OBSTRUCTIONS TO EXTENDING ALMOST X-STRUCTURES

BY
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1. Introduction

This paper has its origin in two unexpected phenomena that I encountered while constructing certain almost-complex manifolds. First, I noticed that, although a smooth, oriented manifold M may admit an almost-complex (or even a complex) structure, the manifold $-M$ obtained from it by reversing its orientation may not. Secondly, I noticed that although two smooth, oriented manifolds M_1 and M_2 may admit almost-complex (or even complex) structures, their connected sum $M_1 + M_2$ may not.

Indeed, complex projective 2-space, CP^2, with the standard smoothness structure and orientation, exemplifies both phenomena, for neither $-CP^2$ nor $CP^2 + CP^2$ admit almost-complex structures. Simple obstruction-theoretic arguments, using the calculations, of, say, [11], confirm this assertion. Moreover, these arguments indicate where the trouble lies. Briefly, it is possible to show that, when open 4-discs are deleted from $-CP^2$ and $CP^2 + CP^2$, the resulting manifolds-with-boundary admit almost-complex structures. According to [11], the obstruction to extending such a structure over $-CP^2$ or $CP^2 + CP^2$ is a linear combination of certain characteristic numbers (Chern, Pontrjagin, Euler), which, *with one exception*, are all additive (that is, they respect orientation and connected sum). Thus, the obstruction is not additive, which means that its vanishing for CP^2 does not imply its vanishing for $-CP^2$ or $CP^2 + CP^2$; indeed, it definitely does not vanish for these latter. The exceptional characteristic number is the Euler characteristic, χ, which, for any smooth, oriented, n-manifolds M_1^n and M_2^n satisfies the following well-known relations:

$$\chi[-M_1^n] = \chi[M_1^n] = -\chi[M_1^n] + \chi[M_1^n]\chi[S^n] \tag{1}$$

$$\chi[M_1^n + M_2^n] = \chi[M_1^n] + \chi[M_2^n] - \chi[S^n], \tag{2}$$

where S^n is the standard, oriented n-sphere. Relation (1) is trivial. Relation (2) is most easily verified by triangulating M_1^n and M_2^n, deleting the interior of an n-simplex from each and matching them along the resulting simplex boundaries, thus obtaining $M_1 + M_2$, and calculating χ on the simplex chain level.

The main point of this paper is to generalize relations (1) and (2) (Theorems 1 and 2) and to show that their "semi-additivity" is (in a sense to be clarified in §2) an unstable phenomenon. Central to the generalization is

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the concept of an almost-X-structure, which, together with other definitions and the main results, is presented in §2.

In §3, we present some applications of these results. For example, we obtain some of the results of Massey [11, Theorem II] on obstructions to the existence of almost-complex structures (Corollary 2 and subsequent remarks). We also obtain a special case of a result of Milnor (proof unpublished, but see [3, pp. 122–127]): namely, every stably almost-complex manifold is complex-cobordant to an almost-complex manifold (equivalently, every stably almost-complex manifold has the same Chern numbers as some almost-complex manifold). Finally, in Corollary 6, we show that, for every $k \geq 1$, there exist two (smooth, closed, compact, connected, oriented) $8k$-manifolds, both of the same oriented-homotopy type, only one of which admits an almost-complex structure. For another application, which will appear in [8], we construct pairs of manifolds of the same oriented homotopy type but not complex-cobordant. These enable us to answer affirmatively the following unpublished conjecture of Milnor: *a rational, linear combination of Chern numbers is an oriented-homotopy-type invariant for almost-complex manifolds if and only if it is a rational linear combination of the Euler characteristic and the Index.*

In §4, we present proofs of the results stated in §2, as well as definitions of more local interest.

2. Main definitions and results

Throughout this paper, we shall deal simultaneously with two situations in both of which feature a topological group G, a principal G-bundle ξ, and a topological space X on which there is defined a fixed, not necessarily effective G-action. We shall be interested in cross-sections or partial cross-sections of the bundle $\xi(X)$ associated to ξ with fibre X.

In one situation, $G = SO_n$, $n \geq 2$, and $\xi = \tau_M$, the principal SO_n tangent bundle of an oriented, smooth n-manifold M^n. In the other, $G = SO$, the stable orthogonal group, and ξ is the principal SO tangent bundle of M^n, $n \geq 2$, which we still denote by τ_M. In both cases, we require M^n to be closed, compact,[2] connected, and we require X to be pathwise connected and $(n-1)$-simple.

Although, by and large, we deal with both cases at once, we shall occasionally want to sort one out. We do so by appropriate use of the words "unstable" (to indicate the first case) or "stable" (to indicate the second).

Definition 2.1. An X-structure on M^n is a cross-section of $\tau_M(X)$. Two X-structures on M^n are said to be homotopic if, as maps, they are homotopic through cross-sections of $\tau_M(X)$. (Compare [10, p. 4] and [14].)

Examples. (a) X is arbitrary, the G-action is trivial. In this case, an X-structure on M^n corresponds simply to a map $M^n \to X$.

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[2] Actually, our methods and results can easily be modified to apply to manifolds-with-boundary. For simplicity however, we restrict ourselves to the unbounded case.

In the other two examples that interest us, X is a homogeneous space G/H, H a closed subgroup of G, with G acting on X by left translation. A cross-section of $\tau_M(X)$ corresponds to a reduction of the group of τ_M to H (cf. [16, p. 44]).

(b) $G = SO_n$ and $H = SO_{n-1}$. Then $X = S^{n-1}$, the standard sphere in Euclidean n-space, $\tau_M(X)$ is the tangent sphere bundle of M, and an X-structure corresponds to a non-singular vector field on M.

(c) $G = SO_{2k}$ and $H = U_k$ (imbedded in SO_{2k} in the standard way). An X-structure on M^{2k}, in this case, is commonly known as an *almost-complex* structure. When $G = SO$ and $H = U \subset SO$, then a cross-section of $\tau_{M^n}(X)$ is called a *stable almost-complex* structure.

In this paper, the word "almost" is also used in a different sense.

DEFINITION 2.2. An almost X-structure on M^n is an X-structure on M^n-interior D^n, where D^n is some closed n-disc smoothly imbedded in M^n.

We show, in the Appendix, that for most questions involving almost X-structures, the particular choice of imbedded disc D^n is irrelevant.

Now, let $N = M^n$-interior D^n, where D^n is as above, and let s be an X-structure on N. The obstruction to extending s over M is a class

$$c_X(M, s) \in H^n(M, N; \pi_{n-1}(X)) \cong \pi_{n-1}(X).$$

Let $\mu_M \in H_n(M, N)$ be the image of the orientation generator of $H_n(M)$ under the inclusion-induced isomorphism $H_n(M) \cong H_n(M, N)$, and let $\langle\ ,\ \rangle$ denote the Kronecker evaluation pairing

$$H^n(M, N; \pi_{n-1}(X)) \times H_n(M, N) \to \pi_{n-1}(X).$$

DEFINITION 2.3. For M and s as above, let

$$c_X[M, s] = \langle c_X(M, s), \mu_M\rangle \in \pi_{n-1}(X).$$

Examples. We refer by letter to the examples given above.

(a) In this case, the classes $c_X[M, s] \in \pi_{n-1}(X)$ are those represented by maps $\partial D^n = S^{n-1} \to X$ that extend over N. For a special case of some interest, take M^n to be a π-manifold (closed, compact, connected, n odd) and suppose that X is $(k-1)$-connected, $2k \geq n$. Then, using surgery, one can easily show that $c_X[M, s] = 0$ [2, p. 87].

(b) In this case, it is a classical result of Hopf [5] that each class

$$c_X[M, s] \in \pi_{n-1}(S^{n-1})$$

equals the Euler characteristic $\chi[M]$ (under the standard identification of $\pi_{n-1}(S^{n-1})$ with Z). In the stable case, non-zero cross-sections of vector-bundle representatives of τ_M always exist, so that it makes sense to define

$$\begin{aligned} \chi[\tau_M] &= \chi[M] && \text{in the unstable case,} \\ &= 0 && \text{in the stable case.} \end{aligned}$$

(c) In this case, Theorem II of [11] calculates $c_X[M^{2k}, s]$ under certain circumstances. For example, when $k \equiv 2 \pmod 4$, the group $\pi_{2k-1}(SO_{2k}) \cong Z$, and the integer $c_X[M^{2k}, s]$ is given by

$$\tfrac{1}{4}\left(\textstyle\sum_{i+j=k} (-1)^i c_i c_j[s] - (-1)^l p_l[\tau_M]\right),$$

where $2l = k$, $p_l[\tau_M]$ is the top Pontrjagin number of M^{2k}, $c_i c_j[s]$, $i, j < k$, is the Chern number obtained by evaluating the cup product of Chern classes of s, $c_i(s) \cup c_j(s)$, on the orientation generator of $H_{2k}(M)$, and $c_k[s] = \chi[\tau_M]$. (Cf. Corollary 2, §3.)

Now, let S^n be the unit sphere in Euclidean $(n+1)$-space, endowed with the standard orientation. Since S^n-point is contractible, S^n admits almost X-structures, and any two such structures with the same domain are homotopic. Moreover, for any two smoothly imbedded discs $D_1 \subset S^n$ and $D_2 \subset S^n$, there is a degree-one homeomorphism $S^n \to S^n$ taking D_1 onto D_2. From these facts it follows easily that $c_X[S^n, s]$ is independent of the almost X-structure s. Henceforth, we write it as $c_X[S^n]$. It can be regarded as the obstruction to the existence of an X-structure on S^n. In the stable case, $c_X[S^n] = 0$ because S^n is a π-manifold.

If M^n, M_1^n, and M_2^n are oriented, smooth n-manifolds, then, as in the introduction, we denote the connected sum of M_1^n and M_2^n by $M_1^n + M_2^n$ and by $-M^n$ the oriented manifold obtained from M^n by reversing its orientation.

LEMMA 1. *Let M^n, M_1^n, and M_2^n be as above, and suppose that they admit almost X-structures. Then, $-M^n$ and $M_1^n + M_2^n$ admit almost X-structures.*

Part of Lemma 1 is an immediate consequence of the following result, proved in §4.1.

LEMMA 2. *Let M^n be as above, and let $N = M^n$-interior D, where D is a closed n-disc smoothly imbedded in M^n. Then $\tau_M \mid N$ is equivalent in G to $\tau_{-M} \mid N$.*

To deduce part of Lemma 1, notice that Lemma 2 implies that $\tau_M(X) \mid N$ is equivalent to $\tau_{-M}(X) \mid N$, so that if the former has a cross-section, so does the latter. Indeed, the equivalence establishes a 1-1 correspondence between the cross-sections of these two bundles.

Now, in general, there are, even up to homotopy, many equivalences between these bundles, so that there is no preferred correspondence between their cross-sections. However, there is a particularly good, non-empty class of equivalences, which we call *admissible equivalences* (see §4.1), such that almost X-structures that correspond to one another under such an equivalence have obstructions that are related by a simple formula.

THEOREM 1. *Let M^n and N be as in Lemma 2, and let s and $-s$ be cross-sections of $\tau_M(X) \mid N$ and $\tau_{-M}(X) \mid N$, respectively, corresponding under some admissible equivalence. Then,*

$$c_X[-M^n, -s] = -c_X[M^n, s] + \chi[\tau_M] c_X[S^n] \tag{3}$$

Next, note that the notation "$M_1^n + M_2^n$" for the connected sum of M_1^n and M_2^n is, *a priori*, ambiguous, since the connected-sum construction depends on the choice of imbedded discs $D_\alpha^n \subset M_\alpha^n$, $\alpha = 1, 2$ (see §4.4). Nevertheless, one can show that, up to orientation-preserving diffeomorphism, the resulting sum is independent of this choice (see [13]).

Now when we are given almost X-structures s_α on M_α^n, $\alpha = 1, 2$, it is natural to form the connected sum $M_1^n + M_2^n$ with respect to the discs $D_\alpha^n \subset M_\alpha^n$ defined by M_α^n-interior D_α^n = domain s_α. We do so and then define (in §4.4) what it means for an almost X-structure on $M_1^n + M_2^n$ to be *compatible* with s_1 and s_2. Such almost X-structures always exist (see §4.4).

THEOREM 2. *Given almost X-structures s_α on M_α^n, $\alpha = 1, 2$, we form the connected sum $M_1^n + M_2^n$ as above and let $s_1 + s_2$ be an almost X-structure on $M_1^n + M_2^n$ compatible with s_1 and s_2. Then,*

$$c_X[M_1^n + M_2^n, s_1 + s_2] = c_X[M_1^n, s_1] + c_X[M_2^n, s_2] - c_X[S^n] \tag{4}$$

Applying Theorems 1 and 2 to example b) above, we immediately obtain (1) and (2), so that (3) and (4) are the desired generalizations.

Finally, it is easy to see that the semi-additivity in (3) and (4) are *unstable* phenomena, for in the stable case $c_X[S^n] = 0$, so that (3) and (4) become additive.

3. Applications

We begin by obtaining a result (Lemma 6) that enables us to characterize c_X under certain circumstances. We apply this result in the case $X = SO_{2k}/U_k$ to calculate c_X (Corollary 2). We conclude by applying the calculation to the construction of certain almost-complex manifolds (Corollaries 4–6) as described in §1. Unless stated otherwise, we deal exclusively with the unstable case and with manifolds of dimension $n \geq 3$.

Consider pairs (M_α, s_α), $\alpha = 1, 2$, of closed, compact, connected, oriented n-manifolds M_α, and X-structures s_α defined on M_α-interior D_α, as in §2. A *map*

$$f : (M_1, s_1) \to (M_2, s_2)$$

is a continuous function $f : (M_1, M_1\text{-interior } D_1) \to (M_2, M_2\text{-interior } D_2)$ covered by some map of principal tangent bundles that pulls s_2 back to s_1 (see §4.2). We let $\mathfrak{M}(X)$ denote the category of all such pairs and maps, and we let $\mathcal{C}$ be an arbitrary, full subcategory of $\mathfrak{M}(X)$ satisfying the following conditions:

(i) If $(S^n, s) \in \mathfrak{M}(X)$, then $(S^n, s) \in \mathcal{C}$.

(ii) If $f : (M_1, s_1) \to (M_2, s_2)$ is a map in $\mathfrak{M}(X)$, and if $(M_2, s_2) \in \mathcal{C}$, then $(M_1, s_1) \in \mathcal{C}$.

(iii) If $(M_\alpha, s_\alpha) \in \mathcal{C}$, $\alpha = 1, 2$, then $(-M_1, -s_1) \in \mathcal{C}$ and $(M_1 + M_2, s_1 + s_2) \in \mathcal{C}$.

(iv) If $(M, s_1) \in \mathcal{C}$ and if s_1 is homotopic to s_2, then $(M, s_2) \in \mathcal{C}$. Of course, $\mathfrak{M}(X)$ itself satisfies these conditions.

Suppose that φ is an arbitrary function from the objects of $\mathcal{C}$ to $\pi_{n-1}(X)$; when we want to emphasize that the domain of φ is the class of objects of $\mathcal{C}$, we write it as $\varphi\colon \mathcal{C}$. It may possess some of the following properties.

(1) *Naturality.* For any map $f : (M_1, s_1) \to (M_2, s_2)$ in $\mathcal{C}$,

$$\varphi(M_1, s_1) = (\text{degree } f)\varphi(M_2, s_1).$$

(2) *Homotopy Invariance.* If $(M, s_1) \in \mathcal{C}$, and if s_1 is homotopic to s_2, then $\varphi(M, s_1) = \varphi(M, s_2)$.

If φ satisfies (1) and (2), $\varphi(S^n, s)$ is independent of s, so that we may write it as $\varphi(S^n)$. In this case, φ may additionally satisfy the following:

(3) *Semi-additivity.* Suppose that $(M_\alpha, s_\alpha) \in \mathcal{C}$, $\alpha = 1, 2$.
Then
(a) $\varphi(-M_1, -s_1) = -\varphi(M_1, s_1) + \chi[M_1]\varphi(S^n)$
(b) $\varphi(M_1 + M_2, s_1 + s_2) = \varphi(M_1, s_1) + \varphi(M_2, s_2) - \varphi(S^n)$.
(4) *Normalization.* $\varphi(S^n) = c_X[S^n]$.

It is not hard to show that $c_X\colon\mathfrak{M}(X)$ satisfies (1)–(4).

So far, we only have objects in $\mathcal{C}$ of the form (S^n, s) at our disposal. Choose any such and suppose that s has domain S^n-interior D. Let M^n be any π-manifold (closed, compact, connected, oriented), let D_1 be any closed n-disc smoothly imbedded in M^n, and let $N = M^n$-interior D_1. By the "Gauss map" construction (see [2], p. 86), we can obtain a map

$$f : (M^n, N) \to (S^n, S^n\text{-interior } D)$$

and a covering bundle map $\bar{f}\colon \tau_M \to \tau_{S^n}$. Then, letting $t\colon N \to N \times X$ be a pull-back of s by $\bar{f}$, f is a map $(M^n, t) \to (S^n, s)$ in $\mathfrak{M}(X)$, so that $(M^n, t) \in \mathcal{C}$. We now have

LEMMA 3. *Let M^n be any closed, compact, connected, oriented π-manifold. Then there is an almost X-structure t on M^n and an integer K, such that $(M^n, t) \in \mathcal{C}$ and, for any $\varphi\colon\mathcal{C}$ satisfying* (1) *and* (2), $\varphi(M, t) = K\varphi(S^n)$. *When n is even, $K = \frac{1}{2}\chi[M^n]$.*

Proof. Let t and s be as above. By (1) and (2)

$$\varphi(M, t) = (\text{degree } f)\varphi(S^n, s) = (\text{degree } f)\varphi(S^n).$$

Specializing to $\varphi = c_X = \chi$ (example (b), §2) when n is even,

$$\chi[M^n] = (\text{degree } f)\cdot 2, \qquad \text{Q.E.D.}$$

Remark. When n is odd and $n \neq 1, 3, 7$, f can be chosen so that degree f is any prescribed integer within a congruence class modulo two. This class, according to [2] and [9], is represented by the semi-characteristic of

$M^n (n \neq 1, 3, 7)$:

$$\chi^*[M^n] = \sum_{i=1}^{r-1} \text{rank } H_i(M^n, Z_2), \qquad n = 2r - 1.$$

LEMMA 4. *Suppose that M^n is parallelizable. Then, there is an almost X-structure s such that $(M^n, s) \in \mathcal{C}$ and such that, for every $\varphi : \mathcal{C}$ satisfying* (1), $\varphi(M^n, s) = 0$.

Proof. When M^n is parallelizable, the trivial map

$$(M^n, N) \to (S^n, S^n\text{-interior } D)$$

can be covered by a tangent bundle map. Apply (1), Q.E.D.

COROLLARY. *$S^p \times S^q$ admits an almost X-structure t such that $(S^p \times S^q, t) \in \mathcal{C}$ and, for all $\varphi : \mathcal{C}$ satisfying* (1) *and* (2),

$$\begin{aligned} \varphi(S^p \times S^q, t) &= 2\varphi(S^{p+q}), \quad p \text{ and } q \text{ even} \\ &= 0, \qquad\qquad \text{otherwise.} \end{aligned}$$

Proof. $S^p \times S^q$ is a π-manifold. When p and q are even, $\chi[S^p \times S^q] = 4$. When p or q is odd, $S^p \times S^q$ is parallelizable. Apply Lemma 3 or Lemma 4, Q.E.D.

LEMMA 5. *Suppose that $\varphi : \mathcal{C}$ satisfies* (1)–(4) *and consider any $(M, s) \in \mathcal{C}$ for which $\varphi(M, s) = A\varphi(S^n)$. Then* (i) *if $A^n \geq 0$, there is a $(P, t) \in \mathcal{C}$ such that $\varphi(M + | A | P, s + | A | t) = 0$;* (ii) *If $A^n < 0$, there is a $(P, t) \in \mathcal{C}$ such that $\varphi(-M + | A | P, -s + | A | t) = 0$.*

(*Explanation.* $| A |$ is the absolute value of the integer A, $\pm M + | A | P$ is the connected sum of $\pm M$ and $| A |$ copies of P, $\pm s + | A | t$ is the almost X-structure on $\pm M + | A | P$ obtained by gluing together $\pm s$ and $| A |$ copies of t, as described in §4.4. Finally, $(\pm M + | A | P, \pm s + | A | t) \in \mathcal{C}$, by property (iii) of $\mathcal{C}$.)

Proof. An easy extension of property (3) yields

$$\varphi(\pm M + | A | P, \pm s + | A | t) = \varphi(\pm M, \pm s) + | A | \varphi(P, t) - | A | \varphi(S^n).$$

If $A \geq 0$, we let

$$(P, t) = (S^1 \times S^{n-1}, t),$$

and if $A < 0$, we let

$$(P, t) = (S^2 \times S^{n-2}, t),$$

where t is as in the above corollary. Then, if $A \geq 0$,

$$\varphi(M + | A | P, s + | A | t) = | A | \varphi(S^n) + 0 - | A | \varphi(S^n) = 0,$$

whereas if $A < 0$ and n even,

$$\varphi(M + | A | P, s + | A | t) = - | A | \varphi(S^n) + 2 | A | \varphi(S^n) - | A | \varphi(S^n) = 0.$$

When n is odd, $\chi[M] = 0$, so that $\varphi(-M, -s) = -\varphi(M, s)$. Therefore, when, in addition, $A < 0$,

$$\varphi(-M + |A| P, -s + |A| t)$$
$$= |A| \varphi(S^n) + 0 - |A| \varphi(S^n) = 0, \quad \text{Q.E.D.}$$

Note that the choice of (P, t) depends only on A.

Now, let $\mathfrak{M}_0(X)$ be the full subcategory of $\mathfrak{M}(X)$ whose objects are all pairs (M, s) for which $c_X[M, s]$ is a multiple of $c_X[S^n]$. It is not hard to show that $\mathfrak{M}_0(X)$ satisfies (i)–(iv). Property (i) is obvious, (ii) follows from the fact that c_X satisfies (1), (iii) follows from the fact that c_X satisfies (3), and (iv) follows from the fact that c_X satisfies (2).

LEMMA 6. *Suppose that* $\varphi:\mathfrak{M}_0(X)$ *satisfies* (1)–(4) *and that null space* $\varphi \supset$ *null space* c_X. *Then,* $\varphi = c_X \mid \mathfrak{M}_0(X)$.

(*Explanation.* By "null space φ," we mean all objects sent to zero by φ. By "$c_X \mid \mathfrak{M}_0(X)$," we mean the restriction of c_X to objects of $\mathfrak{M}_0(X)$.) This lemma is, of course, an analogue of a standard algebraic fact.

Proof. Choose $(M, s) \in \mathfrak{M}_0(X)$ and suppose that $c_X[M, s] = Ac_X[S^n]$. By Lemma 5, we may choose a product of spheres P and an almost X-structure t, depending only on A, so that

$$(P, t) \in \mathfrak{M}_0(X) \quad \text{and} \quad c_X[\pm M + |A| P, \pm s + |A| t] = 0.$$

Thus $\varphi(\pm M + |A| P, \pm s + |A| t) = 0$. Since $\varphi:\mathfrak{M}_0(X)$ satisfies (3) and (4), we may expand this last relation to solve for $\varphi(M, s)$:

$$\varphi(M, s) = A\varphi(S^n) = Ac_X[S^n] = c_X[M, s], \qquad \text{Q.E.D.}$$

To apply Lemma 6, we need some more information about $c_X[S^n]$. Let $\pi_X:G \to X$ be defined by $\pi_X(g) = g \cdot x_0$, where x_0 is some arbitrary point chosen in X. We now state a result, proved in §4.2, that applies to both the stable and unstable cases.

LEMMA 7. *Let* $\hat{k}: S^{n-1} \to G$ *be a characteristic map for* τ_{S^n} (see §4.2 or [16, p. 97]). *Then,* $\pi_X \circ \hat{k}$ *represents* $c_X[S^n]$.

Next, let the pair of spaces (X, Y) be either (SO_n, SO), $n \geq 3$, or $(SO_{2k}/U_k, SO/U)$, $k \geq 1$. In either case, there is a standard inclusion map $i:X \to Y$. In the second case, let $n = 2k$.

LEMMA 8. $c_X[S^n]$ *generates kernel* $(i_*:\pi_{n-1}(X) \to \pi_{n-1}(Y))$.

Proof. When $(X, Y) = (SO_n, SO)$, the result follows from the facts that $c_{SO_n}[S^n]$ is represented by a characteristic map for τ_{S^n} (Lemma 7, together with the observation that $\pi_{SO_n}: SO_n \to SO_n$ is homotopic to the identity) and that this map represents a generator of kernel i_* (23.2 of [16]). In the rest of the proof, we consider the second case. There is a commutative diagram,

$$\begin{array}{ccccc} U_k & \longrightarrow & SO_{2k} & \xrightarrow{\pi_X} & SO_{2k}/U_k \\ \downarrow & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ U & \longrightarrow & SO & \longrightarrow & SO/U \end{array}$$

in which all the maps are standard inclusions or projections. Thus, the induced diagram, in which the rows are exact and the columns are epimorphisms, commutes:

$$\begin{array}{ccccccc} \pi_{2k-1}(U_k) & \longrightarrow & \pi_{2k-1}(SO_{2k}) & \xrightarrow{(\pi_X)_*} & \pi_{2k-1}(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle i'_*} & & \downarrow{\scriptstyle i_*} & & \downarrow \\ \pi_{2k-1}(U) & \longrightarrow & \pi_{2k-1}(SO) & \longrightarrow & \pi_{2k-1}(Y) & \longrightarrow & 0. \end{array}$$

Therefore, by standard diagram chasing, kernel $i_* = (\pi_X)_*$ (kernel i'_*). By the first case considered, kernel i'_* is generated by $c_{SO_n}[S^n]$, which is represented by a characteristic map $\hat{k}$ of τ_{S^n}. Therefore, $\pi_X \circ \hat{k}$ represents a generator of kernel i_*. Apply Lemma 7, Q.E.D.

For the convenience of the reader, we present the following tables (cf. [18, p. 171], for the first two, and [11] for the third). Note that, for $(X, Y) = (SO_n, SO)$, $n \neq 1, 3, 7$, kernel i_* has infinite order or order two according as n is even or odd. In the second case, kernel i_* is the entire group $\pi_{2k-1}(X)$ except when $k \equiv 0 \pmod 4$.

TABLE 1

$(X, Y) = (SO_n, SO), \quad n \neq 1, 2, 3, 7.$

$n - 1 \pmod 8$	0	1	2, 4, 6	5	3, 7
$\pi_{n-1}(X)$	$Z_2 + Z_2$	$Z + Z_2$	Z_2	Z	$Z + Z$
$\pi_{n-1}(Y)$	Z_2	Z_2	0	0	Z

TABLE 2

n	1	2	3	7
$\pi_{n-1}(X)$	0	Z	0	0
$\pi_{n-1}(Y)$	0	Z_2	0	0

TABLE 3

$(X, Y) = (SO_{2k}/U_k, SO/U)$

k (mod 4)	$\pi_{2k-1}(X)$	$\pi_{2k-1}(Y)$
0	$Z + Z_2$	Z_2
1	Z_r, $r = (k-1)!$	0
2	Z	0
3	$Z_{1/2r}$, $r = (k-1)!$	0

The relationship between these considerations and Lemma 6 comes from the following observation. Let (X, Y) and $i: X \to Y$ be as above. The map i determines, for each manifold M, a fibre-preserving map $i_0 : \tau_M(X) \to \tau_M(Y)$ that sends each almost X-structure s to an almost Y-structure $i_0(s)$. Clearly

$$c_Y[M, i_0(s)] = i_*(c_X[M, s]).$$

Since kernel i_* is generated by $c_X[S^n]$, $\mathfrak{M}_0(X)$ consists precisely of those pairs $(M, s) \in \mathfrak{M}(X)$ for which $i_0(s)$ extends over M. When $(X, Y) = (SO_n, SO)$, $n \neq 1, 3, 7$, the first member M of each pair $(M, s) \in \mathfrak{M}_0(X)$ must, then, be a π-manifold. Lemma 3 shows that every π-manifold is so obtained. When $(X, Y) = (SO_{2k}/U_k, SO/U)$, the first member M of each pair $(M, s) \in \mathfrak{M}_0(X)$ admits a stable almost-complex structure (e.g., some extension of $i_0(s)$). Now, it is not hard to show, using the fact that M^{2k}-interior D has the homotopy type of a $(2k-1)$-dimensional complex, that the association $s \to i_0(s)$ determines a 1-1 correspondence on homotopy classes, so that every M admitting a stable almost-complex structure appears as a first member of some pair in $\mathfrak{M}_0(X)$.

We now let $(X, Y) = (SO_n, SO)$, $n \neq 1, 3, 7$, and define $\varphi: \mathfrak{M}_0(X)$ as follows:

$$\begin{aligned} \varphi(M, s) &= \tfrac{1}{2}\chi[M]c_X[SO_n], \quad && n \text{ even}, \\ &= \chi^*[M]c_X[SO_n], && n \text{ odd} \end{aligned}$$

(see the remark following Lemma 3). It is easily verified that $\varphi: \mathfrak{M}_0(X)$ satisfies (2)–(4). When n is even, the classical characterization of χ by Hopf implies immediately that $\varphi: \mathfrak{M}_0(X)$ satisfies (1) and that null space $\varphi \supset$ null space c_X. When n is odd, these facts can be proved by the methods of [2] or [9]. Thus, Lemma 6 implies that $\varphi = c_X \mid \mathfrak{M}_0(X)$, from which it immediately follows that a π-manifold M^n, $n \neq 1, 3, 7$, is parallelizable if and only if $\chi[M^n] = 0$, if n is even, or $\chi^*[M^n] \equiv 0 \pmod 2$, if n is odd (see [2, Theorem2]). However, when n is even, these results are obtained much more directly by the methods of Lemma 3, and when n is odd, the verification of the crucial properties of φ (in [2] or [9]) amounts to showing that $\varphi = c_X \mid \mathfrak{M}_0(X)$. Thus we state only the following:

COROLLARY 1. *Let $X = SO_n$, $n \neq 1, 3, 7$, and define $\varphi:\mathfrak{M}_0(X)$ by*

$$\begin{aligned}\varphi(M, s) &= \tfrac{1}{2}\chi[M], && n \text{ even}\\ &= \chi^*[M] \pmod 2, && n \text{ odd}.\end{aligned}$$

Properties (1)–(4) are satisfied by φ and uniquely characterize it. (Here' $(M, s) \in \mathfrak{M}_0(X)$ if and only if M is a π-manifold and s is an almost-framing of M that extends as a stable framing over M.)

In the remainder of this section, we deal with the case

$$(X, Y) = (SO_{2k}/U_k, SO/U).$$

Unless stated otherwise, $k > 1$.

Recall that $(M^{2k}, s) \in \mathfrak{M}_0(X)$ if and only if $i_0(s)$ (the stable structure determined by s) extends over M. Let s_0 be any extension. There are then defined Chern classes $c_i(s) \in H^{2i}(M)$, $i = 1, \cdots, k-1$, and $c_i(s_0) \in H^{2i}(M)$, $i = 1, \cdots, k$. Since they are stable invariants, $c_i(s) = c_i(s_0)$, $i = 1, \cdots, k-1$. We denote by $c_k[s_0]$, the evaluation of $c_k(s_0)$ on the orientation class of M. We now define $\varphi:\mathfrak{M}_0(X)$ by

$$\varphi(M^{2k}, s) = \tfrac{1}{2}(\chi[M] - c_k[s_0])c_X[S^{2k}].$$

LEMMA 9. *$\varphi:\mathfrak{M}_0(X)$ is well-defined.*

Proof. We must show that $\varphi(M^{2k}, s)$ does not depend on the choice of extension s_0. When $k = 2l$, this is an immediate consequence of the well-known formula

$$\begin{aligned}(-1)^l p_l[M^{2k}] &= 2c_k[s_0] + \textstyle\sum_{i=1}^{k-1} (-1)^i c_i c_{k-i}[s_0]\\ &= 2c_k[s_0] + \textstyle\sum_{i=1}^{k-1} (-1)^i c_i c_{k-i}[s].\end{aligned}$$

Here, $p_l[M^{2k}]$ is the evaluation of the top Pontrjagin class of M^{2k} on the orientation generator, and $c_i c_{k-i}[s_0]$ (resp., $c_i c_{k-i}[s]$) is the evaluation of $c_i(s_0) \cup c_{k-i}(s_0)$ (resp., $c_i(s) \cup c_{k-i}(s)$). Clearly, these values depend only on M and s. Thus, so does the value $\frac{1}{2}(\chi[M] - c_k[s_0])$.

When k is odd and >3, the conclusion follows from the next lemma.

LEMMA 10. *Suppose that k is odd and >3, that $(M^{2k}, s) \in \mathfrak{M}_0(X)$, and that s_0 and s_1 are two stable extensions of s over M^{2k}. Then*

$$c_k[s_0] \equiv c_k[s_1] \pmod{2\cdot \text{order } c_X[S^{2k}]}.$$

Proof. Note that

$$\begin{aligned}2\cdot\text{order } c_X[S^{2k}] &= 2[(k-1)!], && k \equiv 1 \pmod 4\\ &= (\mathrm{k}-1)!, && k \equiv 3 \pmod 4.\end{aligned}$$

Let N = domain s, and let $p: M^{2k} \to S^{2k}$ be the degree-one map obtained by collapsing N to a point. Let $\alpha_i \in \widetilde{K}U(M^{2k})$ be the classes determined by s_i, $i = 0, 1$, respectively. Since the restrictions of α_i to N are equal, there is a $\beta \in \widetilde{K}U(S^{2k})$ such that $\alpha_0 - \alpha_1 = p^*(\beta)$. (To see this, use for example, the exact $\widetilde{K}U$ sequence of the pair (M^{2k}, N).) According to Atiyah and Hirzebruch [1], $c_k(\beta)$ is divisible by $(k - 1)!$. Thus

$$c_k(\alpha_0) \equiv c_k(\alpha_1) \pmod{(k-1)!},$$

which proves the desired result for $k \equiv 3 \pmod 4$.

When $k \equiv 1 \pmod 4$ we make use of the fact that $\widetilde{K}U(S^{2k}) \cong Z$, that $\widetilde{K}O(S^{2k}) \cong Z_2$, and that realification $r: \widetilde{K}U(S^{2k}) \to \widetilde{K}O(S^{2k})$ is, in this case, an epimorphism. The hypotheses imply that the equivalence $\alpha_0 \mid N \approx \alpha_1 \mid N$ extends over M^{2k} as a real equivalence,[3] so that β satisfies $r(\beta) = 0$. Thus, β is an even class in $\widetilde{K}U(S^{2k})$, and so

$$c_k(\alpha_0) \equiv c_k(\alpha_1) \pmod{2\,[(k-1)!]}. \qquad \text{Q.E.D.}$$

When $k = 3$, $\pi_{2k-1}(X) = 0$, so that $\varphi: \mathfrak{M}_0(X)$ is well-defined in this case too.

It is easy to verify that φ satisfies (1)–(4). Now, suppose that $c_X[M, s] = 0$, and let t be an (unstable) extension of s. Then, $i_0(t)$ is an extension of $i_0(s)$. By standard results, (e.g. see [4] or [12, p. 65]), $c_k[i_0(t)] = \chi[M]$, so that $\varphi(M, s) = 0$, that is, null space $\varphi \supset$ null space c_X. Thus we have

Corollary 2. *For any $k \geq 1$, let M^{2k} be a smooth, closed, compact, connected, oriented manifold, admitting an almost X-structure s, $X = SO_{2k}/U_k$, that extends over M as a stable structure s_0. Then*

$$c_X[M^{2k}, s] = \tfrac{1}{2}(\chi[M^{2k}] - c_k[s_0])c_X[S^{2k}].$$

Proof. For $k \neq 1, 3$, this follows from Lemma 6. When $k = 1, 3$, $\pi_{2k-1}(X) = 0$, so that the result holds in these cases as well, Q.E.D.

Remark. When $k \equiv 2 \pmod 4$, this result is equivalent to Theorem II of [11]. (Cf. Example c), §2). When $k \equiv 0 \pmod 4$, Theorem II of [11] can be obtained by considering, for any $(M, s) \in \mathfrak{M}(X)$, the pair $(M + M, s + s) \in \mathfrak{M}_0(X)$ and applying Corollary 2 to it. That $i_0(s + s)$ extends over $M + M$ follows from the equalities

$$c_Y[M + M, i_0(s + s)] = i_*(c_X[M + M, s + s]) = 2i_*(c_X[M, s])$$

and the fact that $\pi_{2k-1}(Y) \cong Z_2$.

We allow $k = 1$ in the remaining results.

Corollary 3. *A stable almost-complex structure t on M^{2k} admits a reduction to an almost-complex structure, if and only if $c_k[t] = \chi[M^{2k}]$.*

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[3] We abuse terminology here by identifying α_i with complex vector bundles of high fibre dimension.

Remark. This result is known, but I know of no proof in the literature.

Proof. The necessity of the relation is standard (see [4] or [12, p. 65]).

Let s be an almost X-structure ($X = SO_{2k}/U_k$) on M^{2k} such that $i_0(s)$ has an extension s_0 homotopic to t. By the remarks preceding Corollary 1, such an s always exists. Then,

$$\begin{aligned} c_X[M, s] &= \tfrac{1}{2}(\chi[M] - c_k[s_0])c_X[S^{2k}] \\ &= \tfrac{1}{2}(\chi[M] - c_k[t])c_X[S^{2k}] = 0, \end{aligned}$$

so that s extends to an almost-complex structure s_1. Let $\alpha_1, \alpha_2 \in \widetilde{KU}(M^{2k})$ be determined by s_1 and t, respectively, and let $N =$ domain s. Then, α_1 and α_2 have the same restriction to N; moreover, $c_k[\alpha_1] = \chi[M^{2k}] = c_k[\alpha_2]$. As in the proof of Lemma 10, there is a $\beta \in \widetilde{KU}(S^{2k})$ with $p^*(\beta) = \alpha_1 - \alpha_2$, where $p: M^{2k} \to S^{2k}$ is the degree-one map obtained by collapsing N to a point. But, then, β satisfies $p^*(c_k(\beta)) = 0$, so that $Ch_k(\beta) = 0$. Since

$$Ch: KU^*(S^{2k}) \to H^*(S^{2k}; Q)$$

is injective, it follows that $\beta = 0$, so that $\alpha_1 = \alpha_2$, Q.E.D.

COROLLARY 4. *Suppose that M^{2k} admits an almost X-structure. When $k \not\equiv 0 \pmod 4$, M^{2k} is oriented-cobordant to an almost-complex manifold. When $k \equiv 0 \pmod 4$, $M^{2k} + M^{2k}$ is oriented-cobordant to an almost-complex manifold.*

Proof. Let s be an almost X-structure on M^{2k}. When $k \not\equiv 0 \pmod 4$, $c_X[M^{2k}, s]$ is a multiple of $c_X[S^{2k}]$, so that Lemma 5 applies. There exists a product of spheres P^{2k} and a non-negative integer B, such that $M^{2k} + BP^{2k}$ admits an almost-complex structure. Clearly, M^{2k} is oriented-cobordant to $M^{2k} + BP^{2k}$. When $k \equiv 0 \pmod 4$, $c_X[M^{2k} + M^{2k}, s + s]$ is a multiple of $c_X[S^{2k}]$. Apply Lemma 5 again. (When $k = 1$, all oriented $2k$-manifolds have complex structures.) Q.E.D.

COROLLARY 5. *Every even-dimensional stably almost-complex manifold is complex-cobordant to an almost-complex manifold.*

Proof. It suffices to show that, given any manifold M_1^{2k} with stable almost-complex structure s_1, there exists a manifold M_2^{2k} with stable almost-complex structure s_2 admitting a reduction to an almost-complex structure, such that the Chern numbers determined by s_1 and s_2 are the same (see [3]).

Given M_1^{2k} and s_1, as above, let $A = \frac{1}{2}(\chi[M_1] - c_k[s_1])$. By Lemma 5 and Corollary 2, there exists a product of spheres P such that $M_2^{2k} = M_1^{2k} + |A| P$ admits an almost-complex structure, whose stabilization we call s_2. It is easy to show that Chern numbers involving non-top-dimensional Chern classes are unchanged by connected-summation with P. Moreover, $c_k[s_2] = \chi[M_2^{2k}] = \chi[M_1^{2k}] - 2A = c_k[s_1]$, Q.E.D.

Remark. Corollary 5 is a special case of a result of Milnor (see [3, pp. 122–127]), no proof of which appears in print.

COROLLARY 6. *For every $k \geq 1$, there are closed, compact, connected oriented $8k$-manifolds, M_1 and M_2 such that*

(i) *M_1 has the same oriented homotopy type as M_2*

(ii) *M_1 admits an almost-complex structure, whereas M_2 does not.*

Proof. Suppose that N^{8k} admits a stable almost-complex structure s, and suppose that $c_i(s) = 0$, unless $i = 2k, 4k$. Then, the Chern classes $c_{2k}(s)$ and $c_{4k}(s)$ can be expressed in terms of the Pontrjagin classes of N^{8k}, $p_k(N^{8k})$ and $p_{2k}(N^{8k})$. Therefore, by Corollary 3, s reduces to an almost-complex structure if and only if

$$l[N^{8k}] = \chi[N^{8k}],$$

where $l[N^{8k}]$ is a certain linear combination of the Pontrjagin numbers $p_k^2[N^{8k}]$ and $p_{2k}[N^{8k}]$ of N^{8k}. Note that this relation is independent of s. Thus, *N^{8k} admits an almost-complex structure if and only if $l[N^{8k}] = \chi[N^{8k}]$.*

Now, consider two (4k-1)-connected 8k-manifolds N_1 and N_2 that have the same oriented-homotopy type and satisfy $l[N_1] \neq l[N_2]$. Examples of such manifolds are constructed in [6], [7], and in [17]. Obstructions to the existence of an almost X-structure s_i on N_i vanish trivially, $i = 1, 2$. Thus $s_i + s_i$ is an almost X-structure on $N_i + N_i$; the remark following Corollary 2 shows that $s_i + s_i$ extends to a stable almost-complex structure on $2N_i = N_i + N_i$. Let

$$A = \chi[N_1] - l[N_1] - 1,$$

and let $P = S^1 \times S^{8k-1}$, if $A \geq 0$, and $P = S^2 \times S^{8k-2}$, if $A < 0$. P, being a π-manifold, admits a trivial stable almost-complex structure, so that $M_i = 2N_i + |A| P$ admits a stable almost-complex structure with $c_l(M_i) = 0$, unless $l = 2k, 4k$. We compute $\chi[M_i]$.

$$\begin{aligned} \chi[M_i] &= \chi[2N_i] - 2A \\ &= 2\chi[N_i] - 2 - 2\chi[N_1] + 2 + 2l[N_1] \\ &= l[M_1], \end{aligned}$$

the last equality coming from the facts that $\chi[N_1] = \chi[N_2]$ and that $2l[N_1] = l[2N_1] = l[2N_1 + |A| P] = l[M_1]$. Therefore, $\chi[M_1] = l[M_1]$, so that M_1 admits an almost-complex structure, whereas $\chi[M_2] = l[M_1] \neq l[M_2]$, so that M_2 does not, Q.E.D.

4. Proofs

4.1. *Proof of Lemma* 2. We must show that $\tau_M | N \approx \tau_{-M} | N$, where $N = M^n - \text{interior } D^n$.

Let β be the unstable, oriented tangent vector bundle of M^n. Since N has the homotopy type of CW complex of dimension strictly less than n, $\beta | N$

admits a cross-section c, which determines a splitting $\alpha \oplus \varepsilon$ of $\beta \mid N$. Here ε is a trivial, oriented line bundle. Let I_α and I_ε be the identity maps of α and ε, respectively. Then $h = I_\alpha \oplus (-I_\varepsilon)$ is an orientation-reversing bundle equivalence $\beta \mid N \to \beta \mid N$, which implies the desired result, Q.E.D.

Remark. Let h_0 be the unstable principal bundle equivalence determined by h, and let h_1 be the stable principal bundle equivalence determined by h. For any trivial, oriented k-plane bundle ε_k over M^n, let I_k be the identity map of $\varepsilon_k \mid N$. Then, the self-equivalence $I_k \oplus h$ of $\varepsilon_k \oplus \beta \mid N$ determines the same stable equivalence h_1.

We may regard β as a sub-bundle of $\varepsilon_k \oplus \beta$ and c as a cross-section of $\varepsilon_k \oplus \beta$. If $k \geq 1$, c extends over M^n as a cross-section of $\varepsilon_k \oplus \beta$. This implies that the equivalence $I_k \oplus h$ extends over $\varepsilon_k \oplus \beta$.

Thus, h_0 has the property that the stable principal bundle equivalence corresponding to it extends over the entire stable tangent bundle.

When n is odd, c extends as a cross-section of β, so that, in this case, h_0 itself extends over the entire unstable tangent bundle.

Definition. Let $h\colon \tau_M \mid N \to \tau_{-M} \mid N$ be a principal bundle equivalence. We call h *admissible* if (i) for even n, the stable equivalence corresponding to h extends over the entire stable tangent bundle; (ii) for odd n, h itself extends over τ_M.

In the stable case, this definition implies that h is admissible if and only if it extends.

Note that our proof of Lemma 2 shows that there always exists an admissible equivalence $\tau_M \mid N \to \tau_{-M} \mid N$.

4. 2. *Pull-backs.* Let $f : Y_1 \to Y_2$ be a continuous map covered by a bundle map $\bar{f} : \xi_1 \to \xi_2$. If there exist partial cross-sections s_α of ξ_α, $\alpha = 1, 2$, such that $s_2 \circ f \mid \text{domain } s_1 = \bar{f} \circ s_1$, then we call s_1 a *pull-back* of s_2. If $\tilde{f}$ is a principal bundle map inducing $\bar{f}$, we may say that s_1 is a pull-back of s_2 determined by $\tilde{f}$. Note that we do not, in general, require that domain $s_1 = f^{-1}$ (domain s_2).

We now specialize to the case in which s_2 is an almost X-structure with domain M_2^n-interior D_2^n, for some closed disc D_2^n smoothly imbedded in M_2^n, and f is a map of pairs

$$(M_1^n, M_1^n\text{-interior } D_1^n) \to (M_2^n, M_2^n\text{-interior } D_2^n)$$

covered by some principal bundle map $\tau_{M_1} \to \tau_{M_2}$. This principal bundle map determines a bundle map $\tau_{M_1}(X) \to \tau_{M_2}(X)$ and a corresponding pull-back

$$s_1 : M_1^n\text{-interior } D_1^n \to \tau_{M_1}(X) \mid M_1^n\text{-interior } D_1^n$$

of s_2. The definition of s_1 is obvious. Note that if s_2 extends over D_2^n, then s_1 extends over D_1^n.

We can describe $c_X[M, s]$ by means of pull-backs. Let s be an almost X-structure defined on M-interior D, as before. Orient D concordantly with M and ∂D concordantly with D. We identify τ_D with $D \times G$. There is a bundle equivalence $D \times G \to \tau_M \mid D$ that determines a pull-back

$$s_0 : \partial D \to \partial D \times X$$

of s. Then, $c_X[M, s]$ is represented by the composition of s_0 with the projection $\partial D \times X \to X$. Note that the class is independent of the initial choice of equivalence $D \times G \to \tau_M \mid D$.

Now, suppose that $s_1 : \partial D \to \partial D \times X$ is the pull-back of s_0 determined by a bundle equivalence $k : \partial D \times G \to \partial D \times G$. Let $\hat{k} : \partial D \to G$ be the map given by $k(x, g) = (x, \hat{k}(x) \cdot g)$, and recall that $\pi_X : G \to X$ is defined by $\pi_X(g) = g \cdot x_0$, for some fixed $x_0 \in X$. Let $c_X[k]$ denote the homotopy class of $\pi_X \circ \hat{k}$. The following easy result is well known:

LEMMA 11. *The composition*

$$\partial D \xrightarrow{s_1} \partial D \times X \to X$$

represents

$$c_X[M, s] - c_X[k].$$

We use this to prove Lemma 7.

Divide S^n into hemispheres D_1 and D_2 oriented concordantly with it, and orient $\partial D_1 = \partial D_2$ concordantly with D_1. Let

$$h_\alpha : D_\alpha \times G \to \tau_{S^n} \mid D_\alpha$$

be bundle maps, $\alpha = 1, 2$, and let $k = h_2^{-1} \circ h_1 \mid \partial D_1 \times G$. The corresponding map

$$\hat{k} : \partial D_1 \to G$$

is called a *characteristic map* for τ_{S^n} (cf. [16, p. 97]); any two such characteristic maps are homotopic.

Proof of Lemma 7. We must show that $\pi_X \circ \hat{k}$ represents $c_X[S^n]$. For any cross-section $s : D_2 \to \tau_{S^n}(X) \mid D_2$, let $s_\alpha : \partial D \to \partial D_\alpha \times X$ be the pull-back of s by $h_\alpha \mid \partial D_\alpha \times G$, $\alpha = 1, 2$. The composition

$$\partial D_1 \xrightarrow{s_1} \partial D_1 \times X \to X$$

represents $c_X[S^n, s] = c_X[S^n]$.

With k as above, it is easily checked that s_2 is the pull-back of s_1 determined by k. Thus, by Lemma 11, the composition

$$\partial D_2 \xrightarrow{s_2} \partial D_2 \times X \to X$$

represents $c_X[S^n] - c_X[k]$. Since s extends over D_2, so does s_2, so $c_X[S^n] - c_X[k] = 0$, as desired, Q.E.D.

4.3. *Proof of Theorem* 1. Let s and $-s$ be almost X-structures of M^n and $-M^n$, respectively, with domain $N = M^n$-interior D^n, and suppose that s is the pull-back of $-s$ determined by some admissible bundle equivalence

$$h : \tau_M \mid N \to \tau_{-M} \mid N.$$

In the stable case, or when n is odd, h extends over τ_M so that, by naturality of obstructions, $c_X(M, s) = c_X(-M, -s)$. Thus,

$$c_X[-M, -s] = \langle c_X(-M, -s), -\mu\rangle = -\langle c_X(M, s), \mu\rangle = -c_X[M, s],$$

as desired.

In the unstable case with n even, we have a commutative diagram of bundle equivalences

$$\begin{array}{ccc} \tau_M \mid \partial D & \xrightarrow{h} & \tau_{-M} \mid \partial D \\ g_1 \uparrow & & \uparrow g_2 \\ \partial D \times SO_n & \xrightarrow{k} & \partial D \times SO_n . \end{array}$$

Indeed, g_1 and g_2 are chosen so that they extend over $D \times SO_n$, and k is chosen to be $g_2^{-1} \circ h \circ g_1$. Let $\hat{k}$, as usual, be given by $k(x, g) = (x, \hat{k}(x) \cdot g)$. *Orient* ∂D *concordantly with* $-M^n$. Note that, since the stable equivalence corresponding to h extends over the entire stable tangent bundle, the homotopy class of $\hat{k}$ lies in kernel $(i_* : \pi_{n-1}(SO_n) \to \pi_{n-1}(SO))$, and so, by Lemma 7, this class is of the form $m_h c_{SO_n}[S^n]$, for some integer m_h depending only on h.

Let s_1 be the pull-back of s determined by g_1, and let s_2 be the pull-back of $-s$ determined by g_2. Then, s_1 is the pull-back of s_2 determined by k. Let σ_α be the homotopy class of the composition

$$\partial D \xrightarrow{s_\alpha} \partial D \times X \to X, \qquad \alpha = 1, 2.$$

Then

$$\sigma_1 = -c_X[M, s], \qquad \sigma_2 = c_X[-M, -s],$$

$$\sigma_1 = \sigma_2 - c_X[k] = \sigma_2 - m_h c_X[S^n].$$

Therefore, we have

$$c_X[-M_1 - s] = -c_X[M, s] + m_h\, c_X[S^n].$$

Since m_h is independent of $\pm s$ and X, we may specialize to the case $c_X = \chi$ (example (b), §2). Thus,

$$\chi[-M] = -\chi[M] + 2m_h .$$

Since $\chi[-M] = \chi[M]$, the desired result follows, Q.E.D.

4.4. *Connected sums*. Let $D^n(r)$ be the closed disc of radius r in Euclidean n-space and $S^{n-1}(r)$ its boundary, both given the standard Euclidean orientation. Let

$$A^n = D^n(2) - \text{interior } D^n(\tfrac{1}{2}),$$

and define $\rho : A^n \to A^n$ by $\rho(x) = R(x)/ \| x \|^2$, where R is reflection through the hyperplane $x_n = 0$. The map ρ is an orientation-preserving diffeomorphism whose differential determines a bundle map $k : A^n \times G \to A^n \times G$. Notice that ρ sends $S^{n-1}(\frac{1}{2})$ to $S^{n-1}(2)$ with degree -1.

Given orientation-preserving imbeddings $i_\alpha : D^n(2) \to$ interior M^n_α, $\alpha = 1, 2$, we form subspaces $N\alpha \subset P_\alpha \subset M^n_\alpha$ by defining

$$P_\alpha = M^n_\alpha\text{-interior } i_\alpha(D^n(\tfrac{1}{2}))$$

and

$$N_\alpha = M^n_\alpha\text{-interior } i_\alpha(D^n(2)).$$

We then take the disjoint union $P_1 \cup P_2$ and identify $i_1(x)$ with $i_2(\rho(x))$, for all $x \in A^n$, obtaining a topological manifold $M^n_1 + M^n_2$. This manifold admits a smoothness structure characterized up to diffeomorphism by the property that the inclusions $P_\alpha \subset M^n_1 + N^n_2$ are smooth imbeddings. We call $M^n_1 + M^n_2$ with such a smoothness structure the *connected sum* of M^n_1 and M^n_2. For further details, see [13].

It is easy to show that $\tau_{M_1+M_2}$ is obtained from $\tau_{M_1} \mid P_1 \cup \tau_{M_2} \mid P_2$ by identifying $\tau_{M_1} \mid i_1(A^n)$ with $\tau_{M_2} \mid i_2(A^n)$ via the bundle map induced by the differential of

$$i_2 \circ \rho \circ i_1^{-1} : i_1(A^n) \to i_2(A^n).$$

An analogous construction yields $\tau_{M_1+M_2}(X)$, for any X. It follows that

$$\tau_{M_1}(X) \mid N_1 \cup \tau_{M_2}(X) \mid N_2 = \tau_{M_1+M_2}(X) \mid N_1 \cup N_2 .$$

Completion of the proof of Lemma 1. Suppose s_α is an almost X-structure on M^n_α with domain N_α, $\alpha = 1, 2$. Then, $s_1 \cup s_2$ can be regarded as a cross-section of $\tau_{M_1+M_2}(X) \mid N_1 \cup N_2$.

Notice that

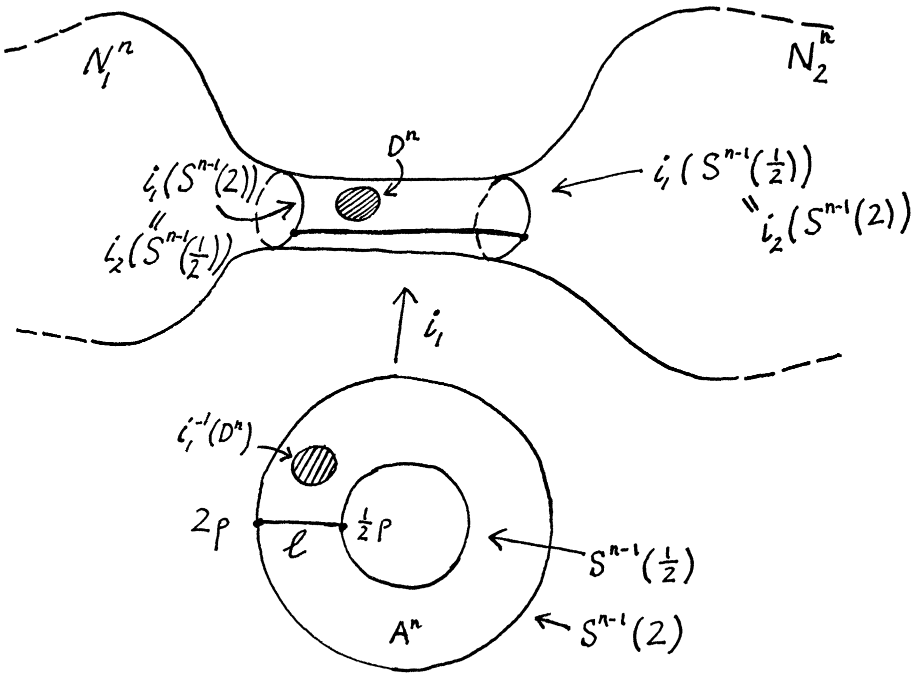
$$M^n_1 + M^n_2 - (N_1 \cup N_2) = i_1(\text{interior } A^n) = i_2(\text{interior } A^n)$$

(cf. the diagram below), and that $\tau_{M_1+M_2}(X) \mid i_1(A^n)$ is trivial. Let $p = (1, 0, \cdots, 0) \in D^n(2)$, and let $l \subset A^n$ be the radial line joining $2p$ and $\frac{1}{2} p$. Since X is connected, $s_1 \cup s_2$ extends to a cross-section of

$$\tau_{M_1+M_2}(X) \mid N_1 \cup N_2 \cup i_1(l).$$

Any closed disc D^n imbedded in the complement of $N_1 \cup N_2 \cup i_1(l)$ has the property that $N_1 \cup N_2 \cup i_1(l)$ is a deformation retract of $M^n_1 + M^n_2$-interior D^n (cf. the diagram below). Thus, $s_1 \cup s_2$ extends to an almost X-structure of $M^n_1 + M^n_2$,

DEFINITION. We call any almost X-structure of $M^n_1 + M^n_2$ extending $s_1 \cup s_2$ *compatible* with s_1 and s_2, and we label it $s_1 + s_2$.



The proof given above shows that for given s_1 and s_2, a compatible almost X-structure on $M_1^n + M_2^n$ always exists.

4.5. *Proof of Theorem* 2. Using the connectedness and (n − 1)-simplicity of X, it is not hard to show that the class $c_X[M_1 + M_2, s_1 + s_2]$ is independent of the particular choice of D^n and extension of $s_1 \cup s_2$.

Indeed, it is clear that when we use the bundle map

$$k_1 : A^n \times G \rightarrow \tau_{M_1+M_2} \mid i_1(A^n)$$

induced by the differential of i_1 to pull back s_1 to a cross-section

$$t_1 : S^{n-1}(2) \rightarrow S^{n-1}(2) \times X$$

and s_2 to a cross-section

$$t_2 : S^{n-1}(\tfrac{1}{2}) \rightarrow S^{n-1}(\tfrac{1}{2}) \times X,$$

then, letting $c_\alpha \in \pi_{n-1}(X)$ be represented by t_α followed by projection onto X, $\alpha = 1, 2$, we have

$$c_X[M_1 + M_2, s_1 + s_2] = c_1 - c_2 .$$

Since k_1, viewed as a bundle map $A^n \times G \rightarrow \tau_{M_1} \mid i_1(A^n)$, extends to

$$D^n(2) \times G \rightarrow \tau_{M_1} \mid i_1(D^n(2)),$$

it follows that $c_1 = c_X[M_1, s_1]$, so that

$$c_X[M_1 + M_2, s_1 + s_2] = c_X[M_1, s_1] - c_2 .$$

Now, let $k_2 : A^n \times G \to \tau_{M_1+M_2} \mid i_2(A^n)$ be the bundle map induced by the differential of i_2, and note that $k_1 = k_2 \circ k$ (where k is as in §4.4). Therefore, t_2 can be obtained by pulling back s_2 by k_2, getting the cross-section

$$t_2' : S^{n-1}(2) \to S^{n-1}(2) \times X,$$

and then pulling t_2' back by k. Since k_2, viewed as a bundle map

$$A^n \times G \to \tau_{M_2} \mid i_2(A^n),$$

extends to

$$D^n(2) \times G \to \tau_{M_2} \mid i_2(D^n(2)),$$

it follows that the composition

$$S^{n-1}(2) \xrightarrow{t_2'} S^{n-1}(2) \times X \to X$$

represents $c_X[M_2, s_2]$.

Therefore, recalling that $k \mid S^{n-1}(\frac{1}{2}) \times G$ covers the map $\rho \mid S^{n-1}(\frac{1}{2})$, which has degree -1, a slight modification of Lemma 11 yields

$$-c_2 = c_X[M_2, s_2] - c_X[k],$$

where $c_X(k)$, as usual, is represented by $\pi_X \circ \hat{k}$ and $k(x, g) = (\rho(x), \hat{k}(x) \cdot g)$. Combining this with our previous equality, we obtain,

$$c_X[M_1 + M_2, s_1 + s_2] = c_X[M_1, s_1] + c_X[M_2, s_2] - c_X[k].$$

We could evaluate $c_X[k]$ directly by examining k. Instead, we make use of the simple observation that $c_X[k]$ is independent of M_1, M_2, s_1, and s_2, so that we may specialize: let $M_\alpha^n = S^n$, $\alpha = 1, 2$, and let s_α be arbitrary. Then, since $M_1^n + M_2^n = S^n$,

$$c_X[M_1 + M_2, s_1 + s_2] = c_X[M_1, s_1] = c_X[M_2, s_2] = c_X[S^n],$$

which, together with the above equality, yields the desired value for $c_X[k]$, Q.E.D.

Appendix

As specified in Definition 2.2, an almost X-structure on a manifold M^n is an X-structure on M^n-interior D^n, where D^n is *some* closed n-disc smoothly imbedded in M^n. The purpose of this section is to sketch a justification for the following assertion: *For purposes of studying:* (i) *the question of existence of almost X-structures,* (ii) *the homotopy classification problem for almost X-structures,* (iii) *the extension problem for almost X-structures* (*e.g., possible values of obstructions to extending*), (iv) *the homotopy classification problem for extensions* *of a fixed almost X-structure;—the particular choice of the imbedded disc $D^n \subset$ interior M^n is irrelevant.*

The principal tool that we use to justify the above statement is the following result due to Palais and Cerf [15].

Let M be a closed, compact, oriented, connected n-manifold, and let $f, g : D^n[0, 2] \to M$ be orientation-pereserving imbeddings. Then, there exists a diffeomorphism $H : M \to M$, diffeotopic to the identity, with $H \circ f = g$.

Thus, there are diffeomorphisms $M \to M$ taking any one disc imbedded in M onto any other. The differentials of such diffeomorphisms determine bundle maps $\tau_M(X) \to \tau_M(X)$ which can be used to pull back almost X-structures defined on the complement of one open disc to almost X-structures defined on the complement of another. Such a pull-back procedure determines a 1-1 correspondence between structures over one complement and structures over another; the correspondence preserves homotopy classes, sends extendible structures to extendible ones, and extensions to extensions. Indeed, corresponding extensions are homotopic as cross-sections $M \to \tau_M(X)$, the homotopy determined by the diffeotopy to the identity. Finally, the naturality of the obstruction classes $c_X(M, s)$ and the fact that the diffeomorphisms used have degree one imply that if s and t are almost X-structures that correspond as described above, then $c_X[M, s] = c_X[M, t]$.

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