = i$, X jumps in the next dt seconds to $i+1$ and $i-1$ with probabilities $w_{i+1}^t\, dt$ and $w_{i-1}^t\, dt$, respectively. So, if $X_0 = i$, the time of its first jump has an exponential (rate $w_{i+1} + w_{i-1}$) distribution and that jump is to $i+1$ with probability $\frac{w_{i+1}}{w_{i+1}+w_{i-1}}$. Whether the jump is to $i+1$ or $i-1$ is independent of the time of this jump. Let $0 = \tau_0$ and let τ_i, $i > 0$, be the time of the ith jump of X_t. We call the discrete time process X_{τ_i}, $i \geq 0$, *Werner walk*. Werner walk can be defined without invoking continuous time: the reinforcements are conditionally exponential. This will be discussed more fully in the next section.

We call a discrete time integer valued process Y_i, $i \geq 0$, transient if $Y_i = k$ for only finitely many i for each k a.s., and recurrent if $Y_i = k$ for infinitely many i for each k a.s. Takeshima shows in [Tak00] that if the unreinforced walk on $\mathbb{Z}$ associated with a bond weighting is recurrent (resp. transient), then, for any $\delta > 0$, the Diaconis walk associated with this initial weighting is recurrent (resp. transient). Recurrence preservation is to be expected, but the transience preservation is surprising, especially since a theorem of Pemantle [Pem88] shows that Diaconis walk on the binary tree, with initial bond weights 1, is transient for small δ and recurrent for large δ. Here, we prove the analog of Takeshima's theorem for Werner reinforcement. Takeshima's proof used extensively that Diaconis walk is a mixture of Markov chains, which is not true of Werner walk. Our proof of transience preservation is easy, while the proof of recurrence preservation is more difficult. We do not know how to prove recurrence preservation without the unexpected (to us) scaling result, Theorem 2.3 of Section 2. We use some results from [DV02] and [DV04] but not the main results, rather only some fairly quickly provable propositions from the beginnings of these papers. The proofs of most of these are sketched.

It is immediate that the (unreinforced) random walk on $\mathbb{Z}$, determined by the bond weights ω_i of $(i, i+1)$, and the random walk, determined by weights $c\omega_i$ for $c > 0$, are the same, and easy to show that no other bond weightings determine this random walk. It is also easy to see that the reinforced walks resulting from Diaconis reinforcement (δ) of the initial weights ω_i and Diaconis reinforcement ($c\delta$) of the initial weights $c\omega_i$ have the same distribution. Thus, the collection of all the Diaconis reinforced walks on $\mathbb{Z}$ (i.e., for all $\delta > 0$) corresponding to initial bond weights $\{\omega_i : i \in \mathbb{Z}\}$ depends only on the unreinforced Markov process corresponding to these weights. In Section 4, we prove that, unusually for a vertex reinforcement scheme, the analogous result holds for Werner walk. See Theorem 4.1. This is not difficult to prove but neither is it immediate.

2. Preliminaries

We use $\exp(\lambda)$ to denote an exponential variable of rate λ, with expectation $\frac{1}{\lambda}$. Throughout, $C, C_1, C_2, \ldots$ will stand for positive constants, each of which we could replace by an explicit number. Two stochastic processes will be said to be the same if they have the same distributions. We begin this section by describing a way to simulate Werner walk and VRJP. The construction below may be taken as a definition, which some readers may prefer to the definition given in the Introduction. A third definition is given at the beginning of Section 3.

The construction we now give will describe Werner walk on graphs with vertices $\{i : i \in I\}$, where I is a set of consecutive integers, and bonds $\{(i, i+1) : i, i+1 \in I\}$. These are the only graphs considered in this paper. We denote this graph again by I.

The parameters of this process are the initial weights, which are positive numbers w_i assigned to each $i \in I$, and a reinforcement constant $\delta > 0$.

The construction of Werner walk $Y_0, Y_1, \ldots$ on I proceeds inductively. Let $N(v)$ be the set of neighbors of $v \in I$, so $N(i) = \{i-1, i+1\}$ unless i is a largest or smallest integer in I. Let $Z_1, Z_2, \ldots$ be i.i.d. exponential(1) random variables. Let $Y_0 = v$ and $W_v^0 = w_v$, $v \in I$. Then $\mathbb{P}(Y_{k+1} = \theta | Z_i, i \leq k+1, Y_i, i \leq k) = 0$, unless θ is a neighbor of Y_k in which case it equals

$$\frac{W_\theta^k}{\sum_{u \in N(Y_k)} W_u^k}.$$

Also $W_\theta^{k+1} = W_\theta^k$ unless $\theta = Y_k$, in which case

$$W_{Y_k}^{k+1} = W_{Y_k}^k + \delta\left(\frac{Z_{k+1}}{\sum_{u \in N(Y_k)} W_u^k}\right).$$

To define the continuous time vertex reinforced jump process (VRJP) X_t, $t \geq 0$, associated with initial weights w_θ, we put

$$\tau_k = \frac{1}{\delta} \sum_{j \in I} (W_j^k - W_j^0), \quad k \geq 0,$$

and $X_t = Y_k$ for $\tau_k \leq t < \tau_{k+1}$. Note that in the above sum, for each k, only finitely many terms are positive.

Our main result, Theorem 3.1, while stated for all $\delta > 0$, will be shown to be implied by the $\delta = 1$ case. *Unless explicitly mentioned otherwise, we study only the* $\delta = 1$ *case in the rest of this paper.*

Even if we were only concerned with Werner walk, we would find it very convenient to study it as a process embedded in a VRJP.

In the rest of this section, we study VRJP on $\{0, 1\}$. Up through Proposition 2.2, we present results which are easy corollaries of results in [DV02], with brief sketches of some proofs. For readers who refer to [DV02], we note that the notation $L(k, t)$ there has been replaced by w_k^t which, as defined in the Introduction, stands for the weight of k at time t. As noted, we assume $\delta = 1$ and always start our VRJP at 0. We usually designate the initial weights at 0 and 1 to be a and b, respectively, and this is always assumed if not indicated otherwise. The VRJP will be denoted by $(X_t,\ t \geq 0)$. We put $\xi^{a,b}(t) = \xi(t) = \inf\{s \geq 0 : w_0^s = t\}$, $t \geq a$. Thus, $w_0^{\xi(t)} = t$. Note that $(w_1^{\xi(t)} - b)$ is the sum of a random number of exponential random variables. We put $m_t = m_t^{a,b} = \frac{w_1^{\xi(t)}}{t}$, $t \geq a$.

PROPOSITION 2.1 (Corollary 2.3 of [DV02]). *m_t, $t \geq 0$, is a martingale.*

Sketch of a proof. Note $m_0 = \frac{b}{a}$. We will show $\mathbb{E} m_t$ satisfies the differential equation $y' = 0$ which yields

$$\mathbb{E} m_t = \frac{b}{a}, \quad t \geq a, \tag{2.1}$$

and the proof we give of (2.1) is not difficult to extend to a full proof of the proposition. Note $X_{\xi(t)} = 0$, and that $w_1^{\xi(t)} = w_1^{\xi(t+dt)}$ unless there is an excursion to 1 between times t and $t + dt$. This happens with probability $dt \cdot w_1^{\xi(t)}$ and if it does happen the excursion is $\exp(t)$ so has expectation t^{-1}. Thus, conditioned on $w_1^{\xi(t)} = c$,

$$\mathbb{E} m_{t+dt} = \frac{c + c\, dt \cdot \frac{1}{t}}{t + dt} = \frac{c}{t} = m_t. \qquad \square$$

Similarly, by showing $\mathbb{E} m_t^2$ satisfies another differential equation, it can be shown that (this is (2.7) of [DV02])

$$\mathbb{E} m_t^2 = \frac{ab^2 + b}{a^3} - \frac{b}{at^2}, \quad t \geq a. \tag{2.2}$$

Thus, m_t, $t \geq a$, is an L^2 bounded martingale, and so we have

$$\begin{aligned} \mathbb{E}m_\infty &= \lim_{t\to\infty} \mathbb{E}m_t = \frac{b}{a}, \\ \mathbb{E}m_\infty^2 &= \lim_{t\to\infty} \mathbb{E}m_t^2 = \frac{ab^2+b}{a^3}. \end{aligned} \tag{2.3}$$

Let $\tau = \inf\{t : X_t = 1\}$, put $\theta(t) = \inf\{s \geq 0 : w_1^{s+\tau} = t\}$, $t \geq b$, and put $M_t = \frac{w_0^{\theta(t)}}{w_1^{\theta(t)}} = \frac{w_0^{\theta(t)}}{t}$. Given $\tau = x$, M_t, $t \geq b$, has the distribution of $m_t^{b,a+x}$, $t \geq b$, and it is easy to check that M_t, $t \geq b$ is an L^2 bounded martingale. We use $m^{b,a+\exp(b)}$ to denote a martingale with the distribution of M_t, $t \geq b$. Thus, using (2.3),

$$\begin{aligned} \mathbb{E}M_\infty &= \mathbb{E}\mathbb{E}(M_\infty|\tau) = \mathbb{E}\frac{a+\tau}{b} = \left[1 + \frac{1}{ab}\right]\frac{a}{b}, \\ \mathbb{E}M_\infty^2 &= \mathbb{E}\mathbb{E}(M_\infty^2|\tau) = \mathbb{E}\frac{b(a+\tau)^2 + (a+\tau)}{b^3} \\ &= \left[1 + \frac{3}{ab} + \frac{3}{(ab)^2}\right]\frac{a^2}{b^2}. \end{aligned} \tag{2.4}$$

Now Corollary 2.4 of [DV02] states that $\lim_{t\to\infty} \frac{w_1^t}{w_0^t} = m_\infty$ a.s. and that $m_\infty \in (0,\infty)$ a.s. Also, given the discussion before (2.4) above, Corollary 2.4 of [DV02] gives $\lim_{t\to\infty} \frac{w_1^t}{w_0^t} = M_\infty$ a.s., from which we get $\frac{1}{m_\infty} = M_\infty$. It will be convenient to write

$$\tilde{m}_t = m_t \cdot \frac{a}{b} \quad \text{and} \quad \tilde{r}_t^{a,b} = \tilde{r}_t = \frac{w_0^t}{w_1^t} \cdot \frac{b}{a}.$$

The following is immediate from (2.3) and (2.4).

$$\begin{aligned} \mathbb{E}\tilde{m}_\infty &= 1, \\ \mathbb{E}\tilde{m}_\infty^2 &= \frac{a^2}{b^2}\left(\frac{ab^2+b}{a^3}\right) = 1 + \frac{1}{ab}, \\ \mathbb{E}\frac{1}{\tilde{m}_\infty} &= \frac{b}{a}\left(\frac{a+\frac{1}{b}}{b}\right) = 1 + \frac{1}{ab}, \\ \mathbb{E}\frac{1}{\tilde{m}_\infty^2} &= \frac{b^2}{a^2}\mathbb{E}M_\infty^2 = 1 + \frac{3}{ab} + \frac{3}{(ab)^2}. \end{aligned} \tag{2.5}$$

In [DV02] the following inequality, related to Doob's L^2 martingale maximal inequality, is proved.

$$\mathbb{E}\sup_{t\geq 0}\left(\frac{w_1^t}{w_0^t} - m_\infty\right)^2 \leq 16\frac{b}{a^3}. \tag{2.6}$$

With the discussion above (2.4) and the fact that $\frac{1}{m_\infty} = M_\infty$, (2.6) gives

$$\mathbb{E}\sup_{t\geq\tau}\left(\frac{w_0^t}{w_1^t} - \frac{1}{m_\infty}\right)^2 = \mathbb{E}\mathbb{E}\left(\sup_{t\geq\tau}\left(\frac{w_0^t}{w_1^t} - M_\infty\right)^2 \Big| \tau\right) \leq 16\mathbb{E}\frac{a+\tau}{b^3} = 16\frac{a+\frac{1}{b}}{b^3}. \tag{2.7}$$

Multiplying (2.6) by $(\frac{a}{b})^2$ gives

$$\mathbb{E}\sup_{t\geq 0}\left(\frac{1}{\tilde{r}_t} - \tilde{m}_\infty\right)^2 \leq 16\cdot\frac{1}{ab}, \tag{2.8}$$

and multiplying (2.7) by $(\frac{b}{a})^2$ gives

$$\mathbb{E}\sup_{t\geq\tau}\left(\tilde{r}_t - \frac{1}{\tilde{m}_\infty}\right)^2 \leq 16\left(\frac{1}{ab} + \frac{1}{(ab)^2}\right). \tag{2.9}$$

PROPOSITION 2.2. *There is a positive constant C such that, for $\varepsilon > 0$,*

$$\mathbb{P}\left(\sup_{t\geq\tau}\left|\tilde{r}_t - \left(1+\frac{1}{ab}\right)\right| > \varepsilon\right) < \frac{C}{\varepsilon^2}\cdot\frac{1+ab}{(ab)^2}.$$

Proof. From (2.5), we know the variance of $\frac{1}{\tilde{m}_\infty}$:

$$\operatorname{var}\frac{1}{\tilde{m}_\infty} = \frac{2+ab}{(ab)^2}.$$

From this, and Chebyshev's inequality, we get

$$\mathbb{P}\left(\left|\frac{1}{\tilde{m}_\infty} - \left(1+\frac{1}{ab}\right)\right| > \varepsilon\right) < \frac{2+ab}{(ab)^2\varepsilon^2}. \tag{2.10}$$

Now,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\geq\tau}\left|\tilde{r}_t - \left(1+\frac{1}{ab}\right)\right| > \varepsilon\right) &< \mathbb{P}\left(\sup_{t\geq\tau}\left|\tilde{r}_t - \frac{1}{\tilde{m}_\infty}\right| > \frac{\varepsilon}{2}\right) \\ &\quad + \mathbb{P}\left(\left|\frac{1}{\tilde{m}_\infty} - \left(1+\frac{1}{ab}\right)\right| > \frac{\varepsilon}{2}\right).\end{aligned}$$

The second term is controlled by (2.10), and the first term by (2.9), giving us

$$\begin{aligned}\mathbb{P}\left(\sup_{t\geq\tau}\left|\tilde{r}_t - \left(1+\frac{1}{ab}\right)\right| > \varepsilon\right) &< 64\cdot\frac{1+ab}{(ab)^2\varepsilon^2} + 4\cdot\frac{2+ab}{(ab)^2\varepsilon^2} \\ &= \frac{1}{\varepsilon^2}\cdot\left(\frac{68}{ab} + \frac{72}{(ab)^2}\right).\end{aligned}$$

□

Note that all the equalities in (2.5) depend only on the product ab. We were able to find the density of $\tilde{m}_\infty$ and this also depended only on ab, which led us to the following theorem.

THEOREM 2.3. *If* $ab = cd$, *then* $\tilde{m}^{a,b}_{a\cdot t}, t \geq 0$ *and* $\tilde{m}^{c,d}_{c\cdot t}, t \geq 0$ *are equal in distribution.*

To prove Theorem 2.3, we will first prove a lemma about the distribution of waiting times between jumps. If the initial weights are a at 0 and b at 1, we let $\delta^{a,b}_k = \delta_k = \tau_k - \tau_{k-1}$ where, as before, τ_k is the time of the kth jump, for $k \geq 1$ and $\tau_0 = 0$. So δ_1 is the waiting time at 0 before the jump to 1, δ_2 is the waiting time at 1 before the jump back to 0, and so on. At time $\delta_1 + \cdots + \delta_k$, then, the weight at vertex 0 is $a + \delta_1 + \delta_3 + \cdots$ and the weight at 1 is $b + \delta_2 + \delta_4 + \cdots$, where the final terms are either δ_{k-1} or δ_k, depending on the parity of k.

LEMMA 2.4. *Assume* $ab = cd$. *Then*

$$\left\{\frac{\delta^{a,b}_1}{a}, \frac{\delta^{a,b}_2}{b}, \frac{\delta^{a,b}_3}{a}, \ldots\right\} \quad \textit{and} \quad \left\{\frac{\delta^{c,d}_1}{c}, \frac{\delta^{c,d}_2}{d}, \frac{\delta^{c,d}_3}{c}, \ldots\right\}$$

are equal in distribution.

Proof. We inductively show that the joint distributions of the first n random variables in each of the two sequences above are the same for all n. Now $\frac{\delta^{a,b}_1}{a}$ is $\exp(ab)$ and $\frac{\delta^{c,d}_1}{c}$ is $\exp(cd)$ and so the case $n = 1$ holds. Suppose that the case $n = k - 1$ holds. We will show that the case $n = k$ holds. We treat the case $k - 1$ even first, and show that the conditional distribution of $\frac{\delta^{a,b}_k}{a}$ given

$$\left(\frac{\delta^{a,b}_1}{a}, \frac{\delta^{a,b}_2}{b}, \ldots, \frac{\delta^{a,b}_{k-1}}{b}\right) = (x_1, \ldots, x_{k-1})$$

is equal to the conditional distribution of $\frac{\delta^{c,d}_k}{c}$ given

$$\left(\frac{\delta^{c,d}_1}{c}, \frac{\delta^{c,d}_2}{d}, \ldots, \frac{\delta^{c,d}_{k-1}}{d}\right) = (x_1, \ldots, x_{k-1}),$$

where $x_i > 0$. The first of these conditional distributions is exponential with rate $(b + \sum_{\substack{j \text{ even} \\ j \leq k-1}} bx_j)a$, the second is exponential $(d + \sum_{\substack{j \text{ even} \\ j \leq k-1}} dx_j)c$, where the final a and c in these rates come from the denominators of $\frac{\delta^{a,b}_k}{a}$ and $\frac{\delta^{c,d}_k}{c}$ respectively. Since $ab = cd$, these two conditional distributions are the same, and since we know that the two vectors above have the same distributions, this proves the $n = k$ case, if $k - 1$ is even. The case where $k - 1$ is odd can be treated similarly. $\square$

Proof of Theorem 2.3. Recall the definition of $\tilde{m}_t$ and consider

$$\tilde{m}^{a,b}_{at} = \frac{\frac{w_1^{\xi(at)}}{b}}{\frac{w_0^{\xi(at)}}{a}}, \quad t \geq 1. \tag{2.11}$$

Denote the left-hand vector in the statement of Lemma 2.4 by A. Given A, the entire process

$$\left(\frac{w_1^{\xi(at)}}{b}, \frac{w_0^{\xi(at)}}{a}\right), \quad t \geq 1,$$

can be reconstructed. Since $w_0^{\xi(s)} = s$, we have $\frac{w_0^{\xi(at)}}{a} = t$ for $1 \leq t < \infty$. Now $w_1^{\xi(s)}$ is constant except for jumps at $s = \delta_1, \delta_1 + \delta_3, \delta_1 + \delta_3 + \delta_5, \ldots$ at which times $w_1^{\xi(s)}$ jumps by $\delta_2, \delta_4, \delta_6, \ldots$ respectively. So $\frac{w_1^{\xi(at)}}{b}$ is constant except for jumps at $\frac{\delta_1}{a}, \frac{\delta_1}{a} + \frac{\delta_3}{a}, \frac{\delta_1}{a} + \frac{\delta_3}{a} + \frac{\delta_5}{a}, \ldots$ at which times it jumps by $\frac{\delta_2}{b}, \frac{\delta_4}{b}, \frac{\delta_6}{b}, \ldots$, respectively. This, together with its analog in which c, d replace a, b, and Lemma 2.4, establishes Theorem 2.3. $\square$

We note that A determines the range of

$$\frac{\frac{w_0^t}{a}}{\frac{w_1^t}{b}} = \tilde{r}_{a,b}^t, \quad t \geq 0.$$

By the range, we mean the random interval which is the set of all values taken on by a sample path of $\tilde{r}_{a,b}^t$. For the numerator of $\tilde{r}_{a,b}^t$ increases linearly from 1 to $1 + \frac{\delta_1}{a}$ on the interval $[0, \delta_1]$, while the denominator does not change. Then on $[\delta_1, \delta_1 + \delta_2]$ the denominator of $\tilde{r}_{a,b}^t$ increases linearly from 1 to $1 + \frac{\delta_2}{b}$, while the numerator does not change, and so on. Thus, the distributions of $\sup_{t\geq 0} \tilde{r}_{a,b}^t$ and $\sup_{t\geq 0} \frac{1}{\tilde{r}_{a,b}^t}$ are determined by the distribution of A. Therefore, if $ab = cd$, Lemma 2.4 implies that $\sup_{t\geq 0} \tilde{r}_{a,b}^t$ and $\sup_{t\geq 0} \tilde{r}_{c,d}^t$ are equidistributed, as are the corresponding infima. Furthermore, these comments and Lemma 2.4 also imply that both $\sup_{t\geq \tau} \tilde{r}_{a,b}^t$ and $\sup_{t\geq \tau} \frac{1}{\tilde{r}_{a,b}^t}$ are equidistributed with $\sup_{t\geq \tau} \tilde{r}_{c,d}^t$ and $\sup_{t\geq \tau} \frac{1}{\tilde{r}_{c,d}^t}$ respectively, if $ab = cd$, where τ is the time of the first jump (also denoted δ_1 in the previous proofs).

3. Recurrence and transience of Werner walk on $\mathbb{Z}$

We call a VRJP recurrent (resp. transient) if its associated Werner walk is recurrent (resp. transient). Let $\mathbb{Z}^+$ be the natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbb{Z}^-$ be $\{0, -1, -2, \ldots\}$. The graphs $\mathbb{Z}^+$, $\mathbb{Z}^-$, and $\mathbb{Z}$ are trees in the sense of [DV04] and thus by Proposition 3 of [DV04], VRJP is either transient or recurrent on each of them, if the initial weights are positive constants, which is always assumed. Then Lemma 2 of [DV04] says that VRJP on a tree is recurrent if and only if $w_v^\infty = \infty$ a.s. for each vertex v of the tree and VRJP is transient if and only if $w_v^\infty < \infty$ a.s. for each vertex v. Furthermore, whether recurrence or transience obtains is not influenced by the initial position.

The main theorem of this paper is the following.

THEOREM 3.1. *Werner walk on $\mathbb{Z}$ is recurrent (resp. transient) for all $\delta > 0$ if the Markov process associated with the initial vertex weights is recurrent (resp. transient).*

Now let $Z_1, Z_2, \dots$ be i.i.d. exponential(1) random variables. Let $Y_0, Y_1, \dots$ and $W_0, W_1, \dots$ be respectively, the steps of the Werner walk on $\mathbb{Z}$ started at 0 with initial weights w_i and reinforcement constant δ, and the Werner walk started at 0 with initial weights $\delta^{-1} w_i$ and reinforcement constant 1, both constructed from the initial data (Z_i), exactly according to the instructions in the definition of Werner walk close to the beginning of Section 2. It is easy to see that $Y_i = W_i$, $i \geq 0$. Also, the (unreinforced) Markov process corresponding to weights w_i and the Markov process corresponding to weights $\delta^{-1} w_i$ have the same distribution. These facts immediately show that if the particular case $\delta = 1$ of Theorem 3.1 holds, then Theorem 3.1 is true in its entirety. Thus as noted earlier, in the proof of Theorem 3.1, that is, in the rest of this section, we may and do study only the $\delta = 1$ Werner walk.

Now let $F = F_t$, $t \geq 0$, be VRJP on $\mathbb{Z}$ with initial weights w_i, started at 0. Let G be VRJP on $\mathbb{Z}^+$ with initial weights w_i, started at 0, and let H be VRJP on $\mathbb{Z}^-$, with initial weights w_i, started at 0.

PROPOSITION 3.2. *F is recurrent if and only if both G and H are recurrent.*

We note that the analog of Proposition 3.2 for the unreinforced Markov processes corresponding to the weights w_i also holds.

Next, we sketch a Poisson construction of Werner walk and VRJP on an interval I from [DV04]. This will be used in the proof of Proposition 3.2.

Let $\langle i, i+1\rangle$ and $\langle i+1, i\rangle$ stand for directed bonds between consecutive integers i and $i+1$ in I. To these directed bonds, independent Poisson processes (rate 1) $\Gamma_0^{\langle i,i+1\rangle}$ and $\Gamma_0^{\langle i+1,i\rangle}$ (i, $i+1 \in I$) are associated. Let $T^0_{\langle\cdot,\cdot\rangle}$ be the first jump time of $\Gamma_0^{\langle\cdot,\cdot\rangle}$.

We describe the Werner walk associated with the initial weights w_i, $i \in I$, started at $v \in I$, and reinforcement constant $\delta = 1$. Let the walk start at v and denote the weight of i at time 0 by L_i^0, $i \in I$, so that $L_i^0 = w_i$. Let $\tau_1 = \min\{\frac{1}{L_j^0} T^0_{\langle v,j\rangle} : j \in N(v)\}$. If $\frac{1}{L_k^0} T^0_{\langle v,k\rangle}$ is this minimum, the walk jumps to k at τ_1. Put $L_i^1 = L_i^0$, $i \in I$, $i \neq v$ and $L_v^1 = L_v^0 + \tau_1$, and also define $\Gamma_1^{\langle i,j\rangle}$ to be the Poisson process $\Gamma_0^{\langle i,j\rangle}$ if i is a neighbor of j, unless $i = v$ and j is a neighbor of v, in which case $\Gamma_1^{\langle v,j\rangle}(t)$ is $\Gamma_0^{\langle v,j\rangle}(L_j^0\tau_1 + t) - \Gamma_0^{\langle v,j\rangle}(L_j^0\tau_1)$, $t \geq 0$. Let $T^1_{\langle i,j\rangle}$ be the first jump time of $\Gamma_1^{\langle i,j\rangle}$. Note that the Poisson processes $\Gamma_1^{\langle i,j\rangle}$ just constructed are still independent. To determine the second jump, mimic the previous procedure, with Γ_1, T_1, and L^1 in the roles of Γ_0, T_0 and L^0.

To define the VRJP associated with this Werner walk, take the definitions used to get from Werner walk to VRJP in our Section 2 definition. Note the collection of all the jump times of all the Poisson processes $\Gamma_0^{\langle i,j\rangle}$ determines the entire Werner walk.

Proof of Proposition 3.2. We construct F using the construction described above, started from 0, and then use the same Poisson processes to construct both G started from 0 and H started from 0. That is, for example, the Poisson process which determines the jumps from 2 to 3 for F is also the determining process for G.

If F is recurrent, and we observe only the jumps that F makes between two consecutive integers of $\mathbb{Z}^+$, we see the jumps of G. Formally, these two processes, which are defined on the same probability space, are the same. A similar statement holds for H. Since F visits 0 infinitely often, so do G and H.

Conversely, if G and H are recurrent, both of their associated Werner walks return to 0 a.s. after each jump from 0 to 1 and -1, respectively, so F returns to 0 a.s. after each jump from 0. □

In view of Proposition 3.2, and the discussion after the statement of Theorem 3.1, Theorem 3.1 follows from the following proposition.

PROPOSITION 3.3. *Werner walk with $\delta=1$ on $\mathbb{Z}^+$ is recurrent if the unreinforced Markov process corresponding to its initial weights is recurrent and transient if the unreinforced process is transient.*

It is easy to show that the unreinforced Markov process $Z_0, Z_1, \dots$ on $\mathbb{Z}^+$ corresponding to weights w_i is recurrent if and only if $\sum_{i=0}^{\infty}\frac{1}{w_i w_{i+1}}=\infty$. This follows from the fact that if

$$f(k)=\sum_{i=0}^{k-1}\frac{1}{w_i w_{i+1}}$$

then $f(\hat{Z}_i)$, $i\ge 0$ is a martingale, where $\hat{Z}$ is Z stopped at 0, or by the standard criterion for transition probabilities (see [HPS72, p. 33]) and a little algebra.

Proposition 3.3 follows from the following two propositions.

PROPOSITION 3.4. *If $\sum_{i=0}^{\infty}\frac{1}{w_i w_{i+1}}<\infty$, then the VRJP on $\mathbb{Z}^+$ with $\delta=1$ and initial weights $(w_i)_{i\ge 0}$ is transient.*

PROPOSITION 3.5. *If $\sum_{i=0}^{\infty}\frac{1}{w_i w_{i+1}}=\infty$, then the VRJP on $\mathbb{Z}^+$ with $\delta=1$ and initial weights $(w_i)_{i\ge 0}$ is recurrent.*

Recall that we use $\exp(a)$ to denote an exponential random variable and $m^{b,a+\exp(b)}$ to denote a martingale with the distribution of M_t, $t\ge b$, as defined in the paragraph between (2.3) and (2.4).

We need the following result from [DV02]. We sketch an alternate proof to the one given there. Let $\sigma(\cdot)$ stand for the σ-field generated by $\cdot$.

LEMMA 3.6. *Let X be VRJP on $\mathbb{Z}^+$ started at* 0 *with initial weights $w_i^0 = w_i$. Let $T_n = \inf\{t \geq 0 : X_t = n\}$, $n \geq 0$. Then for $0 \leq i < n$, the conditional distribution of $\frac{w_i^{T_n}}{w_{i+1}^{T_n}}$ given $\sigma(w_k^{T_n}, i+1 \leq k \leq n) := \mathcal{F}_i$, is the distribution of $m_{w_{i+1}^{T_n}}^{w_{i+1}, w_i + \exp(w_{i+1})}$.*

Proof. We may and do assume that X is VRJP on $\{0, 1, \ldots, n+1\}$ with initial weights w_i started at 0. Construct X via the Poisson construction. Let $\theta(t) = \inf\{s \geq 0 : w_{i+1}^s = t\}$, $t \geq w_{i+1}$. Then $q(t) := \frac{w_i^{\theta(t)}}{w_{i+1}^{\theta(t)}}$, $t \geq w_{i+1}$, has the same distribution as $m_t^{w_{i+1}, w_i + \exp(w_{i+1})}$, $t \geq w_{i+1}$, since only $\Gamma_0^{\langle i, i+1 \rangle}$ and $\Gamma_0^{\langle i+1, i \rangle}$ are involved in this ratio.

Now $w_{i+1}^{T_n}$ is the weight of $i+1$ at the last time before T_n that X jumps from $i+1$ to $i+2$ (unless $i+1 = n$, in which case it is simply the initial weight), and so is determined by Poisson processes associated with bonds to the right of $i+1$, as is the entire σ-field $\mathcal{F}_i$, and so is independent of $\Gamma_0^{\langle i, i+1 \rangle}$ and $\Gamma_0^{\langle i+1, i \rangle}$. So $\frac{w_i^{T_n}}{w_{i+1}^{T_n}}$ samples the process $q(t)$ at an independent time. $\square$

Proof of Proposition 3.4. Let X be a VRJP on $\mathbb{Z}^+$ with initial weights w_i started at 0. We will examine the ratio of the weight of 0 at time T_n as compared to its initial weight, $\frac{w_0^{T_n}}{w_0}$. By expanding this into a telescoping product, we can express it in the following form:

$$\frac{w_0^{T_n}}{w_0} = \frac{\frac{w_0^{T_n}}{w_1^{T_n}}}{\frac{w_0}{w_1}} \cdot \frac{\frac{w_1^{T_n}}{w_2^{T_n}}}{\frac{w_1}{w_2}} \cdots \frac{\frac{w_{n-1}^{T_n}}{w_n^{T_n}}}{\frac{w_{n-1}}{w_n}} \cdot \frac{w_n^{T_n}}{w_n}. \tag{3.1}$$

With the exception of the last, each of these multiplicands are of the form

$$\frac{\frac{w_k^{T_n}}{w_{k+1}^{T_n}}}{\frac{w_k}{w_{k+1}}} = \frac{w_k^{T_n}}{w_{k+1}^{T_n}} \cdot \frac{w_{k+1}}{w_k}.$$

This is similar to our definition of $\tilde{r}$ in the previous section, and we will denote these by $\tilde{R}_k^{T_n} = \tilde{R}_k^n$. We always use s, t, T, etc. for times and k, n, N, etc. for vertices, so the ambiguity in notation should not cause confusion here. The reader familiar with [DV02] will note that $\tilde{R}_k^n$ is the 'adjusted' version of the R_k^n found there. The last term in (3.1) is identically equal to 1, since the weight at vertex n does not increase until after T_n. We rewrite the identity using $\tilde{R}$ notation:

$$\frac{w_0^{T_n}}{w_0} = \tilde{R}_0^n \cdot \tilde{R}_1^n \cdots \tilde{R}_{n-1}^n. \tag{3.2}$$

Now Lemma 3.6 and (2.1) and the fact that $\sigma(\tilde{R}_j, i+1 \le j \le n) = \mathcal{F}_i$ (as defined in the statement of Lemma 3.6) together with

$$\mathbb{E}m_c^{w_{i+1}, w_i + \exp(w_{i+1})} = \mathbb{E}\frac{w_i + \exp(w_{i+1})}{w_{i+1}} = \frac{w_i + \frac{1}{w_{i+1}}}{w_{i+1}}, \quad t \ge 0, c \ge w_{i+1},$$

give

$$\begin{aligned}\mathbb{E}(\tilde{R}_i^n | \tilde{R}_j^n, i+1 \le j \le n) &= \frac{w_{i+1}}{w_i}\left[\frac{w_i + \frac{1}{w_{i+1}}}{w_{i+1}}\right] \\ &= \left(1 + \frac{1}{w_i w_{i+1}}\right)\end{aligned}$$

for $i = n-1,\ n-2, \ldots, 0$. Thus,

$$\begin{aligned}\mathbb{E}\tilde{R}_{n-2}^n \tilde{R}_{n-1}^n &= \mathbb{E}\tilde{R}_{n-1}^n \mathbb{E}\mathbb{E}(\tilde{R}_{n-2}^n | \tilde{R}_{n-1}^n) \\ &= \left(1 + \frac{1}{w_n w_{n-1}}\right)\mathbb{E}\left(1 + \frac{1}{w_{n-1} w_{n-2}}\right) \\ &= \left(1 + \frac{1}{w_n w_{n-1}}\right)\left(1 + \frac{1}{w_{n-1} w_{n-2}}\right).\end{aligned}$$

Continuing, we get

$$\mathbb{E}\frac{w_0^{T_n}}{w_0} = \prod_{i=0}^{n-1}\left(1 + \frac{1}{w_i w_{i+1}}\right).$$

Letting n approach infinity, we get

$$\mathbb{E}w_0^\infty = w_0 \prod_{i=0}^{\infty}\left(1 + \frac{1}{w_i w_{i+1}}\right) < \infty$$

if $\sum_{i=1}^\infty \frac{1}{w_i w_{i+1}} < \infty$. Thus $w_0^\infty < \infty$ a.s. and so, by the discussion at the beginning of this section, the Werner walk is transient. $\square$

Proof of Proposition 3.5. We begin with a version of a standard fact.

LEMMA 3.7. *Let X be a random variable with $\mathbb{E}X \ge \theta$, $\mathbb{E}X^2 \le K\theta^2$, where θ and K are positive constants. Then*

$$\mathbb{P}\left(X \ge \frac{\theta}{2}\right) > \frac{1}{4K}.$$

Proof. Let $A = \{X \ge \frac{\theta}{2}\}$. Then we have

$$\begin{aligned}&\theta \le \mathbb{E}X\mathbb{I}_A + \mathbb{E}X\mathbb{I}_{A^C} < \mathbb{E}X\mathbb{I}_A + \frac{\theta}{2}, \quad \text{so} \\ &\frac{\theta}{2} < \mathbb{E}X\mathbb{I}_A \le (\mathbb{E}X^2)^{\frac{1}{2}}(\mathbb{E}\mathbb{I}_A)^{\frac{1}{2}} \le \sqrt{K}\theta\mathbb{P}(A)^{\frac{1}{2}}.\end{aligned}$$

$\square$

We retain the notation and conventions of the proof of Proposition 3.4. We will show that there is a constant C such that

$$\sup_{n\geq 0}\mathbb{P}\biggl(\frac{w_0^{T_n}}{w_0}>M\biggr)>C, \tag{3.3}$$

if $M\in\mathbb{R}$ and $\sum_{i=0}^{\infty}\frac{1}{w_iw_{i+1}}=\infty$. Since $w_0^{T_n}$ is nondecreasing in n, (3.3) implies $\mathbb{P}(w_0^{\infty}=\infty)\geq C$, from which, for reasons explained at the beginning of this section, the recurrence of X follows.

Taking logarithms of both sides of (3.2) gives $\ln(\frac{w_0^{T_n}}{w_0})=\sum_{i=0}^{n-1}\ln\tilde{R}_i^n$. We put

$$\begin{aligned}\Delta_i^n&=\mathbb{E}(\ln\tilde{R}_i^n|\tilde{R}_k, i<k\leq n),\quad 0\leq i<n,\quad\text{and}\\ \delta_i^n&=\ln\tilde{R}_i^n-\Delta_i^n,\quad 0\leq i<n.\end{aligned}$$

We note that δ_{n-1}^n, $\delta_{n-2}^n,\dots,\delta_0^n$, in this order, is a martingale difference sequence. To prove (3.3), it suffices to prove

$$\liminf_{n\to\infty}\mathbb{P}\Biggl(\sum_{k=n-1}^{0}\delta_k^n+\Delta_k^n>\lambda\Biggr)>C,\quad\lambda>0, \tag{3.4}$$

if $\sum_{i=0}^{\infty}\frac{1}{w_iw_{i+1}}=\infty$, where C does not depend on λ. To this end, we will prove that there are constants C_1, C_2, C_3 such that if $n\geq 0$ and $0\leq i<n$,

$$\Delta_i^n\geq C_1\ln\biggl(1+\frac{1}{w_iw_{i+1}}\biggr), \tag{3.5}$$

$$\mathbb{E}(\delta_i^n)^2\leq C_2\ln\biggl(1+\frac{1}{w_iw_{i+1}}\biggr),\quad\text{if } w_iw_{i+1}>\frac{1}{2},\quad\text{and} \tag{3.6}$$

$$\mathbb{E}(\delta_i^n)^2\leq C_3\biggl[\ln\biggl(1+\frac{1}{w_iw_{i+1}}\biggr)\biggr]^2,\quad\text{if } w_iw_{i+1}\leq\frac{1}{2}. \tag{3.7}$$

Before proving (3.5)–(3.7), we will show that together they imply Proposition 3.5. Let

$$\begin{aligned}\mathcal{A}_n&=\biggl\{k:0\leq k\leq n-1\text{ and } w_kw_{k+1}>\frac{1}{2}\biggr\},\quad\text{and}\\ \mathcal{B}_n&=\biggl\{k:0\leq k\leq n-1\text{ and } w_kw_{k+1}\leq\frac{1}{2}\biggr\}.\end{aligned}$$

Put

$$\begin{aligned}\mu_n&=\sum_{i\in\mathcal{A}_n}\ln\biggl(1+\frac{1}{w_iw_{i+1}}\biggr),\qquad \nu_n=\sum_{i\in\mathcal{B}_n}\ln\biggl(1+\frac{1}{w_iw_{i+1}}\biggr),\\ \sigma_n&=\sum_{i\in\mathcal{A}_n}\mathbb{E}(\delta_i^n)^2,\quad\text{and}\quad\theta_n=\sum_{i\in\mathcal{B}_n}\mathbb{E}(\delta_i^n)^2.\end{aligned}$$

Proposition 3.5 presupposes that $\sum_{i=0}^{\infty} \frac{1}{w_i w_{i+1}} = \infty$, which implies

$$\sum_{i=0}^{\infty} \ln\left(1 + \frac{1}{w_i w_{i+1}}\right) = \infty,$$

and so

$$\lim_{n\to\infty} \mu_n + \nu_n = \infty. \tag{3.8}$$

We put $A_n = \sum_{i\in\mathcal{A}_n} \Delta_i^n$ and $B_n = \sum_{i\in\mathcal{A}_n} \delta_i^n$. Then, by (3.5), we have

$$A_n \geq C_1 \mu_n. \tag{3.9}$$

Now $\mathbb{E}B_n = 0$ since B_n is a sum of a martingale difference sequence, and, for the same reason,

$$\mathbb{E}B_n^2 = \sum_{i\in\mathcal{A}_n} \mathbb{E}(\delta_i^n)^2 \leq C_2 \mu_n, \tag{3.10}$$

by (3.6), and so, by Chebyshev

$$\frac{B_n}{C_1 \mu_n} \to 0 \tag{3.11}$$

in probability if $\lim_{n\to\infty} \mu_n = \infty$. Together, (3.9) and (3.11) give

$$\lim_{n\to\infty} \mathbb{P}\left(A_n + B_n > \left(\frac{C_1}{2}\right)\mu_n\right) = 1 \tag{3.12}$$

if $\lim_{n\to\infty} \mu_n = \infty$.

Now let $G_n = \sum_{i\in\mathcal{B}_n} \Delta_i^n$ and $H_n = \sum_{i\in\mathcal{B}_n} \delta_i^n$. Then using arguments similar to those just given we have, using (3.7) and (3.5),

$$\mathbb{E}H_n^2 = \sum_{i\in\mathcal{B}_n} \mathbb{E}(\delta_i^n)^2 \leq C_3 \nu_n^2, \quad \text{and} \tag{3.13}$$

$$G_n \geq C_1 \nu_n. \tag{3.14}$$

Applying Lemma 3.7 to $H_n + C_1\nu_n$, noting $\mathbb{E}H_n = 0$, and using (3.13) to bound $\mathbb{E}(H_n + C_1\nu_1)^2$, gives

$$\mathbb{P}\left(H_n + C_1\nu_n \geq \frac{C_1\nu_n}{2}\right) > \frac{1}{4(\frac{C_3}{C_1^2} + 1)}. \tag{3.15}$$

Thus, using (3.14) we obtain

$$\begin{aligned} \mathbb{P}\left(G_n + H_n \geq \frac{C_1\nu_n}{2}\right) &\geq \mathbb{P}\left(C_1\nu_n + H_n \geq \frac{C_1\nu_n}{2}\right) \\ &> \frac{1}{4(\frac{C_3}{C_1^2} + 1)}. \end{aligned} \tag{3.16}$$

It follows from (3.12) and (3.16) and

$$\sum_{i=n-1}^{0} \delta_i^n + \Delta_i^n = A_n + B_n + G_n + H_n,$$

that if $\lim_{n\to\infty} \mu_n = \infty$ and $\lim_{n\to\infty} \nu_n = \infty$ then (3.4) holds. To complete the proof of (3.4), we must address the possibility that only one of these two limits is infinity. In case $\lim_{n\to\infty} \mu_n = \infty$ and $\lim_{n\to\infty} \nu_n < \infty$, we can use (3.12), together with (3.13) and (3.14) which give both that G_n is nonnegative and that the sequence $\mathbb{E}H_n^2$, is bounded, to easily prove (3.4). And if $\lim_{n\to\infty} \mu_n < \infty$ and $\lim_{n\to\infty} \nu_n = \infty$, (3.16), (3.10), and (3.9) can be used in the same way to prove (3.4).

To complete the proof of Proposition 3.3 it suffices to prove (3.5), (3.6), and (3.7).

Proof of (3.5). By Lemma 3.6, (3.5) is implied by

$$\mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_t^{w_{i+1},w_i+\exp(w_{i+1})}\right) > C\ln\left(1+\frac{1}{w_i w_{i+1}}\right), \quad t \geq w_{i+1}. \tag{3.17}$$

Now by (2.5) and (2.8), we have

$$\mathbb{E}\sup_{t\geq w_{i+1}}\left(\frac{1}{\frac{w_{i+1}}{w_i} m_t^{w_{i+1},w_i+\exp w_{i+1}}}\right)^2 \leq \mathbb{E}\sup_{t\geq 0}\left(\frac{w_i}{w_{i+1}}\frac{r_t^1}{r_t^0}\right)^2 < \infty,$$

and since $(\ln x)^+ < x$, $x > 0$, and $(\ln x)^- = (\ln \frac{1}{x})^+$, we have

$$\mathbb{E}\sup_{t\geq w_{i+1}}\left[\ln\left(\frac{w_{i+1}}{w_i} m_t^{w_i,w_i+\exp w_{i+1}}\right)^-\right]^2 < \infty.$$

Thus, the dominated convergence theorem gives

$$\lim_{t\to\infty}\mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_t^{w_{i+1},w_i+\exp w_{i+1}}\right)^- = \mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_\infty^{w_{i+1},w_i+\exp w_{i+1}}\right)^-.$$

Furthermore, Fatou's lemma gives

$$\varliminf_{t\to\infty}\mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_t^{w_{i+1},w_i+\exp w_{i+1}}\right)^+ \geq \mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_\infty^{w_{i+1},w_i+\exp w_{i+1}}\right)^+.$$

Since $\ln x$ is a concave function of x, the left-hand side of (3.17) is the expectation of a concave function of a martingale thus nonincreasing.

Now if $a_t + b_t$ is nonincreasing, and $\varliminf_{t\to\infty} a_t \geq a_\infty$ while $\lim_{t\to\infty} b_t = b_\infty$, then $a_t + b_t \geq a_\infty + b_\infty$, $t \geq 0$. Thus, using $\ln x = \ln x^+ - \ln x^-$, the inequality and equality just above give

$$\mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_t^{w_{i+1},w_i+\exp(w_{i+1})}\right) \geq \mathbb{E}\ln\left(\frac{w_{i+1}}{w_i} m_\infty^{w_{i+1},w_i+\exp(w_{i+1})}\right), \quad t \geq w_i.$$

Thus, to prove (3.17) it suffices to prove

$$\mathbb{E}\ln\left(\frac{w_{i+1}}{w_i}m_\infty^{w_{i+1},w_i+\exp(w_{i+1})}\right) > C\ln\left(1+\frac{1}{w_iw_{i+1}}\right) \tag{3.18}$$

which is equivalent to

$$\mathbb{E}\ln\left(\frac{1}{\tilde{m}_\infty^{w_i,w_{i+1}}}\right) > C\ln\left(1+\frac{1}{w_iw_{i+1}}\right), \tag{3.19}$$

which, by Theorem 2.3, noting that $\tilde{m}_{\delta,\delta} = m_{\delta,\delta}$, is equivalent to

$$\mathbb{E}\ln\frac{1}{m_\infty^{\delta,\delta}} \geq C\ln\left(1+\frac{1}{\delta^2}\right), \tag{3.20}$$

where $\delta^2 = w_iw_{i+1}$.

LEMMA 3.8. *Let $\varepsilon > 0$ and let $0 \leq \theta$ and let $\tilde{\mathbb{P}}_{\theta,\varepsilon} = \tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}_{\theta,\varepsilon} = \tilde{\mathbb{E}}$ be probability and expectation associated with VRJP on $\{0,1\}$ with initial weights $w_0^0 = \varepsilon, w_1^0 = \varepsilon + \theta$, which satisfies*

$$\tilde{\mathbb{P}}(X_0 = 0) = \tilde{\mathbb{P}}(X_0 = 1) = \frac{1}{2}.$$

Then if $\theta \geq \frac{1}{\varepsilon}$,

$$\tilde{\mathbb{E}}\ln\lim_{t\to\infty}\left(\frac{w_0^t}{w_1^t}\right) > C\ln\left(1+\frac{1}{\varepsilon^2}\right). \tag{3.21}$$

Furthermore,

$$\tilde{\mathbb{E}}\ln\lim_{t\to\infty}\left(\frac{w_0^t}{w_1^t}\right) \geq 0 \tag{3.22}$$

for all θ.

Proof. We use $\bar{\mathbb{P}}$ and $\bar{\mathbb{E}}$ to denote probability and expectation for VRJP on the graph with vertices a, b, c and bonds (a,b) and (b,c), initial weights $w_a^0 = w_b^0 = \varepsilon$ and $w_c^0 = \theta$, and started with probability $\frac{1}{2}$ at a and probability $\frac{1}{2}$ at b. We claim $(w_0^t, w_1^t), t \geq 0$, under $\tilde{\mathbb{P}}$ and $(w_a^t + w_c^t, w_b^t), t \geq 0$, under $\bar{\mathbb{P}}$ have the same distribution. To see this, identify a and c. Note that whether X is at a or c, it jumps to b at rate equal to the weight of b, and if X is at b it jumps to $\{a,c\}$ at rate the weight of a plus the weight of c. Let $F = \lim_{t\to\infty}\frac{w_a^t}{w_b^t}$, $G = \lim_{t\to\infty}\frac{w_c^t}{w_b^t}$, and $H = \lim_{t\to\infty}\frac{w_0^t}{w_1^t}$. Then the distribution of $F+G$ under $\bar{\mathbb{P}}$ is the distribution of H under $\tilde{\mathbb{P}}$. Also F and G are independent under $\bar{\mathbb{P}}$, which follows from the Poisson construction of VRJP given after Proposition 3.2. Finally, this construction, and the fact that $\bar{\mathbb{P}}(X_0 = a) = \bar{\mathbb{P}}(X_0 = b)$, shows that F and $\frac{1}{F}$ have the same distribution under $\bar{\mathbb{P}}$, so that $\ln F$ has a symmetric distribution about zero under $\bar{\mathbb{P}}$. Also, $G \geq 0$. We claim

$$\bar{\mathbb{P}}\left(G \geq \frac{1}{2\varepsilon^2}\right) > C, \tag{3.23}$$

if $\theta \geq \frac{1}{\varepsilon}$. For, note that under $\bar{\mathbb{P}}$, G has the distribution of $m_\infty^{\varepsilon,\theta}$. Now $\mathbb{E}m_\infty^{\varepsilon,\theta} = \frac{\theta}{\varepsilon}$, by (2.1) and $\mathbb{E}(m_\infty^{\varepsilon,\theta})^2 = \frac{\varepsilon\theta^2}{\varepsilon^3} + \frac{\theta}{\varepsilon^3}$, by (2.3), and so, since $\theta \geq \frac{1}{\varepsilon}$, we have $\mathbb{E}(m_\infty^{\varepsilon,\theta})^2 \leq 2(\mathbb{E}m_\infty^{\varepsilon,\theta})^2$. Thus, by Lemma 3.7, $\mathbb{P}(m_\infty^{\varepsilon,\theta} \geq \frac{1}{2\varepsilon^2}) \geq \mathbb{P}(m_\infty^{\varepsilon,\theta} \geq \frac{\theta}{2\varepsilon}) \geq C_4$. Finally, to prove (3.21), we drop the bar above the $\mathbb{E}$ for cosmetic purposes and note that if $\theta \geq \frac{1}{\varepsilon}$,

$$\begin{aligned}
\mathbb{E}\ln(F+G) &= \mathbb{E}[\ln(F+G) - \ln F] \\
&= \mathbb{E}\mathbb{E}\bigl(\ln(F+G) - \ln F|F\bigr) \\
&= \mathbb{E}\mathbb{E}\bigl(\ln(F+G) - \ln F|F\bigr)\mathbb{I}(F>1) \\
&\quad + \mathbb{E}\mathbb{E}\bigl(\ln(F+G) - \ln F|F\bigr)\mathbb{I}(F\leq 1) \\
&\geq \mathbb{E}\mathbb{E}\bigl(\ln(F+G) - \ln F|F\bigr)\mathbb{I}(F\leq 1) \\
&\geq \mathbb{E}(\ln G + 1)\mathbb{I}(F\leq 1) \\
&= \mathbb{E}\ln(G+1)\cdot\frac{1}{2} \\
&\geq C_4 \ln\left(1 + \frac{1}{2\varepsilon^2}\right) \\
&\geq C\ln\left(1+\frac{1}{\varepsilon^2}\right).
\end{aligned} \tag{3.24}$$

To prove (3.22), we note G is nonnegative, and replace the last two inequalities of (3.24) with "≥ 0." ☐

Finally, we prove (3.20) which will complete the proof of (3.5). Let X_t be VRJP on $\{0,1\}$ with initial weight δ at both 0 and 1 and let $X_0 = 0$. To prove (3.20), we need to show

$$\mathbb{E}\ln\lim_{t\to\infty}\left(\frac{w_0^t}{w_1^t}\right) \geq C\ln\left(1+\frac{1}{\delta^2}\right). \tag{3.25}$$

Let τ be the time of the first jump of X from 0 to 1, so that τ is an exponential(δ) random variable. Let Z be an exponential(δ) random variable independent of the process X_t, $t\geq 0$, and put $Y = \min(Z,\tau)$. Note

$$\mathbb{P}(Y = Z|Y = r) = \mathbb{P}(Y = \tau|Y = r) = \frac{1}{2}, \quad r > 0.$$

Furthermore, on $\{Y = r\}$, $w_0^r = \delta + r$ and $w_1^r = \delta$. Thus, conditioned on $Y = r$, $(X_{t+r}, t\geq 0)$ has exactly the distribution of $X_t, t \geq 0$, under $\tilde{\mathbb{P}}_{r,\delta}$, as defined in the statement of Lemma 3.8. Thus, by (3.21)

$$A_r := \mathbb{E}\left(\ln\lim_{t\to\infty}\frac{w_0^t}{w_1^t}\Big|Y = r\right) \geq C\ln\left(1+\frac{1}{\delta^2}\right), \quad r \geq \frac{1}{\delta},$$

and by (3.22) we have $A_r \geq 0$ if $r \geq 0$. Now Y is $\exp(2\delta)$ and so $\mathbb{P}(Y > \frac{1}{\delta}) = e^{-2}$. Thus,

$$\begin{aligned}
\mathbb{E}\ln\left(\lim_{t\to\infty}\left(\frac{w_0^t}{w_1^t}\right)\right) &= \mathbb{E}A_Y \\
&= \mathbb{E}A_Y\mathbb{I}\left(0 < Y < \frac{1}{\delta}\right) + \mathbb{E}A_Y\mathbb{I}\left(Y \geq \frac{1}{\delta}\right) \\
&\geq \mathbb{E}A_Y\mathbb{I}\left(Y \geq \frac{1}{\delta}\right) \\
&\geq C\ln\left(1 + \frac{1}{\delta^2}\right)\mathbb{P}\left(Y \geq \frac{1}{\delta}\right) \\
&= C\ln\left(1 + \frac{1}{\delta^2}\right). \qquad \square
\end{aligned}$$

Proof of (3.6). To prove (3.6), it suffices to establish

$$\mathbb{E}\big((\ln \tilde{R}_i^n)^2|\tilde{R}_k, i < k \leq n\big) \leq C\frac{1}{w_i w_{i+1}}, \quad \text{if } w_i w_{i+1} > \frac{1}{2} \tag{3.26}$$

since $\mathbb{E}((\ln \delta_i^n)^2|\tilde{R}_k, i < k \leq n) \leq \mathbb{E}((\ln \tilde{R}_i^n)^2|\tilde{R}_k, i < k \leq n)$ while $\log(1 + x) > Cx$ if $0 < x \leq 2$.

To prove (3.26), it suffices by Lemma 3.6 to show

$$\mathbb{E}((\ln \tilde{r}_T^{a,b})^2) \leq C(\varepsilon)\frac{1}{ab}, \quad \text{if } ab \geq \varepsilon \tag{3.27}$$

for every random variable $T \geq \tau$, where τ is the time of the first jump to 1. In this case we could restrict ourselves to T of the form: The first time $w_1^s = t$, but we do not need to consider this restriction.

Proof of (3.27). We shorten $\tilde{r}_t^{a,b}$ to $\tilde{r}_t$ and $\tilde{m}_t^{a,b}$ to $\tilde{m}_t$ in the rest of Section 3. Note that $(\ln x)^2 \leq (x - 1)^2 + (\frac{1}{x} - 1)^2$. Therefore,

$$\mathbb{E}(\ln \tilde{r}_T)^2 \leq \mathbb{E}(\tilde{r}_T - 1)^2 + \mathbb{E}\left(\frac{1}{\tilde{r}_T} - 1\right)^2.$$

Now, for general x, y, we have the following simple inequality,

$$x^2 \leq x^2 + (x - 2y)^2 = 2x^2 - 4xy + 4y^2 = 2(x - y)^2 + 2y^2.$$

Here we let $\tilde{r}_T - 1$ take the role of x and $\frac{1}{\tilde{m}_\infty} - 1$ the role of y.

Taking expectations,

$$\begin{aligned}
\mathbb{E}(\tilde{r}_T - 1)^2 &\leq 2\mathbb{E}\left((\tilde{r}_T - 1) - \left(\frac{1}{\tilde{m}_\infty} - 1\right)\right)^2 + 2\mathbb{E}\left(\frac{1}{\tilde{m}_\infty} - 1\right)^2 \\
&= 2\mathbb{E}\left(\tilde{r}_T - \frac{1}{\tilde{m}_\infty}\right)^2 + 2\mathbb{E}\left(\frac{1}{\tilde{m}_\infty}\right)^2 - 4\mathbb{E}\left(\frac{1}{\tilde{m}_\infty}\right) + 2.
\end{aligned}$$

The first and second moments of $\frac{1}{\tilde{m}_\infty}$ are given in (2.5), and a bound for the first term is given in (2.9). Therefore,

$$\begin{aligned}\mathbb{E}(\tilde{r}_T-1)^2 &\le 32\frac{1+ab}{a^2b^2}+2\left(1+\frac{3}{ab}+\frac{3}{a^2b^2}\right)-4\left(1+\frac{1}{ab}\right)+2\\ &=34\frac{1}{ab}+38\frac{1}{a^2b^2}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\tilde{r}_T}-1\right)^2 &\le 2\mathbb{E}\left(\left(\frac{1}{\tilde{r}_T}-1\right)-(\tilde{m}_\infty-1)\right)^2+2\mathbb{E}(\tilde{m}_\infty-1)^2\\ &=2\mathbb{E}\left(\frac{1}{\tilde{r}_T}-\tilde{m}_\infty\right)^2+2\mathbb{E}(\tilde{m}_\infty)^2-4\mathbb{E}(\tilde{m}_\infty)+2.\end{aligned}$$

This time (2.6) and (2.5) give us the necessary bounds for each of the terms, resulting in

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\tilde{r}_T}-1\right)^2 &\le 2\cdot 16\frac{1}{ab}+2\left(1+\frac{1}{ab}\right)-4+2\\ &=34\frac{1}{ab}.\end{aligned}$$

Putting these back into our initial calculation, we can conclude that if $ab\ge\frac{1}{2}$,

$$\begin{aligned}\mathbb{E}(\ln\tilde{r}_T)^2 &\le 68\frac{1}{ab}+38\frac{1}{a^2b^2}\\ &\le C\frac{1}{ab}.\end{aligned}$$

□

Proof of (3.7). The following lemma together with Lemma 3.6 and the fact that $\ln(x)<\ln(1+x)$ gives (3.7).

LEMMA 3.9. *Let $ab=\varepsilon$ with $\varepsilon\le\frac{1}{2}$, and let $s>b$ be given. If $\theta(s)$ is the time the weight at 1 first reaches s, then,*

$$\mathbb{E}\bigl(\ln\bigl(\tilde{r}_{\theta(s)}\bigr)\bigr)^2<C\left(\ln\frac{1}{\varepsilon}\right)^2.$$

Proof. We show that $\mathbb{E}(\ln(\tilde{r}_{\theta(s)}))^2<C(\ln\frac{1}{\varepsilon})^2$ by showing

$$\mathbb{E}\sup_{t\ge\tau}((\ln\tilde{r}_t)^+)^2<C\left(\ln\frac{1}{\varepsilon}\right)^2\quad\text{and}\tag{3.28}$$

$$\mathbb{E}\sup_{t\ge\tau}((\ln\tilde{r}_t)^-)^2<C\left(\ln\frac{1}{\varepsilon}\right)^2,\tag{3.29}$$

where τ is the first jump to 1. Note $\theta(s) > \tau$. To prove (3.28), we use Proposition 2.2, which gives us

$$\mathbb{P}\left(\sup_{t\geq\tau}\left|\tilde{r}_t - \left(1+\frac{1}{\varepsilon}\right)\right| > \frac{k}{\varepsilon}\right) < \frac{C(1+\varepsilon)}{k^2} \leq \frac{C}{k^2}, \quad k \geq 1, \tag{3.30}$$

so

$$\mathbb{P}\left(\sup_{t\geq\tau}\tilde{r}_t > 1 + \frac{k+1}{\varepsilon}\right) < \frac{C}{k^2}, \quad k \geq 1.$$

Now we use summation by parts to prove (3.28). Let $Z := (\sup_{t\geq\tau}\tilde{r}_t)^+$. Let $a_k = 1 + \frac{k+1}{\varepsilon}$. We have

$$\begin{aligned} \mathbb{E}(\ln Z)^2 &= \int_1^\infty (\ln x)^2\, d\mathbb{P}(Z \leq x) \\ &\leq (\ln a_0)^2 + \sum_{k=0}^\infty (\ln a_{k+1})^2 \mathbb{P}\big(Z \in [a_k, a_{k+1})\big) \\ &= (\ln a_0)^2 + \sum_{k=0}^\infty (\ln a_{k+1})^2 \big(\mathbb{P}(Z \geq a_k) - \mathbb{P}(Z \geq a_{k+1})\big). \end{aligned} \tag{3.31}$$

Summation by parts gives us, for sequences f_k and g_k such that $\lim f_n g_n$ exists:

$$\sum_{k=0}^\infty f_k(g_{k+1} - g_k) = \lim_{n\to\infty}(f_n g_n) - f_0 g_0 - \sum_{k=0}^\infty g_{k+1}(f_{k+1} - f_k).$$

Letting $f_k = (\ln a_{k+1})^2$ and $g_k = -\mathbb{P}(Z \geq a_k)$ and noticing that

$$\lim(\ln a_{n+1})^2\big(\mathbb{P}(Z \geq a_n)\big) \leq \lim\left(\ln\left(1 + \frac{n+2}{\varepsilon}\right)\right)^2\left(\frac{C}{n^2}\right) = 0,$$

we get from (3.31),

$$\begin{aligned} \mathbb{E}(\ln Z)^2 &\leq (\ln a_0)^2 + (\ln a_1)^2\mathbb{P}(Z \geq a_0) + (\ln a_2)^2\mathbb{P}(Z \geq a_1) \\ &\qquad + \sum_{k=1}^\infty \mathbb{P}(Z \geq a_{k+1})\big((\ln a_{k+2})^2 - (\ln a_{k+1})^2\big) \\ &\leq 3(\ln a_2)^2 + \sum_{k=2}^\infty \frac{C}{k^2}(\ln a_{k+1})^2 \\ &= 3(\ln a_2)^2 + \sum_{k=2}^\infty \frac{C}{k^2}\left(\ln\left(1 + \frac{k+2}{\varepsilon}\right)\right)^2 \\ &\leq 3(\ln a_2)^2 + \sum_{k=2}^\infty \frac{C}{k^2}\left(\ln\frac{k+3}{\varepsilon}\right)^2 \quad \left(\text{since } \varepsilon \leq \frac{1}{2}\right) \\ &= 3(\ln a_2)^2 + \sum_{k=2}^\infty \frac{C}{k^2}\left(\ln(k+3) + \ln\frac{1}{\varepsilon}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= 3(\ln a_2)^2 + \sum_{k=2}^{\infty} \frac{C(\ln(k+3))^2}{k^2} + \sum_{k=2}^{\infty} \frac{2C\ln(k+3)\ln\frac{1}{\varepsilon}}{k^2} \\
&\quad + \sum_{k=2}^{\infty} \frac{C(\ln\frac{1}{\varepsilon})^2}{k^2} \\
&= 3\left(\ln\left(1+\frac{3}{\varepsilon}\right)\right)^2 + C_1 + C_2\ln\frac{1}{\varepsilon} + C_3\left(\ln\frac{1}{\varepsilon}\right)^2 \\
&\le C\left(\ln\frac{1}{\varepsilon}\right)^2.
\end{aligned}
$$

To prove (3.29), since

$$
\left(\ln \tilde{r}_t \mathbb{I}(\tilde{r}_t < 1)\right)^2 = \left(\ln\frac{1}{\tilde{r}_t}\mathbb{I}\left(\frac{1}{\tilde{r}_t} > 1\right)\right)^2,
$$

(3.29) follows from (3.28) and the fact that $\sup_{t\ge 0}\tilde{r}_t = \sup_{t\ge\tau}\tilde{r}_t$, and

$$
\mathbb{P}\left(\sup_{t\ge\tau}\frac{1}{\tilde{r}_t} > \lambda\right) \le \mathbb{P}\left(\sup_{t\ge 0}\tilde{r}_t > \lambda\right), \quad \lambda > 0. \tag{3.32}
$$

The comments just after the proof of Theorem 2.3 imply that to prove (3.32), we may assume $a = b = \sqrt{\varepsilon}$, which we do. Since $\frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon}} = 1$,

$$
\mathbb{P}^{\sqrt{\varepsilon},\sqrt{\varepsilon}}\left(\sup_{t\ge\tau}\frac{1}{\tilde{r}_t} > \lambda\right) = \mathbb{P}^{\sqrt{\varepsilon},\sqrt{\varepsilon}}\left(\sup_{t\ge\tau}\frac{w_1^t}{w_0^t} > \lambda\right),
$$

where the superscripts denote the initial weights on 0 and 1 respectively, while

$$
\mathbb{P}^{\sqrt{\varepsilon},\sqrt{\varepsilon}}\left(\sup_{t\ge\tau}\frac{w_1^t}{w_0^t} > \lambda \,\middle|\, \tau = x\right) = \mathbb{P}^{\sqrt{\varepsilon},x+\sqrt{\varepsilon}}\left(\sup_{t\ge 0}\frac{w_0^t}{w_1^t} > \lambda\right).
$$

Thus, (3.32) follows from

$$
\mathbb{P}^{\sqrt{\varepsilon},\sqrt{\varepsilon}}\left(\sup_{t\ge 0}\frac{w_0^t}{w_1^t} > \lambda\right) \ge \mathbb{P}^{\sqrt{\varepsilon},x+\sqrt{\varepsilon}}\left(\sup_{t\ge 0}\frac{w_0^t}{w_1^t} > \lambda\right), \quad \text{if } x > 0. \tag{3.33}
$$

A proof of (3.33) follows by considering VRJP on the graph with vertices α, β, and γ and bonds (α,β) and (γ,β), started at β, with initial weights $\sqrt{\varepsilon}$ at α and β and x at γ. Then

$$
\mathbb{P}\left(\sup_{t\ge 0}\frac{w_\beta^t}{w_\alpha^t} > \lambda\right)
$$

is the left-hand side of (3.33), while

$$
\mathbb{P}\left(\sup_{t\ge 0}\frac{w_\beta^t}{w_\alpha^t + w_\gamma^t} > \lambda\right)
$$

is the right-hand side of (3.33). $\square$

4. Comparisons of Diaconis and Werner walks

Recall that two processes are said to be the same if they have the same distributions. In this section, we prove the following theorem.

THEOREM 4.1. *Let $\{w_i, i \in \mathbb{Z}\}$ be a set of positive numbers and let c and d be positive numbers. Define $\theta_i = cw_i$ if i is odd and $\theta_i = dw_i$ if i is even. Let F^δ and G^δ be Werner walks with reinforcement constant δ, started at* 0, *with initial weights $\{w_i, i \in \mathbb{Z}\}$ and $\{\theta_i, i \in \mathbb{Z}\}$, respectively. Then $\{F^\delta, \delta > 0\} = \{G^\delta, \delta > 0\}$.*

Proof. We will prove the theorem by showing that the Werner walk corresponding to $w_i, i \in \mathbb{Z}$, with reinforcement parameter δ is the same as the Werner walk corresponding to $\theta_i, i \in \mathbb{Z}$, with reinforcement parameter $cd\delta$.

Once again the Poisson construction described after the statement of Proposition 3.2 provides the simplest proof. Let $\Gamma_0^{\langle i,i+1\rangle}$ and $\Gamma_0^{\langle i+1,i\rangle}, i \in \mathbb{Z}$, be a collection of independent Poisson processes. Use these processes both to construct Werner walk (described just above) started at 0, with initial weights θ_i (call this walk Y), and to construct Werner walk started at 0 with initial weights w_i, denoted X. Construct X and Y exactly as prescribed in Section 3, with the obvious modification to cover the cases where the reinforcement parameter is not equal to 1. Now, using the notation right after the statement of Proposition 3.2 with superscripts X and Y added,

$$\tau_1^X = \min\left\{\frac{1}{w_1}T^0_{\langle 0,1\rangle}, \frac{1}{w_{-1}}T^0_{\langle 0,-1\rangle}\right\}$$

and the direction of the first jump of X is determined by which of the two random variables is this minimum. Also

$$\tau_1^Y = \min\left\{\frac{1}{cw_1}T^0_{\langle 0,1\rangle}, \frac{1}{cw_{-1}}T^0_{\langle 0,-1\rangle}\right\} = \frac{1}{c}\tau_1^X,$$

and the direction of the first jump of Y is similarly determined. Thus $\{X_1 = 1\} = \{Y_1 = 1\}$. Furthermore, $L_j^{1,Y} = cL_j^{1,X}$ if j is odd, and $L_j^{1,Y} = dL_j^{1,X}$ if j is even. This is immediate unless $j = 0$, since the weights are unchanged, while if $j = 0$ we have

$$L_0^{1,Y} = dw_0 + \delta c\, d\tau_1^Y = d\left[w_0 + \delta c\frac{\tau_1^X}{c}\right] = dL_0^{1,X}.$$

Now, conditioned on, say, $\min\{\frac{1}{w_1}T_{\langle 0,1\rangle}, \frac{1}{w_{-1}}T_{\langle 0,-1\rangle}\} = \frac{1}{w_1}T_{\langle 0,1\rangle}$ and on the value of $T_{\langle 0,1\rangle}$, we can repeat the entire reasoning above, and in fact we can iterate forever, inductively showing the two Werner walks are exactly the same, not only in the sense of having the same distribution but for each point in the probability space being the same. □

Diaconis walks on $\mathbb{Z}$ and on trees are random walks in a random environment (RWRE), by which we mean they have the distribution of a mixture of nearest neighbor Markov processes where the transition probabilities $p_{i,i+1}, i \in \mathbb{Z}$, are independent (but not necessarily identically distributed). Werner walk is not a RWRE. Suppose however, that we start a recurrent Werner walk X on $\mathbb{Z}$ at 0. Then using the results of Section 2, we easily get that $\lim_{t\to\infty} \frac{w_{i+1}^t}{w_i^t}$ exists and is positive and finite which immediately implies $\lim_{t\to\infty} \frac{w_i^t}{w_0^t} := \overline{w}_i$ is positive and finite. We call the random Markov process with weights equal to $\overline{w}_i$ the limiting Werner walk, a reasonable terminology since, if $\mathcal{G}_k = \sigma(X_0, X_1, \dots, X_k)$ then if τ_i is the time of the ith jump of the associated VRJP,

$$\mathbb{P}(X_{k+1} = j+1 | X_k = j, \mathcal{G}_k) = \frac{w_{j+1}^{\tau_k}}{w_{j-1}^{\tau_k} + w_{j+1}^{\tau_k}} = \frac{\frac{w_{j+1}^{\tau_k}}{w_0^{\tau_k}}}{\frac{w_{j-1}^{\tau_k}}{w_0^{\tau_k}} + \frac{w_{j+1}^{\tau_k}}{w_0^{\tau_k}}}.$$

Thus, this limiting walk is a mixture of Markov chains, and while the transition probabilities

$$p_{j,j+1} := \frac{\overline{w}_{j+1}}{\overline{w}_{j+1} + \overline{w}_{j-1}}, \quad j \in \mathbb{Z}$$

are not independent, by the Poisson construction and the fact that

$$p_{j,j+1} = \lim_{t\to\infty} \frac{\frac{w_{j+1}^t}{w_j^t}}{\frac{w_{j-1}^t}{w_j^t} + \frac{w_{j+1}^t}{w_j^t}}$$

we see that $p_{j,j+1}$ depends only on $\Gamma_0^{\langle j-1,j\rangle}$, $\Gamma_0^{\langle j,j-1\rangle}$, $\Gamma_0^{\langle j,j+1\rangle}$, and $\Gamma_0^{\langle j+1,j\rangle}$. Thus $\{p_{2i-1,2i}, i \in \mathbb{Z}\}$ are independent. This is of no help whatsoever in studying Werner walk not known to be recurrent, in contradistinction to the Diaconis walk situation. However it is potentially useful if all initial weights are 1, especially since the distributions of $p_{i,i+1}$ are explicitly derived in [DV02]. We note in passing that when a Diaconis walk is recurrent the (independent) $p_{j,j+1}$, which in this case govern the motion of the process from the beginning, can be recovered as $\lim_{t\to\infty} \frac{B_{j+1}^{\tau_k}}{B_j^{\tau_k} + B_{j+1}^{\tau_k}}$, where $B_i^{\tau_k}$ is the weight of $(i-1,i)$ after the kth jump.

Finally, we discuss some similarities and differences of Werner walk and Diaconis walk on $\mathbb{Z}$ and on the binary tree, in the setting where all initial bond weights or vertex weights equal 1. Diaconis walk on $\mathbb{Z}$ is a mixture of positive recurrent Markov chains while the limiting Werner walk, with the transition probabilities $p_{j,j+1}$ defined above, is also a mixture of positive recurrent Markov chains. This follows from the fact that the limiting weights $\overline{w}_i$ described above are shown in [DV02] to decrease more or less geometrically both as i increases from 0 and as i decreases from 0, together with the fact that an unreinforced Markov process on $\mathbb{Z}$ described by positive vertex weights w_i, $i \in \mathbb{Z}$, is positive recurrent if and only if

$$\sum_{i=-\infty}^{\infty} w_i w_{i+1} < \infty.$$

The proof is easy, and we omit it.

On the binary tree (or more generally any b-ary tree, $b \geq 2$), both Werner walk and Diaconis walk are transient for small δ and recurrent for large δ ([DV04] and [Pem88]). But Pemantle's Diaconis walk result is much stronger: an explicit cutoff is provided. Davis and Volkov could not even prove that there are not $0 < \delta_1 < \delta_2 < \delta_3$ such that δ_1-Werner walk and δ_3-Werner walk are recurrent and δ_2-Werner walk is transient. Furthermore, below Pemantle's cutoff Diaconis walk is a mixture of positive recurrent Markov processes. It is probably true that for large enough δ the limiting Werner walk is also such a mixture. Also, on the b-ary tree, if δ is small enough (depending on b), Collevecchio has shown in [Col06a], [Col09] that both Diaconis walk and Werner walk approach ∞ at a limiting constant speed. In addition, Collevecchio shows that a central limit theorem holds for both processes for small enough δ. We remark that in [Col06b] Collevecchio employed techniques used in [Col06a] and [Col09] to very good effect in the study of (ordinary) random walk on Galton–Watson trees.

Remark. After this paper was submitted, the preprint *Continuous time vertex reinforced jump processes on Galton–Watson Trees* by Anne-Laure Basdevant and Arvind Singh came to our attention. One of the results of their paper gives for VRJP the analog of the Pemantle result for Diaconis walk on trees which was mentioned just above in the last paragraph of this paper, by finding an explicit cutoff for recurrence/transience.

References

[Col06a] A. Collevecchio, *Limit theorems for reinforced random walks on certain trees*, Probab. Theory Related Fields **136** (2006), 81–101. MR 2240783

[Col06b] A. Collevecchio, *On the transience of processes defined on Galton–Watson trees*, Ann. Probab. **34** (2006), 870–878. MR 2243872

[Col09] A. Collevecchio, *Limit theorems for vertex-reinforced jump processes on regular trees*, Electron. J. Probab. **14** (2009), 1936–1962. MR 2540854

[CD87] D. Coppersmith and P. Diaconis, *Random walks with reinforcement*, unpublished manuscript, 1987.

[DV02] B. Davis and S. Volkov, *Continuous time vertex-reinforced jump processes*, Probab. Theory Related Fields **123** (2002), 281–300. MR 1900324

[DV04] B. Davis and S. Volkov, *Vertex-reinforced jump processes on trees and finite graphs*, Probab. Theory Related Fields **128** (2004), 42–62. MR 2027294

[HPS72] P. Hoel, S. Port, and C. Stone, *Introduction to stochastic processes*, Houghton Mifflin, Boston, 1972. MR 0358879

[Pem88] R. Pemantle, *Phase transition in reinforced random walk and RWRE on trees*, Ann. Probab. **16** (1988), 1229–1241. MR 0942765

[Pem92] R. Pemantle, *Vertex-reinforced random walk*, Probab. Theory Related Fields **92** (1992), 117–136. MR 1156453

[PV99] R. Pemantle and S. Volkov, *Vertex-reinforced random walk on Z has finite range*, Ann. Probab. **27** (1999), 1368–1388. MR 1733153

[Tak00] M. Takeshima, *Behavior of 1-dimensional reinforced random walk*, Osaka J. Math. **37** (2000), 355–372. MR 1772837

[Tar04] P. Tarrès, *Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points*, Ann. Probab. **32** (2004), 2650–2701. MR 2078554

Burgess Davis, Departments of Mathematics and Statistics, Purdue University, 150 N. University St., West Lafayette, IN 47907, USA

E-mail address: bdavis@purdue.edu

Noah Dean, University of Maine at Machias, 116 O'Brien Avenue, Machias, ME 04654, USA

E-mail address: noah.dean@maine.edu