PIERCING LOCALLY SPHERICAL SPHERES WITH TAME ARCS

BY

L. D LOVELAND

We define a 2-sphere S in S^3 to be *locally spherical at a point* p of S if for each $\varepsilon > 0$ there is a 2-sphere S' and a component $\operatorname{Int} S'$ of $S^3 - S'$ such that $p \in \operatorname{Int} S'$, diam $(S' \cup \operatorname{Int} S') < \varepsilon$, and $S' \cap S$ is a continuum M. A *locally spherical* 2-sphere is one that is locally spherical at each of its points. It is not known that a locally spherical 2-sphere is tamely imbedded in S^3; however several additional conditions have been imposed on M to insure the tameness of S. For example, Burgess [3] showed that S is tame if M is a simple closed curve, and Loveland [12] obtained the same conclusion by requiring that M satisfy Property $(*, M, S)$. This property roughly means that S can be side approximated missing M and implies that M is tame [13]. It is not known that a locally spherical 2-sphere S is tame even when M is required to be tame [6, page 78]; however, it is suspected that Property $(*, M, S)$ is satisfied if M is tame [8], [13].[1] Eaton [7], after reading the first draft of this paper, showed that S is tame if S is locally spherical and M irreducibly separates S.

We show that S is pierced by a tame arc at a point p of S if S is locally spherical at p, and we use this result to show that a locally spherical 2-sphere is tame provided each component of $S^3 - S$ is an open 3-cell. The same techniques show that S can be pierced by a tame arc at each of its points if S is locally spanned in each component of $S^3 - S$ (see the statement following Corollary 1 for definitions). Spheres that are locally spanned in their complementary domains are not known to be tame [4], [12].

The "locally spherical" property is closed related to several local properties identified in [14]; in fact we make use of several results and proofs given there to prove slightly stronger results than those mentioned in the previous paragraph. In Lemma 1 we show that "locally spherical" implies "locally capped"; a 2-sphere S is *locally capped in a component* V of $S^3 - S$ *at a point* p of S if for each $\varepsilon > 0$ there is a disk R on S and an open ε-disk (the interior of a disk of diameter less than ε) D in V such that $p \in \operatorname{Int} R$, $\operatorname{Bd} D \subset S - R$, and R lies on the boundary of an ε-component of $V - D$. A *locally capped* 2-sphere S is one that is locally capped in each component of $S^3 - S$ and at each point of S. In [14] we asked if a locally capped 2-sphere S is tame, and we give an affirmative answer here provided it is known that each component of $S^3 - S$ is an open 3-cell (Theorem 4).

LEMMA 1. *If a 2-sphere S in S^3 is locally spherical at a point $p \in S$, then S is locally capped at p.*

Received August 10, 1967.

[1] J. W. Cannon has recently confirmed the suspicion that $(*, M, S)$ follows when M is tame.

Proof. Let V be a component of $S^3 - S$ and let $\varepsilon > 0$. There exists a 2-sphere S', a component $\operatorname{Int} S'$ of $S^3 - S'$, and a disk E on S such that $p \in \operatorname{Int} E$, $E \cup S' \cup \operatorname{Int} S' \subset N(p, \varepsilon/2)$, and $S' \cap S$ is a continuum in E. Let q be a point in $S - E$, and let J be a simple closed curve such that $J \cap S = \{p, q\}$ and J intersects both components of $S^3 - S$. Let R be a disk on S such that $p \in \operatorname{Int} R \subset R \subset \operatorname{Int} S'$; then J links both $\operatorname{Bd} R$ and $\operatorname{Bd} E$. Without loss in generality we assume that $J \cap S'$ is finite and that J pierces S' at each point of intersection. Now we choose a component D of $S' - S$ such that $D \subset V$ and $D \cap J$ consists of an odd number of points, and we note that D is an open disk in $N(p, \varepsilon/2)$. An argument similar to the proof of Lemma 1 of [14] shows that the continuum $\operatorname{Bd} D$ separates p from q on S; thus R lies on the boundary of an ε-component of $V - D$.

A *crumpled cube* in S^3 is the union of a 2-sphere and one of its complementary domains, and a point p of the boundary of a crumpled cube C is called a *piercing point* of C if there exists a homeomorphism h of C into S^3 such that $h(\operatorname{Bd} C)$ can be pierced by a tame arc at $h(p)$.

THEOREM 1. *If the boundary S of a crumpled cube C in S^3 is locally capped in* Int C *at a point $p \in S$, then p is a piercing point of C.*

Proof. Since there exists a homeomorphism h of C into S^3 such that $S^3 - h(\operatorname{Int} C)$ is a 3-cell [10], [11] and since p is a piercing point of C if and only if $h(p)$ is a piercing point of $h(C)$, we assume that $S^3 - \operatorname{Int} C$ is a 3-cell. We shall establish Theorem 1 by showing that S is arcwise accessible at p by a tame arc from $S^3 - C$ [15].

Let $D_1, D_2, D_3, \cdots$ be a null sequence of disks and let A be an arc such that p is an endpoint of A, $A - p \subset S^3 - C$, A is locally tame modulo p, $D_i \cap C = \operatorname{Bd} D_i$, and $D_i \cap A$ is a point p_i. Such objects exist since S is tame from $S^3 - C$. Since A is locally tame modulo an endpoint, A lies on a 2-sphere. Then it will follow that A is tame once we show the existence of arbitrarily small 2-spheres surrounding p and intersecting A at a point [9].

Let J be a simple closed curve containing A and intersecting S in two points p and q, let N be a neighborhood of p not containing the other endpoint of A, let $V = \operatorname{Int} C$, and let G be a disk such that $p \in \operatorname{Int} G \subset G \subset N \cap S$. Let R be a disk in $\operatorname{Int} G$ such that $p \in \operatorname{Int} R$ and let D be an open disk such that $\operatorname{Bd} D \subset \operatorname{Int} G - R$ and R lies on the boundary of a component of $V - D$ in N. There is an integer i such that $D_i \subset N$ and $\operatorname{Bd} D_i \subset \operatorname{Int} G$. Let H be a disk such that $J \cap D \subset \operatorname{Int} H \subset H \subset D$, and let E be a disk in D_i such that $p_i \in \operatorname{Int} E$. We omit the details justifying that $\operatorname{Bd} H$ and $\operatorname{Bd} D_i$ are homotopic in $N - (J \cup E)$. Once this is known, Dehn's lemma [16], as adjusted by Bing [1] for nonpiecewise linear maps, implies the existence of a 2-sphere S' such that $S' \subset N$, $E \subset S'$, and $A \cap S' = p_i$.

Remark. The hypothesis in Theorem 1 that S is locally capped in Int C at p can be weakened. The essential thing is to be able to shrink an arbitrarily small simple closed curve on S to a point in a small subset of $C - p$. Thus p is a piercing point of C if for each $\varepsilon > 0$ there exists a disk R on S such that $p \in \text{Int } R$, $\text{diam } R < \varepsilon$, and $\text{Bd } R$ can be shrunk to a point in an ε-subset of $C - p$. The converse is also true [15]. In fact p is a piercing point of C if the boundary of the above disk R can be shrunk to a point in the union of an ε-subset of $C - p$ with a neighborhood N of $\text{Bd } R$ where $A \cap N = \emptyset$. The following result is a consequence of this observation.

COROLLARY 1. *If a 2-sphere S in S^3 is locally spanned in a component V of $S^3 - S$, then each point of S is a piercing point of $S \cup V$.*

A 2-sphere S is *locally spanned in V* if for each $\varepsilon > 0$ and for each $p \in S$ there exists an ε-disk R on S such that $p \in \text{Int } R$ and for each $\alpha > 0$ there is an ε-disk D in V such that $\text{Bd } R$ can be shrunk to a point in $N(\text{Bd } R, \alpha) \cup D$. Such spheres are not known to be tame from V [4], [12].

THEOREM 2. *A 2-sphere S in S^3 is pierced by a tame arc at p if S is locally capped at p.*

Proof. From Theorem 1 we see that p is a piercing point of the closure of each component of $S^3 - S$. According to McMillan [15] this implies that S is pierced by a tame arc at p.

COROLLARY 2. *A 2-sphere S in S^3 can be pirced by a tame arc at a point p if S is locally spherical at p.*

Remark. When the definition of locally spherical is extended to a 2-manifold M in S^3 in the obvious way, it follows from Theorem 5 of [2] and Corollary 2 that M can be pierced by a tame arc at $p \in M$ if M is locally spherical at p.

It was shown in [14], based on some techniques developed by Burgess [5], that a 2-sphere S in S^3 is locally tame modulo two points if each component of $S^3 - S$ is an open 3-cell and S is locally annular. A 2-sphere S is *locally annular* in a component V of $S^3 - S$ at a point $p \in S$ if for each $\varepsilon > 0$ and for each simple closed curve J that pierces S at p, there is an open annulus A in $V \cap N(p, \varepsilon)$ such that $J \cap \bar{A} = \emptyset$, one component of $\text{Bd } A$ is a simple closed curve K in V that links J, and $\text{Bd } A - K \subset S$. We give no proof for Lemma 2 because one is easily obtained.

LEMMA 2. *If a 2-sphere S in S^3 is locally capped in a component V of $S^3 - S$ at a point p, then S is locally annular in V at p.*

THEOREM 3. *If S is the boundary of a crumpled cube C in S^3*, Int C *is an open 3-cell, and S is locally capped in* Int C, *then S is tame from* Int C.

Proof. It follows from Lemma 2 and Theorem 4 of [14] that S contains a point p such that S is locally tame from Int C at each point of $S - p$. Then Theorem 3 follows from Theorem 1, [8], and [6].

THEOREM 4. *If a 2-sphere S in S^3 is locally capped and each component of $S^3 - S$ is an open 3-cell, then S is tame.*

COROLLARY 3. *If a 2-sphere S in S^3 is locally spherical and each component of $S^3 - S$ is an open 3-cell, then S is tame.*

Added in proof. Corollary 3 has been generalized by Eaton [7].

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UTAH STATE UNIVERSITY
LOGAN, UTAH