STRUCTURE AND TERMINALITY OF THE MAXIMAL UNIPOTENT SUBGROUPS OF STEINBERG GROUPS[1]

BY

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In [2] we proved that, with possible exceptions in the case of low rank or characteristic 2, the maximal unipotent subgroups of Chevalley groups are terminal nilpotent groups. In the present paper we shall investigate the structure of the maximal unipotent subgroups of the Steinberg groups in order to show that they also are usually terminal.

Notation in this paper derives largely from Steinberg's original paper [3]. In particular, let U^1 be a maximal unipotent subgroup of a Steinberg group of type A_l^1, D_l^1, or E_6^1, and let π, π^1, S, U, U_S, and U_S^1 be defined as in [3] (with the change from German to Roman U). Let Π^{1+} be the set of all $S \in \Pi^1$ which consist of positive roots.

The proof of terminality in [2] used to good advantage the structure of the system of positive roots of the associated Lie algebra. In this case the Steinberg group has associated with it no sets S of type (3), then the set Π^{1+} can be given the structure of a system of positive roots of a Lie algebra, which proves very useful in the arguments to follow.

Therefore, we shall first consider the cases where there are no sets S of type (3). That is, if the Steinberg group in question is of type A_l^1, we assume l to be odd. Define $S + S'$ to be the set of all roots of the form $r + r'$, where $r \in S$, $r' \in S'$, and $S, S' \in \Pi^{1+}$. We note that either $S + S'$ is empty or $S + S' \in \Pi^{1+}$. Then there is a 1-1 map τ such that:

(i) in A_l^1, l odd, τ maps Π^{1+} onto the set of positive roots of $C_{(l+1)/2}$,

(ii) in D_l^1, τ maps Π^{1+} onto the set of positive roots of B_{l-1},

(iii) in E_6^1, τ maps Π^{1+} onto the set of positive roots of F_4, and in each case $\tau(S + S') = \tau(S) + \tau(S')$ for all $S, S' \in \Pi^{1+}$. We shall call τ a "root system isomorphism." Given $S, S' \in \Pi^{1+}$, there may exist an $S'' \in \Pi^{1+}$ such that $S + S'' = S'$. If such an S'' exists, it is uniquely determined by S and S', and so we shall denote it by $S' - S$.

In U^1 we have the central series defined by $U_m^1 = U^1 \cap U_m$, where U_m is as defined in [1]. From Corollary 4.5 and Lemma 4.6 of [3], it follows that

$$U_m^1 = \langle U_S^1 \mid ht\tau(S) \geq m \rangle.$$

We now investigate, in the Lie ring associated with this central series, the commutator relations between U_S^1 and $U_{S'}^1$, for $S, S' \in \Pi^{1+}$.

Received September 6, 1967.

[1] This research was supported in part by the National Science Foundation.

If $S = \{r\}$ and $S' = \{r'\}$ are both of type (1), we have

$$\begin{aligned}[x_r(t), x_{r'}(v)] &= 0 && \text{if } S + S' \text{ is empty}\\ &= x_{r+r'}(N_{r,r'}\, tv) && \text{if } S + S' \in \Pi^{1+}\end{aligned}$$

where $N_{r,r'} = \pm 1$.

If $S = \{r\}$ is of type (1) and $S' = \{r', \overline{r'}\}$, $r' < \overline{r'}$, is of type (2), we have

$$\begin{aligned}[x_r(t), x_{r'}(v)x_{\overline{r'}}(\bar{v})] &= 0 && \text{if } S + S' \text{ is empty}\\ &= x_{r+r'}(N_{r,r'}\, tv)x_{r+\overline{r'}}(N_{r,r'}\, t\bar{v}) && \text{if } S + S' \in \Pi^{1+}\end{aligned}$$

where $N_{r,r'} = \pm 1$.

If $S = \{r, \bar{r}\}$, $r < \bar{r}$, and $S' = \{r', \overline{r'}\}$, $r' < \overline{r'}$, are both of type (2), we have

$$\begin{aligned}&[x_r(t)x_{\bar{r}}(\bar{t}),\, x_{r'}(v)x_{\overline{r'}}(\bar{v})]\\ &\quad= 0 && \text{if } S + S' \text{ is empty}\\ &\quad= x_{r+r'}(N_{r,r'}\, tv)x_{\bar{r}+\overline{r'}}(N_{r,r'}\, \bar{t}\bar{v}) && \text{if } S + S' \in \Pi^{1+} \text{ and } r + r' \in \Pi^1\\ &\quad= x_{r+\overline{r'}}(N_{r,\overline{r'}}\, t\bar{v})x_{\bar{r}+r'}(N_{r,\overline{r'}}\, \bar{t}v) && \text{if } S + S' \in \Pi^{1+} \text{ and } r + \overline{r'} \in \Pi^1\end{aligned}$$

where Nr,r' , $N_{r,\overline{r'}} = \pm 1$.

In this last case, it may happen that $r + \overline{r'} = \bar{r} + r'$, so that

$$x_{r+\overline{r'}}(N_{r,\overline{r'}}\, t\bar{v})x_{\bar{r}+r'}(N_{r,\overline{r'}}\, \bar{t}v) = x_{r+\overline{r'}}(2N_{r,\overline{r'}} \operatorname{Re}(t\bar{v})),$$

where $\operatorname{Re}(t)$ is defined to be $(t + \bar{t})/2$, so that $S + S'$ is of type (1).

Now, for $S = \{r\}$, write $x_r(t) = x_S(t)$, and for $S = \{r, \bar{r}\}$, write $x_r(t)x_{\bar{r}}(\bar{t}) = x_S(t)$. Then we always have

$$\begin{aligned}(*)\quad &[x_S(t), x_{S'}(v)]\\ &\quad= 0 && \text{if } S + S' \text{ is empty}\\ &\quad= x_{S+S'}(N_{S,S'}\, 2\operatorname{Re}(t\bar{v})) && \text{if } S + S' \in \Pi^{1+} \text{ and } S, S' \text{ are of type (2) while } S + S' \text{ is not}\\ &\quad= x_{S+S'}(N_{S,S'}\, tv) && \text{if } S + S' \in \Pi^{1+} \text{ otherwise.}\end{aligned}$$

where $N_{S,S'} = \pm 1$.

THEOREM 1. *Let U^1 be a maximal unipotent subgroup of a Steinberg group of type A_l^1 for odd l, D_l^1, or E_6^1, and let the characteristic of the field be different from 2. Then*

(i) *U^1 is generated by the U_{S_i}, for $\{S_i\}$ a fundamental system of sets of Π^1;*

(ii) *the U_m^1 form the lower central series of U^1.*

In particular, the class of U^1 is the height of the highest root in the system isomorphic to Π^{1+}.

Proof. By Corollary 4.5 and Lemma 4.6 of [3], it suffices to show that in the Lie ring associated with the central series U_m^1, every one of the graded summands U_m^1/U_{m+1}^1 is generated by m-fold simple commutators of the elements $x_{S_i}(t)$. To show this, it suffices to show that every $x_S(t)$, as an element of the Lie ring, is such an m-fold commutator for $m = \text{ht}\ r$, where $r = \tau(S)$.

Let $r_i = \tau(S_i)$. By [1, p. 20], we can write r as a sum

$$r = r_{i_1} + r_{i_2} + \cdots + r_{i_m},$$

where every partial sum $r_{i_1} + r_{i_2} + \cdots + r_{i_k}$ is a root. Then by repeated application of $(*)$, we have

$$x_S(t) = [[[\cdots[x_{S_{i_1}}(1), x_{S_{i_2}}(1)], \cdots], x_{S_{i_{m-1}}}(1)], x_{S_{i_m}}(t/c)]$$

where $c = \pm 2^j$ for some j. Since we are assuming the field characteristic is $\neq 2$, we have $c \neq 0$. Since $r_{i_1} + r_{i_2} + \cdots + r_{i_k}$ is a root, the set

$$(\cdots(S_{i_1} + S_{i_2}) + \cdots) + S_{i_k}$$

is always nonempty.

THEOREM 2. *Let U^1 be a maximal unipotent subgroup of a Steinberg group over a field of characteristic $\neq 2$ of any of the following types*:

$$A_l^1 \text{ for } l \geq 7, \quad l \text{ odd}; \qquad D_l^1 \text{ for } l \geq 5; \qquad E_6^1.$$

Then U^1 is terminal.

Proof. Let h be the class of U^1. We assume by way of contradiction that there is a nilpotent $\tilde{U}^1$ such that $\tilde{U}^1/\Gamma_{h+1}(\tilde{U}^1) \cong U^1$ and $\Gamma_{h+2}(\tilde{U}^1) = 1$.

We form the Lie rings associated with the lower central series of U^1 and $\tilde{U}^1$. The homomorphism of $\tilde{U}^1$ onto U^1 induces a homomorphism of the Lie ring of $\tilde{U}^1$ onto the Lie ring of U^1 with kernel $\Gamma_{h+1}(\tilde{U}^1)$.

In this homomorphism, let $y_S(t)$ be the element of $\Gamma_m(\tilde{U}^1)/\Gamma_{m+1}(\tilde{U}^1)$ mapping onto the element $x_S(t)$ in the Lie ring of U^1, m being the height of $\tau(S)$. Since the $y_{S_i}(t)$ generate the first summand $\tilde{U}^1/\Gamma_2(\tilde{U}^1)$ of the Lie ring of $\tilde{U}^1$, if we show that each $y_{S_i}(t)$ commutes with $y_{S^*}(v)$ for the set S^* such that ht $\tau(S^*) = h$, since $y_{S^*}(v)$ is an arbitrary element of $\Gamma_h(\tilde{U}^1)/\Gamma_{h+1}(\tilde{U}^1)$, it will follow that $\Gamma_{h+1}(\tilde{U}^1) = 0$, a contradiction.

Now suppose we can find an S' such that $\tau(S^*) - \tau(S')$ is a positive root and such that neither of $\tau(S_i) + \tau(S')$, $\tau(S_i) + \tau(S^*) - \tau(S')$ is a root.

On account of the $y_S(t)$ being uniquely determined by the $x_S(t)$, the commutator formula $(*)$ holds with x replaced by y and "$= 0$" replaced by "$\in \Gamma_{h+1}(\tilde{U}^1)$". Then, using the fact that $v = \bar{v}$ and the Jacobi identity, we have

$$\begin{aligned}[y_{S_i}(t), y_{S^*}(v)] &= [y_{S_i}(t), [y_{S'}(1/c), y_{S^*-S'}(v)]] \\ &= -[y_{S'}(1/c), [y_{S^*-S'}(v), y_{S_i}(t)]] \\ &\quad - [y_{S^*-S'}(v), [y_{S_i}(t), y_{S'}(1/c)]],\end{aligned}$$

where $c = \pm 1$ or ± 2.

By our assumptions on S', each of the interior commutators in the last two lines belongs to $\Gamma_{h+1}(\tilde{U}^1)$, and so each of the last two lines is 0, implying that $y_{S_i}(t)$ commutes with $y_{S^*}(v)$.

It therefore suffices to give examples of S' for each of A_l^1, D_l^1, E_6^1. We do this by finding r' in the root system isomorphic to Π^{1+} such that if r^* is the root of highest height, $r^* - r'$ is a root and $r_i + r'$, $r^* + r_i - r'$ are not roots.

For the case A_l^1, l odd, the root system isomorphic to Π^{1+} is that of C_n, $n = (l+1)/2$. Examples of r' for C_n, $n \geq 4$, were given in [2]. Thus, U^1 will be terminal when $l \geq 7$.

For the case E_6^1, the root system isomorphic to Π^{1+} is that of F_4. Again, examples of r' for F_4 were given in [2].

For the case D_l^1, the root system isomorphic to Π^{1+} is that of B_n, $n = l - 1$. In the case B_n, $n \geq 4$, examples of r' were given in [2] for all r_i except r_1. To show that $y_{S_1}(t)$ commutes with $y_{S^*}(v)$, we show that

$$(**) \qquad [y_{S_1}(t), y_{S^*}(v)] = -[y_{S_1}(t), y_{S^*}(v)]$$

which implies that

$$[y_{S_1}(t), y_{S^*}(2v)] = 0,$$

and since we assume the field characteristic is not 2, this shows that $y_{S_1}(t)$ commutes with an arbitrary $y_{S^*}(v)$.

The computation to prove $(**)$ is practically the same as the one appearing in [2], but since the commutator relation $(*)$ here is more complicated than the one in that paper, we give the argument again, to show it indeed does follow from the relation $(*)$. Note that any S for which ht $\tau(S) \geq h - 2$ is of type (1). Thus, the second alternative of $(*)$ does not arise in what is to follow:

$$\begin{aligned}[y_{S_1}(t), y_{S^*}(v)] &= [y_{S_1}(t), [y_{S_2}(1), y_{S^*-S_2}(v/N_{S_2,S^*-S_2})]] \\ &= -[y_{S_2}(1), [y_{S^*-S_2}(v/N_{S_2,S^*-S_2}), y_{S_1}(t)]] \\ &\quad - [y_{S^*-S_2}(v/N_{S_2,S^*-S_2}), [y_{S_1}(t), y_{S_2}(1)]].\end{aligned}$$

The first of the last two terms is 0, since $S_1 + (S^* - S_2)$ is empty. We proceed with the second term:

$$\begin{aligned}&[y_{S_1}(t), y_{S^*}(v)] \\ &\quad = [[y_{S_1}(t), y_{S_2}(1)], [y_{S_1}(t), y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\})]] \\ &\quad = -[y_{S_1}(t), [y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\}), [y_{S_1}(t), y_{S_2}(1)]]] \\ &\qquad - [y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\}), [[y_{S_1}(t), y_{S_2}(1)], y_{S_1}(t)]].\end{aligned}$$

Here the last term is 0, since $(S_1 + S_2) + S_1$ is empty. Now

$$\begin{aligned}&[y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\}), [y_{S_1}(t), y_{S_2}(1)]] \\ &\qquad = -[y_{S_1}(t), [y_{S_2}(1), y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\})]] \\ &\qquad\quad - [y_{S_2}(1), [y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\}), y_{S_1}(t)]],\end{aligned}$$

of which the first term is 0, since $S^* - S_1$ is empty. The second term is

$$[y_{S_2}(1), [y_{S_1}(t), y_{(S^*-S_2)-S_1}(v/\{tN_{S_2,S^*-S_2} N_{S_1,(S^*-S_2)-S_1}\})]] = [y_{S_2}(1), y_{S^*-S_2}(v/N_{S_2,S^*-S_2})] = y_{S^*}(v).$$

Thus (**) is established, and the proof is complete.

We now turn our attention to the case A^1_{2k}. Here matters are complicated by the fact that Π^{1+} consists of sets of types (2) and (3), and an attempt to define $S + S'$ yields sets which can be non-empty and yet not elements of Π^{1+}. As before, in U^1 define the central series whose mth term is $U^1 \cap U_m$. We let $r_1, \cdots, r_{2k}$ now denote a fundamental system of roots of A_{2k} (as opposed to a fundamental system in the root system isomorphic to Π^{1+}. No confusion will result, as we no longer have a root system isomorphism.)

We define

(a) $$x_i(t) = x_{r_i}(t)x_{r_{2k-i+1}}(\bar{t}) \quad \text{for } i = 1, \cdots, k-1$$

(b) $$x_k(t) = x_{r_k}(t)x_{r_{k+1}}(\bar{t})x_{r_k+r_{k+1}}(N_{r_k,r_{k+1}}\, t\bar{t}/2)$$

LEMMA. *Let U^1 be a maximal unipotent subgroup of a group of type A^1_{2k}, and let the characteristic of the field be different from* 2. *Then*

(i) *U^1 is generated by all elements of the forms* (a) *and* (b);

(ii) *the U^1_m form the lower central series of U^1.*

In particular, the class of U^1 is $2k$.

Proof. Again, we show that in the Lie ring associated with the central series U^1_m, every one of the summands U^1_m/U^1_{m+1} is generated by m-fold simple commutators of the elements of forms (a) and (b).

In case m is even, U^1_m/U^1_{m+1} is generated by elements of the forms

(c) $$x_{r_{k-(m/2)+1}+\cdots+r_{k+(m/2)}}(t), \qquad t = -\bar{t}$$

(d) $$x_{r_j+\cdots+r_{j+m}}(t)x_{r_{2k-j-m+1}+\cdots+r_{2k-j+1}}(\bar{t}), \quad \text{where}$$
$$j \le k-1, \quad j < 2k-j-m+1, \quad \text{and} \quad j+m \ne k$$

(e) $$x_{r_j+\cdots+r_k}(t)x_{r_{k+1}+\cdots+r_{2k-j+1}}(\bar{t})x_{r_j+\cdots+r_{2k-j+1}}(w), \quad \text{where}$$
$$w = N_{r_j+\cdots+r_k,r_{k+1}+\cdots+r_{2k-j+1}}\, t\bar{t}/2$$

In case m is odd, U^1_m/U^1_{m+1} is generated by elements of the forms (d) and (e).

Now any element of the form (c) is the m-fold left-normed commutator

$$[x_k(1), x_k(\tfrac{1}{2}), x_{k-1}(1), x_{k-1}(\tfrac{1}{2}), \cdots, x_{k-(m/2)+1}(1), x_{k-(m/2)+1}(\pm t/2)].$$

If $j + m < k$, an element of the form (d) is the m-fold left-normed commutator

$$[x_j(1), x_{j+1}(1), \cdots, x_{j+m-1}(1), x_{j+m}(\pm t)].$$

If $j + m > k$, an element of the form (d) is the m-fold left-normed commutator

$$[x_k(1),\ x_k(\tfrac{1}{2}),\ x_{k-1}(1),\ x_{k-1}(\tfrac{1}{2}),\ \cdots,\ x_{2k-j-m+1}(1),\ x_{2k-j-m+1}(\tfrac{1}{2}),$$
$$x_{2k-j-m}(1),\ x_{2k-j-m-1}(1),\ \cdots,\ x_{j+1}(1),\ x_j(\pm t)].$$

An element of the form (e) modulo U^1_{m+1}, is the m-fold left-normed commutator

$$[x_j(1),\ x_{j+1}(1),\ \cdots,\ x_{k-1}(1),\ x_k(\pm t)].$$

THEOREM 3. *Let U^1 be a maximal unipotent subgroup of a group of type A^1_{2k}, $k \geq 4$, and let the characteristic of the field be different from 2. Then U^1 is terminal.*

Proof. Let h and $\tilde{U}^1$ be as described in the first two paragraphs of the proof of Theorem 2. In the Lie ring homomorphism, we let $y_i(t)$ be the element of $\tilde{U}^1/\Gamma_2(\tilde{U}^1)$ mapping onto the element of $U^1/\Gamma_2(U^1)$ which corresponds to $x_i(t)$, and let $y(t)$ be the element of $\Gamma_h(\tilde{U}^1)/\Gamma_{h+1}(\tilde{U}^1)$ mapping onto the element $x_{r_1+\cdots+r_{2k}}(v)$ of $\Gamma_h(U^1)$.

Let $y'(1) = [y_k(1), y_k(\frac{1}{2}), \cdots, y_2(1), y_2(\frac{1}{2}), y_1(1)]$. Then we have

$$\begin{aligned}[y_i(t), y(v)] &= [y_i(t), [y'(1), y_1(\pm v/2)]] \\ &= -[y'(1), [y_1(\pm v/2), y_i(t)]] - [y_1(\pm v/2), [y_i(t), y'(1)]].\end{aligned}$$

Each of the interior commutators in the last two lines is an element of $\Gamma_{h+1}(\tilde{U}^1)$, unless $i = 1$ or 2, so if $i \neq 1, 2$, we have that $y_i(t)$ commutes with $y(v)$, and so with all of $\Gamma_{h+1}(\tilde{U}^1)$.

For the cases $i = 1, 2$, let $y''(1)$ be

$$[y_k(1),\ y_k(\tfrac{1}{2}),\ \cdots,\ y_4(1),\ y_4(\tfrac{1}{2}),\ y_3(\tfrac{1}{2}),\ y_2(\tfrac{1}{2}),\ y_1(\tfrac{1}{2})].$$

Then we have

$$\begin{aligned}[y_i(t),\ y(v)] &= [y_i(t),\ [y''(1),\ [y_3(1),\ y_2(1),\ y_1(\pm v)]]] \\ &= -[y''(1),\ [[y_3(1),\ y_2(1),\ y_1(\pm v)],\ y_i(t)]] \\ &\quad - [[y_3(1),\ y_2(1),\ y_1(\pm v)],\ [y_i(t),\ y''(1)]],\end{aligned}$$

and so again $y_i(t)$ commutes with $y(v)$, which completes the proof.

Acknowledgement. The author wishes to thank J. L. Alperin for a very helpful conversation during the preparation of this manuscript.

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