ON THE GENUS OF 3-MANIFOLDS

BY
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Introduction[1]

We shall be concerned here with the geometric structure of closed (i.e. compact and without boundary) orientable connected 3-dimensional manifolds. "Manifold" will be used henceforth to mean "closed orientable connected 3-manifold". Since 3-manifolds can be triangulated [1], there is no real distinction between manifolds and combinatorial 3-manifolds. Consequently, manifolds have Heegaard splittings [11, p. 219, Satz]; thus we can speak of the genus of a manifold M, denoted $g(M)$ [10, §16].

In [7], Milnor proved that every manifold M other than S^3 is isomorphic (i.e. piecewise linearly homeomorphic) to a finite connected sum of manifolds M_i indecomposable with respect to the connected sum operation; no M_i is S^3. (See [7] for a definition of "connected sum" together with some of its properties. We denote the connected sum of manifolds N_1 and N_2 by $N_1 \# N_2$.) This decomposition of M is unique up to the ordering of the M_i; and further, each M_i is either isomorphic to $S^2 \times S^1$ or else $\pi_2(M_i) = 0$.

The main theorem of this paper (Theorem 1) states that if $M_1 \# \cdots \# M_n$ is such a decomposition of M into indecomposable manifolds M_i, then

$$g(M) = \sum_{i=1}^{n} g(Mi).$$

This is related to a question mentioned by Papakyriakopoulos [10, §16]; namely, given a manifold M, find its genus.

This theorem is a consequence of a result of Haken [4]. As in [4], we define a polyhedral sphere S in M to be *incompressible* if S does not bound a 3-cell in M. Then a simplified form of Haken's result can be stated as follows (see the lemma of [4]):

Theorem 0. *Let M be the union of handlebodies H_1 and H_2 such that $\partial H_i = H_1 \cap \dot{H}_2 = T$, a closed orientable 2-manifold. Suppose M contains an incompressible sphere. Then there is an incompressible sphere S in M such that $S \cap T$ is a single simple closed curve L. L is not contractible on T.*

Related to the study of decompositions of manifolds is Kneser's conjecture [10, §17] which asks if a free product decomposition of the fundamental group

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of a 3-manifold N (N possibly with boundary), $\pi(N) \cong A * B$, can be "realized" by a connected sum decomposition: $N = N_1 \# N_2$, where $\pi(N_1) \cong A$, $\pi(N_2) \cong B$.[2] (Note: by van Kampen's theorem, the fundamental group of a connected sum is the free product of the fundamental groups of the factors.) In his thesis, Stallings showed the answer is "yes" for N orientable; and in fact he showed more than this, since he considers homomorphisms of $\pi(N)$ onto an arbitrary free product, and also takes into account the non-orientable case. (See [13], p. 25, Theorem.) What is shown in this paper (Theorem 2) is that if N is a manifold and $\pi(N) \cong A * B$, then there are manifolds N_1 and N_2 such that $\pi(N_1) \cong A$, $\pi(N_2) \cong B$, $N \approx N_1 \# N_2$, and $g(N) = g(N_1) + g(N_2)$.

In [8], Papakyriakopoulos proved that, modulo the Poincaré conjecture, every manifold with free fundamental group is a 3-sphere with handles. We prove a related result (Theorem 3) without hypothesizing the Poincaré conjecture.

The following group-theoretic results are used in this paper:

PROPOSITION 1. (Gruško's theorem for finitely generated groups). *If F is a finitely generated free group, and $\phi : F \to A * B$ is a homomorphism onto the free product of A and B, then there exists a free factorization, $F \cong F_A * F_B$, such that $\phi(F_A) \subset A$ and $\phi(F_B) \subset B$. In particular, a finitely generated group is the free product of finitely many finitely generated groups, each of which is indecomposable with respect to free product. Also, if $A * B$ is finitely generated,* $\text{rank}(A * B) = \text{rank}(A) + \text{rank}(B)$. (See [13, p. 23]; [5, p. 58].)

PROPOSITION 2. *Let G be a group and suppose*

$$G \cong A_1 * \cdots * A_n \cong B_1 * \cdots * B_m$$

where A_i and B_j are non-trivial indecomposable groups. Then $m = n$ and B_1, $\cdots$, B_n can be rearranged to yield $B_{j_1}, \cdots, B_{j_n}$ where $B_{j_i} \cong A_i$ [6, p. 245]. *This is a corollary of the Kurosh subgroup theorem.*

Proofs

THEOREM 1. *Let the manifold M be isomorphic to $M_1 \# \cdots \# M_n$, where each M_i is indecomposable. Then*

$$g(M) = \textstyle\sum_{i=1}^{n} g(M_i).$$

Proof. There is no loss of generality in assuming that $g(M) > 0$. (See [9, p. 256, Theorem 2.1].) If $g(M) = 1$, we have two cases.

Case 1. $M \approx S^2 \times S^1$. Then M is indecomposable, and the theorem follows [7, p. 2, Lemma 2].

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[2] $\pi(N)$ means $\pi_1(N, x)$ for some $x \in N$. The fundamental group is of interest here only as an abstract group; so the basepoint is not important.

Case 2. M is a lens space. We claim that every polyhedral sphere S in M bounds a cell. (This implies that M is indecomposable by [7, Lemma 1].) But the universal cover of M is S^3. S lifts to a collection of disjoint spheres in S^3; by [9, p. 256], at least one of these bounds a cell not containing any of the others. Hence S bounds a cell.

Suppose the theorem is true for manifolds whose genus is at most k. Let M be a manifold of genus $k + 1 > 1$. Thus $M = H_1 \cup H_2$, H_i a handlebody of genus $k + 1$, $\partial H_i = T = H_1 \cap H_2$. If M is indecomposable, there is nothing to prove. Otherwise, M contains an incompressible sphere, S. By Theorem 0, we can assume $S \cap T$ is a simple closed curve L not contractible on T. By Lemma 7.2 of [9] and Dehn's lemma [8], we can assume that L separates T. Consequently, cutting along S and attaching 3-cells, we obtain two manifolds, M' and M'', $M \approx M' \# M''$. Clearly $g(M') + g(M'') \leq g(M)$; for by construction, M' has a Heegaard splitting of genus p and M'' a splitting of genus q, such that $p + q = k + 1$.

Now $g(M')$ and $g(M'')$ are both $\leq k$. Hence

$$M' \approx M_1' \# \cdots \# M_r' \quad (M_i' \text{ indecomposable})$$

and

$$M'' \approx M_1'' \# \cdots \# M_s'' \quad (M_i'' \text{ indecomposable}).$$

Furthermore

$$g(M') = \textstyle\sum_{i=1}^r g(M_i'), \qquad g(M'') = \sum_{i=1}^s g(M_i'').$$

Now $M \approx M_1' \# \cdots \# M_s''$. But

$$g = \textstyle\sum_{i=1}^r g(M_i') + \sum_{i=1}^s g(M_i'') \leq k + 1 = g(M).$$

On the other hand, M has a Heegaard splitting of genus g (constructed by fitting together the Heegaard splittings of the two factors in the obvious way). Thus $g(M) \leq g$. We conclude $g(M) = g$.

Now if $M \approx M_1 \# \cdots \# M_n$ is any other factorization of M into indecomposable manifolds, we have by [7], that $n = r + s$, and the M_i can be rearranged so that they are isomorphic to $M_1' \cdots, M_s''$. Hence

$$g(M) = \textstyle\sum_{i=1}^n g(M_i).$$

This completes the proof.

THEOREM 2. (Kneser's Conjecture for orientable closed 3-manifolds). *Let M be a manifold such that $\pi(M) \cong A * B$. Then M is isomorphic to $M_1 \# M_2$ where $\pi(M_1) \cong A$, $\pi(M_2) \cong B$, and $g(M) = g(M_1) + g(M_2)$.*

Proof. We can assume that $A, B \neq 1$.

$M \approx M_1 \# \cdots \# M_n$ where M_i is either $S^2 \times S^1$ or else $\pi_2(M_i) = 0$. By [3], $G_i = \pi(M_i)$ is not a non-trivial free product. Suppose $M_1, \cdots, M_k$ are not homotopy-spheres, and $M_{k+1}, \cdots, M_n$ are. Then $\pi(M) = G_1 * \cdots * G_k$. Since $\pi(M)$ is finitely generated, so are A and B. By Proposition 1,

$$A = A_1 * \cdots * A_r \quad \text{and} \quad B = B_1 * \cdots * B_s,$$

where A_i and B_i are indecomposable with respect to free product. By Proposition 2, $k = r + s$, and the G_i can be rearranged into the sequence $G_{i_1}, \cdots, G_{i_k}$ so that

$$G_{i_1} \cong A_1, \cdots, G_{i_r} \cong A_r, G_{i_{r+1}} \cong B_1, \cdots, G_{i_k} \cong B_s.$$

Take

$$M_1 \approx M_{i_1} \# \cdots \# M_{i_r} \quad \text{and}$$
$$M_2 \approx M_{i_{r+1}} \# \cdots \# M_{i_k} \# M_{k+1} \# \cdots \# M_n.$$

Then

$$\pi(M_1) \cong A \quad \text{and} \quad \pi(M_2) \cong B, \qquad M \approx M_1 \# M_2.$$

The fact that $g(M_1) + g(M_2) = g(M)$ follows from Theorem 1. And so we have Theorem 2.

By a remark in Milnor, orientable compact 3-manifolds with boundary can be decomposed uniquely into a connected sum of indecomposable manifolds; likewise, Epstein's result mentioned above applies to manifolds with boundary. Of course, the conclusion "$g(M) = g(M_1) + g(M_2)$" must be dropped, since it no longer makes sense.

THEOREM 3. *Let M be a manifold such that $\pi(M)$ is free of rank n and M is of genus $n + k$. Then $M \approx M_1 \# M_2$ where M_1 is a 3-sphere with n handles and M_2 is a homotopy 3-sphere of genus k.*

Proof. $M \approx M_1 \# \cdots \# M_n$ where $\pi(M_i) = Z$ for $1 \leq i \leq n$ (by repeated application of Theorem 2 and Proposition 2).

$M_1 \approx M_{11} \# M_{12} \# \cdots \# M_{1q}$ where M_{1p} is indecomposable ($p = 1, \cdots, q$) and either (i) M_{1p} is $S^2 \times S^1$ or (ii) $\pi_2(M_{1p}) = 0$. In case (ii), M_{1p} is a homotopy-sphere, or else $\pi(M_{1p})$ is finite and non-trivial, or else $\pi(M_{1p})$ has one end. (See [12, p. 325, Satz VI].) Since Z is indecomposable with respect to free product and since it has two ends, we conclude by Proposition 2 that exactly one M_{1p}, say M_{11}, is $S^2 \times S^1$; and $M_{12}, \cdots, M_{1q}$ are homotopy-spheres.

Repeated application of this argument shows that M is a composition of n copies of $S^2 \times S^1$ and a homotopy-sphere, M_2. Theorem 1 then implies that $g(M_2) = k$.

COROLLARY. *If $\pi(M)$ is free of rank n and M has a Heegaard splitting of genus $n + 1$, then M is a 3-sphere with n handles.*

Proof. $g(M) \leq n + 1$. So M is the composition of a 3-sphere with n handles and a homotopy-sphere of genus at most one. Such a homotopy-sphere is S^3 [2, p. 31, Theorem 3].

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