ON THE CONCEPT OF TORSION AND DIVISIBILITY FOR GENERAL RINGS

BY

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1. Introduction

In an attempt to generalize the concept of torsion for general rings, a right R-module M is called a torsion module if $\mathrm{Hom}_R(M, Q) = 0$, where Q is Utumi's ring of right quotients of R. Dually, M is called reduced if $\mathrm{Hom}_R(Q, M) = 0$. A module is then called torsion-free if no nonzero submodule is torsion. Dually a module is called divisible if no nonzero factor module is reduced. Many of the usual theorems are shown to be preserved in §2 and §4, and the new definitions are compared with earlier ones by L. Levy, A. Hattori, E. Matlis, and others in §3 and §5. The new definition of torsion does indeed generalize the classical one for integral domains. The same is true for divisibility, provided the integral domain R has a quotient field Q such that $\mathrm{Hom}_R(Q, D) = 0$ for every classically divisible R-module D. The question of when this condition holds is only partially answered.

I shall recall Utumi's ring of right quotients of R. Let I be the minimal injective extension of the right R-module R associated with the ring R, and let $H = \mathrm{Hom}_R(I, I)$ be the ring of endomorphisms of I_R. We write these endomorphisms on the left of their arguments and obtain a bimodule ${}_HI_R$. Again, let $Q = \mathrm{Hom}_H(I, I)$ be the ring of endomorphisms of the left H-module ${}_H\mathrm{I}$. Q is called Utumi's ring of right quotients of R.

In this paper, the letter Q always means, unless the contrary is stated, the Utumi's ring of right quotients of the ring R.

2. A generalized concept of torsion modules and torsion-free modules

Let R be an associative ring with unity, and M, a right R-module on which 1 acts as the identity. Let Q_R be Utumi's maximal ring of right quotients of R (See [9]). In the following, a module means a right R-module unless otherwise stated. We also omit the subscript R or the superscript R in Hom, $\otimes$.

DEFINITION. A module M is a torsion module if and only if

$$\mathrm{Hom}(M, Q_R) = 0.$$

DEFINITION. A module M is torsion-free if and only if no non-zero submodule of M is a torsion module.

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It is not difficult to verify the following remarks.

Remark 1. A factor module of a torsion module is a torsion module.

Remark 2. If Q_R is injective, then any submodule of a torsion module is a torsion module.

Remark 3. If $A = \oplus \sum A_j$ where A_j's are all submodules of a given module, then A is a torsion module iff each A_j is a torsion module.

Remark 4. A submodule of a torsion-free module is torsion-free.

Remark 5. Q is a torsion-free module.

PROPOSITION 1. *An extension of a torsion module N by a torsion module M/N is a torsion module M.*

Proof. Since $0 \to N \to M \to M/N \to 0$ is an exact sequence, it induces an exact sequence

$$0 \to \mathrm{Hom}(M/N, Q) \to \mathrm{Hom}(M, Q) \to \mathrm{Hom}(N, Q).$$

Now $\mathrm{Hom}(M/N, Q) = 0$ and $\mathrm{Hom}(N, Q) = 0$, therefore $\mathrm{Hom}(M, Q) = 0$. That is, M is a torsion module.

PROPOSITION 2. *If M/L is a torsion-free module and N is a torsion submodule of M, then $N \subset L$.*

Proof. Assume $L + N \neq L$, then by remark 1, we have that

$$0 \neq (L + N)/L = N/(N \cap L)$$

is a torsion submodule of M/L. This contradicts the fact that M/L is an torsion-free module. Hence $N \subset L$.

PROPOSITION 3. *Let $T(M)$ = sum of all torsion submodules of M, then $T(M)$ is the largest torsion submodule of M. Moreover $M/T(M)$ is the "largest' torsion-free factor module of M. (We define $M/N_1 \geqq M/N_2$ if $N_1 \subset N_2$.)*

Proof. Let S be the direct sum of all torsion submodules S_i of M, then by Remark 3, S is a torsion module. Now $T(M)$ is a homomorphic image of S, hence $T(M)$ is a torsion submodule of M. Clearly, it is the largest torsion submodule of M. Assume that $M/T(M)$ is not a torsion-free factor module of M, then there exists a non-zero torsion submodule $K/T(M)$ of $M/T(M)$, such that $T(M) \subset_{\neq} K \subset M$. Now, since $T(M)$ is a torsion module and $K/T(M)$ is also a torsion module, by proposition 1, K is a torsion module. This contradicts the fact that $T(M)$ is the largest torsion submodule of M. Therefore $M/T(M)$ is torsion-free.

Assume M/L is a torsion-free factor module of M, we know $T(M)$ is a torsion submodule of M. By Proposition 2, then $T(M) \subset$ L. That is $M/T(M) \geqq M/L$ and therefore $M/T(M)$ is the "largest" torsion-free factor module of M.

PROPOSITION 4. *M is a torsion module if and only if no non-zero factor module of M is torsion-free.*

Proof. Let M be a torsion module and M/L be a torsion-free factor module of M, by Proposition 2, $M \subset L$, that is $M/L = 0$. Conversely, assume M is not a torsion module; then $0 \subset T(M) \subset_{\neq} M$ where $T(M)$ is the sum of all torsion submodules of M. By Proposition 3., $M/T(M) \neq 0$ is a torsion-free factor module of M.

PROPOSITION 5. *A projective module is a torsion-free module.*

LEMMA. *If $A = \oplus \sum A_i$ where A_i's are all submodules of a given module, then A is torsion-free if and only if each A_i is torsion-free.*

Proof. Assume A_i is not a torsion-free module, for some i, then there exists a non-zero torsion submodule B_i of A_i. Clearly, B_i is also a non-zero torsion submodule of A and hence A is not torsion-free. Conversely, let B be any torsion submodule of A. Let π_i be a canonical epimorphism from $\otimes \sum A_i$ to A_i, for each i. Denote $\pi_i B$ by B_i. Then $B_i \subset A_i$ and

$$B \xrightarrow{\pi_i} B_i \rightarrow 0$$

is exact. This induces an exact sequence $0 \rightarrow \mathrm{Hom}(B_i, Q) \rightarrow \mathrm{Hom}(B, Q)$. Since $\mathrm{Hom}(B, Q) = 0$, we have $\mathrm{Hom}(B_i, Q) = 0$ for each i. Now A_i is torsion-free, for each i. Therefore $B_i = 0$ for each i. Hence $B = \sum k_i \pi_i B = \sum k_i 0 = 0$ where k_i is the canonical injection from A_i into $\oplus \sum A_i$. Thus A is a torsion-free module. The proof of Proposition 5 follows the lemma and Remark 4 immediately.

DEFINITION. A module N extending M is called an "essential extension" provided every non-zero submodule of N has non-zero intersection with M. (In other words, M is a "large submodule" in N.)

PROPOSITION 6. *If Q_R is injective, then an essential extension of a torsion-free module is torsion-free.*

Proof. Let C be an essential extension of a torsion-free module A. Assume C has a non-zero torsion submodule K, so that we have $K \cap C = K \neq 0$. Since C is a large module in A, hence $K \cap A \neq 0$. Now if Q_R is injective, by Remark 2, we have $K \cap A$ is a torsion submodule of A, a contradiction to A being torsion-free. Therefore C is torsion-free.

PROPOSITION 7. *If Q_R is injective, then a module M is torsion-free if and only if M is a subdirect product of submodules of Q_R.*

Proof. First assume M is torsion-free. Let ϕ_α range over all non-zero elements of $\mathrm{Hom}(M, Q)$. If $\pi_\alpha : \prod \phi_\alpha M \rightarrow \phi_\alpha M$ is the canonical epimorphism, there exists a unique homomorphism $k : M \rightarrow \prod \phi_\alpha M$ such that $\pi_\alpha k = \phi_\alpha$.

$$\begin{tikzcd} & \Pi\phi_\alpha M \arrow[dr, "\pi_\alpha"] & \\ M \arrow[ur, dashed, "k"] \arrow[rr, "\phi_\alpha"'] & & \phi_\alpha M \end{tikzcd}$$

The kernel K of k is clearly given by

$$K = \bigcap_{\phi_\alpha \in \mathrm{Hom}(M,Q)} \mathrm{Ker}\, \phi_\alpha .$$

I claim that $K = 0$.

Assume $K \neq 0$, then $0 \neq f_\alpha \in \mathrm{Hom}(K, Q)$ which may be extended to $\phi_\alpha \in \mathrm{Hom}(M, Q)$. But then $K \subset \mathrm{Ker}\phi_\alpha$, hence $f_\alpha K = \phi_\alpha K = 0$, a contradiction to $f_\alpha \neq 0$. Thus $K = 0$ and hence k is a monomorphism. Therefore M is a subdirect product of submodules of Q_R.

Conversely, suppose M is a non-zero submodule of the product $P = \prod_{i \in I} A_i$ where $\{A_i \mid i \in I\}$ is the family of all submodules of Q_R. Let T be any torsion submodule of M, then the mapping $T \to M \to P \to A_i$ must be 0, for all $i \in I$. Using the universal property of the direct product, we deduce that the monomorphism $T \to M \to P$ is 0, hence $T = 0$. Thus M is torsion-free.

COROLLARY. *If Q_R is injective, then every torsion-free module M admits a monomorphism into a direct product of copies of Q_R.*

PROPOSITION 8. $\mathrm{Hom}(A, C) = 0$ *for all torsion modules A if and only if C is torsion-free.*

Proof. For any $f \in \mathrm{Hom}(A, C)$, fA is a torsion submodule of C. Since C is torsion-free, hence fA is also torsion-free. Therefore $fA = 0$. Thus $f = 0$. Conversely, let K be any torsion submodule of C, then by assumption, there exists an inclusion map $i_K \in \mathrm{Hom}(K, C) = 0$ and hence $K = 0$. Therefore C is torsion-free.

PROPOSITION 9. *Let M be a right S module where $R \subset S \subset Q$ and S is any ring of right quotients of R; then M_S is a torsion module if and only if M_R is torsion.*

LEMMA. *If Q is Utumi's maximal ring of right quotients of R, then for any $q \in Q$, $qD = 0$ for some dense right ideal D implies $q = 0$.*

Proof. By [3], $Q \cong \bigcup_D \mathrm{Hom}\,(D, R)/\theta$ where D ranges over all dense right ideals of R and θ is the equivalence relation that holds between $f \in \mathrm{Hom}\,(D, R)$ and $f' \in \mathrm{Hom}\,(D', R)$ if and only if $(f - f')(D \cap D') = 0$. Now, let $q = \theta f$ with $f \in \mathrm{Hom}\,(D, R)$. If $fD = 0$ then $q = \theta f = 0$.

Proof of Proposition 9. If M_R is a torsion module, then $\mathrm{Hom}_R\,(M, Q) = 0$. We have $\mathrm{Hom}_S\,(M, Q) \subset \mathrm{Hom}_R\,(M, Q) = 0$. That is $\mathrm{Hom}_S\,(M, Q) = 0$. Therefore M_S is torsion. Conversely, for any $f \in \mathrm{Hom}_R\,(M, Q)$ and $m \in M$

we have $f(m) \in Q$. Let $q' = f(ms) - f(m)s$. Since $S \subset Q$, then

$$D = s^{-1}R = \{r \in R \mid sr \in R\}$$

forms a dense right ideal in R. Thus $q'd = f(ms)d - f(m)sd = 0$, for all $d \in D$. Hence, by the previous lemma, we have $q' = 0$ and so $f(ms) = f(m)s$. Hence $\mathrm{Hom}_R(M, Q) \subset \mathrm{Hom}_S(M, Q) = 0$ and therefore M_S is torsion if and only if M_R is torsion.

A module N_R is a rational extension of M_R if

$$(\forall n \neq 0 \in N)(\forall n' \in N)(\exists r \in R)(nr \neq 0 \quad \text{and} \quad n'r \in M)$$

or equivalently

$$\mathrm{Hom}_K(K/M, N) = 0 \quad \text{for all } K$$

such that $M \subseteq K \subseteq N$.

A module C_R is called rationally complete if it has no proper rational extension.

PROPOSITION 10. *Let Q_R be injective. If M_R is rationally complete and torsion-free, then M is an S-module where S is any ring of right quotients of R, $R \subset S \subset Q$. Moreover, M_S is torsion-free.*

Proof. First, we want to prove that Q/R is a torsion module. Let π be an epimorphism from Q to Q/R. Let $D = q^{-1}R = \{r \in R \mid qr \in R\}$, then D forms a dense right ideal in R. Hence $qD \subset R$ and for any $f \in \mathrm{Hom}(Q/R, Q)$, $f(\pi q)D = f\pi(qD) \subset f\pi(R) = f(0) = 0$ for any $\pi q \in Q/R$. Since $f(\pi q) \in Q$, hence $f(\pi q) = 0$ for any $\pi q \in Q/R$. Then $\mathrm{Hom}(Q/R, Q) = 0$. That is Q/R is a torsion module. Since Q is injective, then by Remark 2, S/R is a torsion submodule of Q/R. By Proposition 8, since M is torsion-free, we have $\mathrm{Hom}(S/R, M) = 0$. Similarly, we also have $\mathrm{Hom}_R(E, M) = 0$, for any submodule E of S/R. Therefore $0 \leqq S/R(M)$ where $R \subset S \subset Q$ (See [3]). Since M is rationally complete, every homomorphism from R to M can be extended to a homomorphism from S to M. Let $0 \neq \hat{m} \in \mathrm{Hom}(R, M)$ such that $\hat{m}r = mr$. This can be extended to $\tilde{m} \in \mathrm{Hom}(S, M)$ so that $\tilde{m}s = ms$. Hence M is an S-module. By Proposition 9, since M_R is torsion-free then M_S is also torsion-free.

PROPOSITION 11. *A simple module S is torsion-free if and only if*

$$0 \to S \xrightarrow{f} Q$$

is exact, for some f.

Proof. For any $f \in \mathrm{Hom}(S, Q)$, since S is simple, then either $\mathrm{Ker}\, f = 0$ or $\mathrm{Ker}\, f = S$. Assume that $\mathrm{Ker}\, f = S$, for all f, then we have $fS = 0$ in Q, that is S is a torsion R-module, a contradiction to S is torsion-free, so that $\mathrm{Ker}\, f = 0$, for some f. Therefore

$$0 \to S \xrightarrow{f} Q$$

is exact for some f. Conversely, assume that N is any torsion submodule of S. Since S is simple then either $N = 0$ or $N = S$. Assume $N = S$, then for any $f \in \mathrm{Hom}(S, Q) = 0$, we have $\mathrm{Ker}\, f = S$. This contradicts the fact that

$$0 \to S \xrightarrow{f} Q$$

is exact for some f. Hence S is torsion-free.

PROPOSITION 12. *M_R is a torsion module if and only if $M \otimes {}_R Q$ is a torsion module.*

LEMMA. $\mathrm{Hom}_R(Q_R, Q_R) = Q_R = \mathrm{Hom}_Q(Q_Q, Q_Q)$.

Proof. If we consider $\mathrm{Hom}_Q(Q, Q)$ to be an abelian group, then we know that $\mathrm{Hom}_Q(Q, Q) = Q$. Now $\mathrm{Hom}_Q(Q_Q, Q_Q)$ forms a right R-module Q_R. Hence $\mathrm{Hom}_Q({}_R Q_Q, Q_Q) = Q_R$, Clearly, $\mathrm{Hom}_Q({}_R Q_Q, Q_Q) \subset \mathrm{Hom}_R(Q_R, Q_R)$. Conversely, I want to show that $\mathrm{Hom}_R(Q_R, Q_R) \subset \mathrm{Hom}_Q({}_R Q_Q, Q_Q)$. For any $f \in \mathrm{Hom}_R(Q_R, Q_R)$, let $q'' = f(qq') - f(q)q'$ where q, q', q'' all belong to Q. Let $D = q'^{-1}R = \{r \in R \mid q'r \in R\}$. This is a dense right ideal in R. Then for any $d \in D$,

$$q''d = [f(qq') - f(q)q']\cdot d = f(qq'd) - f(qq'\,d) = 0.$$

Hence by the lemma used in Proposition 9, we have $q'' = 0$, that is $f(qq') = f(q)q'$. Hence $\mathrm{Hom}_R(Q_R, Q_R) \subset \mathrm{Hom}_Q({}_R Q_Q, Q_Q)$. Therefore this lemma holds.

Proof of Proposition 12. Consider M_R, ${}_R Q_R$ and Q_R; we have

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(Q, Q)) = \mathrm{Hom}_R(M \otimes {}_R Q, Q). \tag{1}$$

By lemma, we have

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(Q, Q)) = \mathrm{Hom}_R(M, Q) \tag{2}$$

Combining (1) and (2), we have $\mathrm{Hom}_R(M \otimes {}_R Q, Q) = \mathrm{Hom}_R(M, Q)$. Therefore M is a torsion module if and only if $M \otimes {}_R Q$ is a torsion module.

3. A comparison of the definitions of torsion modules

Let R be an integral domain. We say that M is a torsion module in the classical sense [1], provided that

$$M = tM = \{m \in M \mid \exists\ 0 \neq s \in R \text{ such that } ms = 0\}.$$

THEOREM 1. *M is a torsion module if and only if M is a torsion module in the classical sense.*

Proof. Assume M is a torsion module in the classical sense, then for any $m \in M$, there exists $0 \neq s \in R$ such that $ms = 0$. For any $f \in \mathrm{Hom}(M, Q)$ and for any $m \in M$, we have $f(ms) = f(0) = f(m)s = 0$ in Q. Since Q is a field, $0 \neq s$ implies $f(m) \neq 0$. That is $f = 0$ for all m. Hence

Hom $(M, Q) = 0$. Therefore M is a torsion module. Conversely, assume that M is not a torsion module in the classical sense, then $tM \subsetneq M$ is a proper torsion submodule of M. Hence there exists $m \in M$, such that $ms \neq 0$ for all $0 \neq s \in R$. Equivalently, there exists $m \in M$ such that for any $s \in R(ms = 0$ implies $s = 0)$. Define a homomorphism f from mR into Q, such that $f(ms) = s$, for any $s \in R$. Since Q_R is injective, $0 \neq f$ can be extended to $0 \neq \phi \in$ Hom (M, Q), because $mR \subset M$, so Hom $(M, Q) \neq 0$. That is M is not a torsion module.

Let R be a ring with unity. Denote the semi-group of regular elements (i.e. non-zero divisors) of R by $R^{\#}$. We call an element m of an R-module M a torsion element if there exists $s \in R$ such that $ms = 0$. We denote the set of all torsion elements of M by $t\,M$.L. Levy [4] has shown that tM forms a submodule of M if and only if R has a classical ring of right quotients, equivalently, R satisfies Ore's condition.

THEOREM 2. *Assume R_R is finite-dimensional,*[2] *if every large right ideal in R has a regular element, then M is a torsion module if and only if M is a torsion module in Levy's sense.*

Proof. See [2]. If R_R is finite-dimensional and every large right ideal has a regular element; then $J(R_R) = 0$, Q_R is injective, every regular element in R is a regular element in Q and $Q = Q(Cl)$ (The classical rings of right quotients of R). For any $f \in \mathrm{Hom}_R\,(M, R)$ and for any $m \in M$, there exists $s \in R$ such that $ms = 0$, then $f(m)s = f(ms) = f(0) = 0$. Since s is regular in Q, hence $f(m) = 0$ for all m. Therefore $f = 0$, that is Hom $(M, Q) = 0$. Conversely, assume that M is a torsion module. For any $m \in M$, construct an annihilator ideal A of m such that $A = \{r \in R \mid mr = 0\}$. We want to show that A is a large right ideal in R. For, let B be any right ideal of R such that $A \cap B = 0$. Define $f : mB \to Q$ such that $f(mb) = b$. We have that $mb = 0$ implies $b \in A \cap B = 0$. Hence f is a mapping. Clearly, f is a homomorphism from mB into Q. Since $mB \subset M$ and Q is injective, we have that Hom $(M, Q) \to$ Hom $(mB, Q) \to 0$ is exact. By assumption Hom $(M, Q) = 0$ and hence Hom $(mB, Q) = 0$, that is $f(mb) = b = 0$ for all b. Hence $B = 0$. Therefore A is a large right ideal in R. Hence A has a regular element $s \in R$. That is $ms = 0$ for some $s \in R^{\#}$. Thus $m \in tM$. Therefore, M is a torsion module in Levy's sense.

Let R be a ring with unity. R. E. Johnson and E. T. Wong [7] have defined the singular submodule $J(M_R)$ of an R-module M such that

$$J(M_R) = \{m \in M \mid mD = 0 \quad \text{for some large right ideal} \quad D \quad \text{of} \quad R\}.$$

Recall that $T(M)$ is the largest torsion submodule of M.

THEOREM 3. *If $J(R_R) = 0$ then $T(M) = J(M_R)$.*

Proof. For any $m \in T(M) =$ Sum of all torsion submodule M_i of M.

[2] A module M_R is called *finite-dimensional* (Goldie) if there do not exist infinitely many non-zero submodules whose sum is direct.

Hence $m = \sum m_i$ where $m_i \in M_i$ and all but a finite number of m_i are 0. By using the argument in the proof of Theorem 2, we have for any $m_i \in M_i$ that there exists a large right ideal A_i such that $m_i A_i = 0$. Let $L = \bigcap_{i=1}^n A_i$. We know that L is again a large right ideal in R. Moreover $mL = 0$. That is $m \in J(M_R)$ and hence $T(M) \subset J(M_R)$. Conversely, let

$$L^* = \{q \in Q \mid qL = 0 \quad \text{where} \quad L \quad \text{is a large right ideal in} \quad R\}.$$

Now, $L^* \cap R = \{r \in R \mid rL = 0 \text{ where } L \text{ is a large right ideal in } R\} = J(R_R) = 0$, Since R is a large module in $Q, L^* \cap Q = L^* = 0$. For any $f \in \operatorname{Hom}(J(M_R), Q)$ and for any $m \in M$, and any $t \in L$ we have $f(m)t = f(mt) = f(0) = 0$. Hence $f(m) \in L^* = 0$. Therefore $\operatorname{Hom}(J(M_R), Q) = 0$ and $J(M)_R$ is a torsion submodule of M. Hence $J(M_R) \subset T(M)$.

THEOREM 4. *Let R be a non-commutative integral domain, and assume that R_R is finite dimensional. Then M is a torsion module if and only if M is a torsion module in Hattori's sense.*

Proof. See [5, Proposition 18]. Since R has a classical ring of right quotients $Q(Cl)$, then M is a torsion module in Levy's sense if and only if M is *a* torsion module in Hattori's sense. Also from Theorem 2, since every non-zero element in R is a regular element in R, M is a torsion module in Levy's sense if and only if M is a torsion module.

4. A generalized concept of reduced modules and divisible modules

Let R be an associative ring with unity, which acts as the identity in any right R module M. Let Q be Utumi's maximal ring of right quotients of R. We say that a module M is reduced if and only if $\operatorname{Hom}_R(Q, M) = 0$. We define a module M to be divisible if and only if no non-zero factor module of M is reduced.

It is not difficult to verify the following remarks:

Remark 1^*. Any submodule of a reduced module is reduced.

Remark 2^*. If Q_R is projective, then any factor module of a reduced module is a reduced module.

Remark 3^*. If $C = \prod C_i =$ the direct product of modules C_i, then C is a reduced module if and only if each C_i is reduced.

Remark 4^*. Any factor module of a divisible module is a divisible module.

Remark 5^*. Q_R is a divisible module.

PROPOSITION 1^*. *An extension of a reduced module N by a reduced module M/N is als a reduced module.*

Proof. Since $0 \to N \to M \to M/N \to 0$ is an exact sequence, it induces an exact sequence

$$0 \to \operatorname{Hom}(Q, N) \to \operatorname{Hom}(Q, M) \to \operatorname{Hom}(Q, M/N) \to \cdots$$

By assumption, we have Hom $(Q, N) = 0$ and Hom $(Q, M/N) = 0$. Hence Hom $(Q, M) = 0$, namely M is reduced.

PROPOSITION 2^*. *If L is a divisible submodule of M and M/N is a reduced module then $L \subset N$.*

Proof. Assume $L \cap N \neq L$, then by Remark 1^*, $0 \neq L/L \cap N = (L + N)/N$ is a reduced submodule of M/N. Hence $L/L \cap N$ is a non-zero reduced factor module of L which contradicts that L is divisible. Hence $L \subset N$.

LEMMA. *If $D = \bigcap_i \{S_i \subset M \mid M/S_i$ is a reduced factor module of $M\}$, then there exists a monomorphism from M/D into $\prod_i M/S_i$.*

Proof. Let $\pi_i : \prod_i M/S_i \to M/S_i$ and $\phi_i : M \to M/S_i$ canonically. Then there exists a unique $\phi : M \to \prod_i M/S_i$ such that $\pi_i \phi = \phi_i$.

$$\begin{tikzcd} & \Pi_i M/S_i \arrow[d, "\pi_i"] \\ M \arrow[ur, dotted, "\phi"] \arrow[dr, "\phi_i"'] & \\ & M/S_i \end{tikzcd}$$

We have

$$\begin{aligned} \operatorname{Ker} \phi &= \{m \in M \mid \phi(m) = 0\} \\ &= \{m \in M \mid (\forall_i)(\pi_i \phi(m) = 0)\} \\ &= \{m \in M \quad (\forall_i)(\phi_i m = 0)\} \\ &= \textstyle\bigcap_i \ \operatorname{Ker} \ \phi_i = D. \end{aligned}$$

We want to construct a monomorphism from M/D into $\prod_i M/S_i$. Define $h : M/D \to \prod_i M/S_i$ such that $h(m + D) = \phi m$. If $m + D = 0$ in M/D, then $m \in D$, hence $\phi m = 0$. Therefore h is a mapping. Clearly h is a homomorphism. Moreover Ker $h = \{m + D \mid \phi m = 0\} = 0$ in M/D. Hence h is a monomorphism.

PROPOSITION 3^*. *Let D be the intersection of all submodules S_i of M such that M/S_i is a reduced module; then M/D is a reduced module and is the "largest" reduced factor module of M. (We define $M/N_1 \geqq M/N_2$ if $N_1 \subset N_2$.) Also D is the largest divisible submodule of M.*

Proof. By Remark 3^*, M/S_i is a reduced module for each i which implies $\prod_i M/S_i$ is reduced. By the lemma, M/D is a submodule of $\prod_i M/S_i$. Hence by Remark 1^*, M/D is reduced. Since $D \subset S_i$, for each reduced factor module M/S_i, and according to our definition of "$\geqq$" (greater than), then $M/D \geqq M/S_i$ for each reduced factor module of M/S_i. Hence M/D is the "largest" reduced factor module of M. Next, we want to show that D

is divisible. Assume $0 \neq D/K$ to be a non-zero reduced factor module of D. By the above, we have shown $M/D = M/K/D/K$ is a reduced factor module of M. Since D/K and $M/K/D/K$ are reduced modules, by Proposition 1^*, M/K is a reduced factor module of M. Since M/D is the "largest" reduced factor module of M, we have $M/K \leqq M/D$ from which we can deduce $D \subset K$. This contradicts $K \subset_{\neq} D$. Hence D is a divisible module. We have shown before that M/D is a reduced factor module of M, and for any divisible submodule L of M, by Proposition 2^*, $L \subset D$. Hence D is the largest divisible submodule of M.

PROPOSITION 4^*. *M is a reduced module if and only if no non-zero submodule of M is a divisible module.*

Proof. Let L be any divisible submodule of M. Since $M = M/0$ is a reduced factor module of M, by Proposition 2^*, we have $L \subset 0$. Hence $L = 0$. Conversely, assume M is not reduced. Let $D = \bigcap_i \{S_i \mid M/S_i$ is a reduced factor module of $M\}$. Clearly, $0 \neq D \subset M$ because $M/0$ is not reduced. By Proposition 3^*, we have that D is a nonzero divisible submodule of M.

COROLLARY. *If M is a non-zero reduced module then M is not a divisible module*

PROPOSITION 5^*. *Any injective module is a divisible module.*

Proof. Let M be an injective module. Assume M is not divisible, then there exists a submodule $N \subset_{\neq} M$ such that Hom $(Q, M/N) = 0$. For any $\psi \in$ Hom (Q, M) we have $\pi \circ \psi \in$ Gom $(Q, M/N)$ where π is an epimorphism from M to M/N. Since $\pi \psi Q = 0$ in M/N, then ψ Q $\subset N$ for any $\psi \in$ Hom (Q, M). Let ϕ be the homomorphism from R into M such that $\phi : r \longrightarrow mr$ for some arbitrarily chosen $m \in M$. Since M is injective, ϕ can be extended to a homomorphism ρ from Q into M such that $\phi = \rho \cdot k$ where k is a monomorphism from R into Q. Thus $mR = \phi R = \rho k R \subset \rho Q \subset N$. Since m is arbitrary, we have $M \subset N$, which is a contradiction of $N \subset_{\neq} M$. Therefore M is divisible.

DEFINITION. A module E is called an essential cover of $M = E/K$ if and only if any submodule $F \subset E$ such that $F \neq E$ implies $F + K \neq E$.

PROPOSITION 6^*. *If Q is projective, an essential cover E of a divisible module E/K is a divisible module.*

Proof. Assume E is not divisible; then there exists a non-zero reduced factor module E/L of E. Since E is an essential cover of E/K. We have $L \subset_{\neq} E$ implies $L + K \subset_{\neq} E$. Now Q_R is projective so, by Remark 2^*, $E/L/L + K/L = E/L + K \neq 0$ is a reduced factor module of E/K. Hence, E/K is not a divisible module which contradicts the fact that E/K is divisible. Thus E is divisible.

PROPOSITION 7*. *If Q_R is projective, then a module M is divisible if and only if M is a sum of factor modules of Q_R.*

Proof. Assume that M is divisible. Let $S = \sum_{\phi_i \epsilon \mathrm{Hom}(Q,M)} \mathrm{im}_{\phi_i}$; I claim $S = M$. Assume $S \neq M$, then M/S is not a reduced factor module of M. Hence there exists $0 \neq f_i \epsilon \mathrm{Hom}\ (Q, M/S)$. By assumption Q is projective, hence f_i can be lifted to $0 \neq \phi_i \epsilon \mathrm{Hom}\ (Q, M)$, hence $\mathrm{im}\ \phi_i \not\subset S = \mathrm{Ker}\ \pi$, where $\pi : M \rightarrow M/S$. This is a contradiction to $\mathrm{im}\ \phi_i \subset S$. Therefore $S = M$.

Conversely, assume $M = \sum_{\phi_i \epsilon \mathrm{Hom}(Q,M)} \mathrm{im}\ \phi_i$, I claim M is divisible.
Let H be any reduced factor module of M, then the mapping

$$H \leftarrow M \leftarrow \oplus \sum\nolimits_{\phi_i \epsilon \mathrm{Hom}(Q,M)} \mathrm{im}\ \phi_i \leftarrow \mathrm{im}\ \phi_i$$

must be 0 for all $i \epsilon I$. Using the universal property of the direct sum, we deduce that the epimorphism

$$H \leftarrow M \leftarrow \oplus \sum\nolimits_{\phi_i \epsilon \mathrm{Hom}(Q,M)} \mathrm{im}\ \phi_i$$

is 0, hence $H = 0$. Thus M is divisible.

COROLLARY. *If Q_R is projective then every divisible module is a homomorphic image of a direct sum of copies of Q_R.*

Proof. By Proposition 7* and the fact that the sum of modules A_i where A_i's are all submodules of a given module is a homomorphic image of $\oplus \sum A_i$.

PROPOSITION 8*. *A module D is divisible module if and only if* $\mathrm{Hom}\ (D, U) = 0$ *for all reduced module U.*

Proof. Assume D is divisible, then for any $f \epsilon \mathrm{Hom}\ (D, U)$, where U is reduced, we have fD is a divisible and $fD \subset U$. By Remark 1*, fD is also reduced. Therefore $fD = 0$. Hence $\mathrm{Hom}\ (D, U) = 0$, for all reduced modules U. Conversely, let fD be any reduced factor module of D. By assumption, $\mathrm{Hom}\ (D, fD) = 0$. Thus $fD = 0$. By definition of divisibility, we have D is divisible.

PROPOSITION 9*. *If M is a right Q module, them M_R is a divisible module.*

Proof. Assume M_R is not divisible, then there exists a submodule $N \subset_{\neq} M$ such that $\mathrm{Hom}\ (Q, M/N) = 0$. For any $\psi \epsilon \mathrm{Hom}\ (Q, M)$, we have $\pi \circ \psi \epsilon \mathrm{Hom}\ (Q, M/N)$ where π is an epimorphism from M to M/N. Since $\pi \psi Q = 0$ in M/N, $\psi Q \subset N$ for any $\psi \epsilon \mathrm{Hom}\ (Q, M)$. Define $\phi \epsilon \mathrm{Hom}\ (Q, M)$ such that $\phi : q \rightarrow mq$, for some arbitrarily chosen $m \epsilon M$. Then one can easily see that $\phi \epsilon \mathrm{Hom}_R\ (Q, M)$. Hence $M = MQ \subset N$, a contradiction.

PROPOSITION 10. *If $A = \oplus \sum A_i$ is a direct sum of submodules A_i, A is a divisible module if and only if each A_i is a divisible module.*

Proof. Assume A is divisible. Now the exact sequence

$$0 \to A_i \xrightarrow{k_i} A$$

is direct for each A_i, that is there exists an epimorphism $\pi_i : A \to A_i$ such that $\pi_i k_i = 1_{A_i}$. Since A is divisible, by Remark 4*, A_i is divisible for each i. Conversely, assume each A_i is divisible. We want to show that A is divisible. Let A/B be any reduced factor module of A, and consider the epimorphism $\pi : A \to A/B$. Now we have $\pi k_i : A_i \to A/B$, such that

$$A_i \xrightarrow{k_i} A \xrightarrow{\pi} A/B$$

where k_i is a monomorphism. Let $B_i = \text{Ker } \pi k_i$; then there exists an epimorphism $\phi_i : A_i \to A_i/B_i$ for each i, and $A_i/B_i = \text{im } \phi_i \subset A/B$. Since $0 \to A_i/B_i \to A/B$ is an exact sequence, it induces an exact sequence

$$0 \to \text{Hom}\,(Q, A_i/B_i) \to \text{Hom}\,(Q, A/B).$$

Since A/B is reduced, A_i/B_i is reduced for each i. By assumption, each A_i is divisible. Hence $A_i \subset B_i$ for each i. For any $a \in A = \oplus \sum A_i$, we have $a = \sum k_i a_i$ with $a_i \in A_i$ where all but a finite number of a_i are 0, and the expression is unique. Now $a_i \in B_i = \text{Ker } \pi\ k_i$, hence $\pi k_i(a_i) = \pi(k_i a_i) = 0$ and thus $k_i a_i \in \text{Ker } \pi = B$ for each i. Hence $a = \sum k_i a_i \in B$ from which we can deduce that $A \subset B$, that is $A/B = 0$. Therefore A is divisible.

PROPOSITION 11*. *A simple module S is divisible if and only if*

$$Q \xrightarrow{f} S \to 0$$

is an exact sequence for some f.

Proof. Assume a simple module S is divisible; then for any $f \in \text{Hom}\,(Q, S)$, either $fQ = 0$ or $fQ = S$. Assume $fQ = 0$ for all f. Then $\text{Hom}\,(Q, S) = 0$ which contradicts the assumption that S is divisible. Hence $fQ = S$ for some f. Conversely, assume the sequence

$$Q \xrightarrow{f} S \to 0$$

is exact. For this particular $f \in \text{Hom}\,(Q, S)$, we have $fQ = S$ and, by using Remark 5* and Remark 4*, that S is divisible.

PROPOSITION 12*. *Let R be an integral domain. A module M is a reduced module if and only if* $\text{Hom}_R\,(Q, M)$ *is a reduced R-module.*

LEMMA. $Q \otimes_R Q = Q_R$.

Proof. Since Q_R is flat, (see [1, p. 130]) $Q_R = Q \otimes_R R \subset Q \otimes_R Q$. Conversely, for any $a \otimes b \in Q \otimes_R Q$. We have

$$\begin{aligned} aq \otimes b &= a(r/s) \otimes b = a(r/s) \otimes (s/s)b = a(rs/s) \otimes (b/s) \\ &= ar \otimes (b/s) = a \otimes (rb/s) \\ &= a \otimes qb \quad \text{where} \quad q \in Q \quad \text{and} \quad r, s \in R^{\#} = R - \{0\}. \end{aligned}$$

Hence $Q \otimes {}_R Q = Q \otimes {}_Q Q$.

Proof of Proposition 12^*.

$$\mathrm{Hom}_R\,(Q_R\,, \mathrm{Hom}_R\,(Q, M)) \cong \mathrm{Hom}_R\,(Q \otimes {}_R Q, M) \cong \mathrm{Hom}_R\,(Q, M)$$

hence $\mathrm{Hom}_R\,(Q, \mathrm{Hom}\,(Q, M)) = 0$ if and only if $\mathrm{Hom}\,(Q, M) = 0$. Hence if R is an integral domain then $\mathrm{Hom}_R\,(Q, M)$ is reduced if and only if M is reduced.

PROPOSITION 13. *Given rings R and S, let A_S be a right S module and ${}_SC_R$ be a bimodule. If C is an R-divisible module, then $A \otimes_S C$ is also R-divisible.*

Proof. Since C is an R-divisible module, we have by Proposition 8^*, that for all reduced right R-module T, $\mathrm{Hom}\,(C, T) = 0$ Now,

$\mathrm{Hom}_R\,(A \otimes {}_S C, T) = \mathrm{Hom}_S\,(A, \mathrm{Hom}\,(C, T)) = \mathrm{Hom}_S\,(A, 0) = 0$ for all reduced modules T. By Proposition 8^*, $A \otimes {}_S C$ is an R-divisible module.

5. A comparison of the definition of reduced modules and divisible modules

Let R be an integral domain. We say that a module M over R is divisible in the classical sense if and only if $Md = M$ for any $0 \neq d \in R$. We say that a module M is reduced in the classical sense if and only if M has no non-zero divisible submodules See [8].

THEOREM 1. *If M is a reduced module in the classical sense, then M is a reduced module.*

Proof. For any $f \in \mathrm{Hom}\,(Q, M), fQ \subset M$. Since R is an integral domain, Q is divisible in the classical sense. Hence by [1], fQ is divisible in the classical sense. By the definition of a reduced module M in the classical sense, we have $fQ = 0$. Hence $\mathrm{Hom}\,(Q, M) = 0$ and therefore M is a reduced module.

THEOREM 2. *If R is a Dedekind domain and M is a reduced module, then M is a reduced module in the classical sense.*

Proof. Let N be any divisible (in the classical sense) submodule of M. Assume $N \neq 0$. Define $f : R \rightarrow N$ such that $r \rightarrow nr$ for some arbitrary chosen $n \neq 0 \in N$. It is easy to see that $f \in \mathrm{Hom}_R\,(R, N)$. If R is a Dedekind domain, by [1], N is injective and hence f can be extended to $0 \neq \chi \in \mathrm{Hom}\,(Q, N)$. Now $0 \rightarrow N \rightarrow M$ is exact, and induces $0 \rightarrow \mathrm{Hom}\,(Q, N) \rightarrow \mathrm{Hom}\,(Q, M)$ which is exact. By assumption, M is reduced, thus $\mathrm{Hom}\,(Q, N) = 0$. Let

$q = 1$, then $0 = \chi \cdot 1 = f \cdot 1 = 1 \cdot n$, which is a contradiction to $n \neq 0$. Hence $N = 0$ and therefore M is reduced in the classical sense.

LEMMA. *Let R be an integral domain; then there exists a largest divisible submodule in the classical sense D of M. Moreover M/D is a reduced module.*

Proof. Let M be any module over R. It is easily seen that the sum D of all classically divisible submodules of M is classically divisible. Assume M/D is not a reduced module in the classical sense, then it has a divisible submodule in the classical sense N/D, such that $D \subset N \subset M$. Since N/D is a divisible module in the classical sense, given any $n \in N$, and any $0 \neq r \in R$, there exists $n' \in N$ such that $n'r = n \bmod D$. Thus $n'r - n \in D$. Since D is a divisible module in the classical sense, for $n'r - n \in D$ with $r \in R$, there exists $d' \in D$ such that $n'r - n = d'r$. Thus $(n' + d')r = n$. Therefore, for any n in N, and any $0 \neq r \in R$, there exists $(n' + d') \in N$ such that $(n' + d') \in N$ such that $(n' + d')r = n$. Hence N is a divisible submodule in the classical sense of M. By the maximality of D, $N = D$, that is $N/D = 0$. Hence M/D has no non-zero divisible submodule in the classical sense. Thus M/D is a reduced module in the classical sense. Now by Theorem 1, we have that M/D is a reduced module.

THEOREM 3. *Let R be any integral domain. Any divisible module M is a divisible module in the classical sense.*

Proof. By a proposition in §4, we know that M is a divisible module if and only if $\mathrm{Hom}_R(M, E) = 0$ for all reduced modules E. In particular, let $E = M/D$ as in the lemma, then $\mathrm{Hom}_R(M, M/D) = 0$. Let π be the epimorphisms of M onto M/D, then $\pi \in \mathrm{Hom}_R(M, M/D) = 0$ and hence $\pi M = M/D = 0$. Thus $M = D$. Therefore M is a divisible module in the classical sense.

THEOREM 4. *Let R be an integral domain, Q be the quotient field of R, and assume that Q is countably generated, then every divisible module in the classical sense is a divisible module.*

Proof. Refer to [6]. If R is an integral domain, then Q is divisible in the classical sense and torsion-free. There exists a countable set of generators $\{q_n\}$ for Q over R and elements $\{a_{n+1}\}$ of R such that $q_1 = 1$ and $a_{n+1} q_{n+1} = q_n$. Let

$$A = q_1 R + q_2 R + \cdots + q_{n-1} R$$
$$= R + (1/a_2)R + (1/a_2 a_3)R + \cdots + (1/a_2 a_3 \cdots a_{n-1})R.$$

Clearly $A \subset A + (1/a_2 a_3 \cdots a_n)R = A + q_n R$. We want to show that for any $f \in \mathrm{Hom}_R(A, D)$, f can be extended to $g \in \mathrm{Hom}(A + q_n R, D)$. We define

$$g(a - r/a_2 a_3 \cdots a_n) = f(a) - x_n r$$

for some $x_n \in D$, where x_n is determined as follows:

Pick $x_n \in D$ such that $x_n a_n = f(1/a_2 \cdots a_{n-1})$. This can be done since D is divisible in the classical sense.

We must verify that g is a mapping. Hence suppose

$$a = r/a_2 a_3 \cdots a_n .$$

Then $f(a)a_n = f(r/a_2 a_3 \cdots a_{n-1}) = x_n a_n r$. Since Q is torsion-free in the classical sense, $f(a) = x_n$, as required. It is easily verified that g is an R-homomorphism.

Writing $A = A_{n-1}$, we see that any element of $\mathrm{Hom}_R\ (A_{n-1}, D)$ can be extended to an element of $\mathrm{Hom}_R\ (A_n, D)$.

Since $Q = \bigcup_n A_n$, it follows that any element of $\mathrm{Hom}_R\ (R, D)$ can be extended to an element of $\mathrm{Hom}_R\ (Q, D)$.

Finally we want to show D is a divisible module. Assume that D is not divisible, then there exists a submodule $N \subset_{\neq} D$ such that $\mathrm{Hom}\ (Q, D/N) = 0$. Thus for any $\chi \in \mathrm{Hom}\ (Q, D)$, we have $\pi\chi \in \mathrm{Hom}\ (Q, D/N) = 0$ where π is an epimorphism from D to D/N. That is $\pi\chi Q = 0$ in D/N, so that $\chi Q \subset N$. Now, we define $\phi \in \mathrm{Hom}\ (R, D)$ such that $\phi : r \to dr$ for some arbitrary chosen $0 \neq d \in D$. Hence ϕ can be extended to $\chi \in \mathrm{Hom}\ (Q, D)$. Thus $dR = \phi R = \chi R \subset N$ and so $d \cdot 1 = \chi \cdot 1 \in N$. Since d is arbitrary, $D \subset N$, which is a contradiction to $N \subset_{\neq} D$. Hence D is a divisible module.

THEOREM 5. *Let R be an integral domain with quotient field Q. A module M is h-reduced module in Matlis' sense* [6] *if and only if M is a reduced module.*

Proof. Let $h(M)$ denote the sum of all submodules M_i of M, each of which is a homomorphic image of an injective module I_i. We will show $h(M)$ is the unique largest submodule of M which is a homomorphic image of an injective module.

$$\begin{tikzcd} 0 \arrow[r] & F_i \arrow[r] \arrow[d] & F_i \otimes Q \arrow[dl, dotted] \\ & I_i \arrow[d] & \\ & M_i & \end{tikzcd}$$

Let F_i be a free module mapping onto I_i, then $F_i \otimes Q$ is torsion-free and divisible in the classical sense and is a direct sum of copies Q. Therefore M_i is a homomorphic image of $\oplus \sum Q$. Now, let $V = \oplus \sum (F_i \otimes Q)$, we have an epimorphism $V \to h(M)$. But V is torsion-free and divisible in the classical sense, hence injective. Therefore $h(M)$ is the unique largest submodule of M which is a homomorphic image of an injective module.

Next, I claim the sequence $0 \to \mathrm{Hom}\ (K, M) \to \mathrm{Hom}\ (Q, M) \to h(M) \to 0$ is exact, where $K = Q/R$. The exact sequence $0 \to R \to Q \to K \to 0$ induces an exact sequence

$$0 \to \text{Hom }(K, M) \to \text{Hom }(Q, M) \xrightarrow{\alpha} M$$

where if $f \in \text{Hom }(Q, M)$, $\alpha(f) = f(1)$. $\text{Hom}_R\ (Q, M)$ is torsion-free and divisible in the classical sense, hence injective, and so im $\alpha \subset h(M)$. Conversely, let $x \in h(M)$, then there exists an injective module I and a homomorphism $g : I \to M$ such that $g(y) = x$ for some $y \in I$. Now, we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "h"'] & Q \arrow[dl, dotted, "k"] \\ & I & \end{tikzcd}$$

where $h(1) = y$.

Since I is injective, there exists a map $k : Q \to I$ such that $ki = h$. Let $f = gk$, then $f \in \text{Hom }(Q, M)$ and $\alpha(f) = (gk)(1) = g(k(1)) = g(y) = x$. Therefore $x \in \text{im } \alpha$ and so $h(M) \subset \text{im } \alpha$, so we have that

$$0 \to \text{Hom }(K, M) \to \text{Hom }(Q, M) \to h(M) \to 0$$

is exact.

In view of the above, if M is reduced, that is $\text{Hom }(Q, M) = 0$, then $h(M) = 0$. Therefore M has no non-zero submodule which is a homomorphic image of an injective module. That is, M is h-reduced in Matlis' sense. Conversely, if M is h-reduced in Matlis' sense, then $h(M) = 0$. For any $f \in \text{Hom }(Q, M) = \text{Hom }(Q, h(M))$, $fQ \subset h(M) = 0$. Hence $\text{Hom }(Q, M) = 0$, that is, M is a reduced module.

THEOREM 6. *Let R be a Noetherian integral domain with the property that every non-zero prime ideal is maximal. Then every divisible module in the classical sense is a divisible module.*

Proof. See [6 Theorem 3.3]. Every divisible module in the classical sense D is a homomorphic image of an injective module. Thus $D = h(D) \neq 0$. Therefore D is not h-reduced in Matlis' sense. Thus, by the previous theorem, D is not h-reduced.

Let $0 \neq D/K$ be any factor module of D. Then by [1], D/K is a divisible module in the classical sense. Hence from the above proof, we know D/K is not a reduced module. Hence D has no non-zero reduced factor modules, and so by definition, D is a divisible module.

COROLLARY. *If R is a Dedekind domain, then a divisible module in the classical sense is a divisible module.*

Let R be an associative ring with unity, which acts as the identity in every module, and let Q be Utumi's maximal ring of right quotients of R.

DEFINITION. A left module ${}_R B$ is called a "bad module" if and only if $Q \otimes {}_R B = 0$.

DEFINITION. A right module G_R is called a "good module" if and only if $G \otimes {}_R B = 0$ for any bad module ${}_R B$

LEMMA. *For any bad module B, and for any module X,* $\mathrm{Hom}_Z (B, X)$ *is a reduced module, where Z is the ring of integers.*

Proof. For,

$$\mathrm{Hom}_R (Q, \mathrm{Hom}_Z (B, X)) = \mathrm{Hom}_Z (Q \otimes B, X) = \mathrm{Hom}_Z (0, X) = 0.$$

Hence $\mathrm{Hom}_Z (B, X)$ is a reduced module.

THEOREM 7. *Every divisible module D is a good module.*

Proof. For any bad module B, and for any module X, we have that

$$\mathrm{Hom}_Z (D \otimes B, X) = \mathrm{Hom}_R (D, \mathrm{Hom}_Z (B, X)).$$

From Proposition 8^*, §4, since D is a divisible module and $\mathrm{Hom}_Z (B, X)$ is a reduced module, $\mathrm{Hom}_R (D, \mathrm{Hom}_Z (B, X)) = 0$, that is, $\mathrm{Hom}_Z (D \otimes B, X) = 0$. Now let $X = D \otimes B$. Then we have that $\mathrm{Hom}_Z (D \otimes B, D \otimes B) = 0$. Therefore $D \otimes B = 0$, for any bad module B. That is, D is a good module.

THEOREM 8. *A module B is a bad module if and only if*

$$B^* = \mathrm{Hom}_Z (B, K/Z)$$

is a reduced module, where K is the field of rational numbers.

Proof. If B is a bad module, from the lemma used in proving Theorem 7, $\mathrm{Hom}_Z (B, K/Z)$ is a reduced module. Conversely, if $B^* = \mathrm{Hom}_Z (B, K/Z)$ is a reduced module, since Q is a divisible module then

$$\begin{aligned}(Q \otimes B)^* = \mathrm{Hom}_Z (Q \otimes B, K/Z) &\cong \mathrm{Hom}_R (Q, \mathrm{Hom}_Z (B, K/Z)) \\ &= \mathrm{Hom}_B (Q, B^*) = 0.\end{aligned}$$

From [10], we have $Q \otimes B = 0$. That is, B is a bad module.

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