COVERING OF MANIFOLDS WITH OPEN CELLS

BY
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1. Introduction

We prove a fundamental theorem for manifolds.

THEOREM 1. *A connected n-dimensional topological (piecewise linear, differentiable) manifold without boundary can be covered with $n + 1$ open topological (piecewise linear, differentiable) cells.*

It is well known that a manifold can be covered with $n + 1$ coordinate systems [4].

Applying engulfing theorems, we can strengthen Theorem 1. If x is a real number, let $\langle x \rangle$ denote the least integer greater than or equal to x.

THEOREM 2. *Let $k \leq n - 3$. A k-connected n-dimensional topological (piecewise linear, differentiable) manifold without boundary can be covered with $\langle (n + 1)/(k + 1) \rangle$ open topological (piecewise linear, differentiable) cells.*

The statement of Theorem 2 for the cases $n = 3, k = 1$ and $n = 4, k = 2$ is for closed manifolds equivalent to the topological (piecewise linear, differentiable) version of the Poincaré conjecture in dimensions $n = 3$ and $n = 4$ respectively. The case $n = 2, k = 1$ is well known: A 1-connected 2-dimensional topological (piecewise linear, differentiable) manifold without boundary is homeomorphic (piecewise linearly homeomorphic, diffeomorphic) to the 2-sphere if it is compact, and to the 2-dimensional euclidean space if it is not compact [7].

Theorem 1 and 2 hold also for analytic manifolds and coverings with open analytic cells. This follows from Theorem B of [6].

The theorems are proved by a certain technique which was motivated by the proof of the topological Poincaré conjecture in dimensions $n \geq 5$ in [5]. Our arguments apply simultaneously to topological, piecewise linear, and differentiable manifolds. The differentiable version of the theorems can also be obtained directly from the piecewise linear results in the fashion of [1], introducing smooth triangulations and approximating piecewise linear homeomorphisms by diffeomorphisms.

Theorem 2 also improves a result in [10].

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2. Preliminaries

By an n-dimensional topological manifold M^n without boundary, we mean a separable Hausdorff space such that each point of M^n has an open neighbor-

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hood homeomorphic to an open subset of the n-dimensional euclidean space $\mathbf{R}^n$. An n-dimensional piecewise linear manifold without boundary is a countable simplicial complex such that the link of each vertex is a piecewise linear $(n-1)$-sphere. And, an n-dimensional differentiable manifold without boundary is an n-dimensional topological manifold without boundary with a fixed differentiable structure.

The image $f(\mathbf{R}^n)$ of $\mathbf{R}^n$, where $f : \mathbf{R}^n \rightarrow M^n$ is a homeomorphism (piecewise linear homeomorphism, diffeomorphism) into the n-dimensional topological (piecewise linear, differentiable) manifold M^n, is called an open topological (piecewise linear, differentiable) n-cell in M^n.

A topological space X is said to be k-connected, $k \geq 0$, if it is path connected and if the homotopy groups $\pi_i(X) = 0$ for $1 \leq i \leq k$.

For the proof of Theorem 1, the following lemma will be sufficient.

LEMMA 1. *Let M^n be a connected n-dimensional topological (piecewise linear, differentiable) manifold without boundary. Let $V \subset M^n$ be an open subset, and consider points $p_1, \cdots, p_N \in M^n$. Then there is a homeomorphism (piecewise linear homeomorphism, diffeomorphism) h of M^n onto itself with $\{p_1, \cdots, p_N\} \subset h(V)$ and such that h is the identity on $M^n - C$ for some compact subset $C \subset M^n$.*

Proof. Not hard. For the differentiable case, compare for example [3].

To prove Theorem 2, the following special case of a more general engulfing theorem will be needed. (See [5] for topological, [9] for piecewise linear, and [1] for differentiable manifolds. Or, the engulfing theorem in [5] holds actually in all three categories of topological, piecewise linear, and differentiable manifolds [2].)

LEMMA 2. *Let M^n be an n-dimensional topological (piecewise linear, differentiable) manifold without boundary, let $U \subset \mathbf{R}^n$ be an open subset, and let $g : U \rightarrow M^n$ be a homeomorphism (piecewise linear homeomorphism, diffeomorphism) into M^n. Suppose that $P \subset U$ is an at most k-dimensional (not necessarily finite!) polyhedron in $\mathbf{R}^n$ with $g(P)$ closed in M^n. Let $V \subset M^n$ be an open subset with $g(P) - V$ compact. Assume that M^n is k-connected, V is $(k-1)$-connected, and that $k \leq n-3$. Then there is a homeomorphism (piecewise linear homeomorphism, diffeomorphism) h of M^n onto itself with $g(P) \subset h(V)$ and such that h is the identity on $M^n - C$ for some compact subset $C \subset M^n$.*

Finally, the following lemma will be applied. If K is a simplicial complex, $|K|$ denotes the underlying point set. The complementary complex of a subcomplex of a simplicial complex is the set of all those simplexes of the simplicial complex which do not have a face in the subcomplex.

LEMMA 3. *Let H be the simplicial complex determined by a simplicial subdivision of $\mathbf{R}^n$, let K be a finite subcomplex of a subdivision H' of H, and let $A \subset \mathbf{R}^n$ be a closed subset with $A \cap |K|$ is empty. Let L be a full subcomplex of K and let L^c be the complementary complex of L in K. Suppose that U and*

V are open subsets of $\mathbf{R}^n$ with $|L| \subset U$ and $|L^c| \subset V$. Then there exists a piecewise linear "stretching" homeomorphism ("stretching" diffeomorphism) s of $\mathbf{R}^n$ onto itself such that

(1) $|K| \subset s(U) \cup V$,
(2) *s moves each point of $\mathbf{R}^n$ only within a simplex of H, and*
(3) *s is the identity on A.*

Proof. We omit the proof of the differentiable version. It can be found in [2]. If dim $(L) = 0$—which is used to prove Theorem 1—the proof simplifies essentially.

First, we consider the simplicial complex K. Since L is full in K, each point x of $|K| - (|L| \cup |L^c|)$ lies in a uniquely determined segment $[a, b]$ contained in a simplex of K, where $a \in |L|$ and $b \in |L^c|$. There is a stretching homeomorphism s' of $|K|$ onto itself defined by mapping each segment piecewise linearly onto itself such that $|K| = s'(U \cap |K|) \cup (V \cap |K|)$.

Next, we extend s' to a homeomorphism s of $\mathbf{R}^n$ onto itself such that the required properties hold. Consider the mth, $m \geq 1$, barycentric subdivision $\beta_K^m(H')$ of H' relative to K such that for all simplexes Δ of $\beta_K^m(H')$ with $\Delta \cap |K|$ is not empty, then $\Delta \cap A$ is empty. We note that K is full in $\beta_K^m(H')$. The extension s of s' is defined as follows. Let $x \in \mathbf{R}^n$. We distinguish three cases:

(a) If $x \in \Delta$, $\Delta \in \beta_K^m(H')$ with $\Delta \cap |K|$ is empty, then $s(x) = x$.
(b) If $x \in \Delta$, $\Delta \in K$, then $s(x) = s'(x)$.
(c) If $x \in \Delta$, $\Delta \in \beta_K^m(H')$ with $\Delta = \Delta_1 \circ \Delta_2$ where $\Delta_1 \in K$, $\Delta_2 \in \beta_K^m(H')$ with $\Delta_2 \cap |K|$ is empty, $x \notin \Delta_1$, and $x \notin \Delta_2$, then x has the unique representation $x = \lambda \circ a + (1 - \lambda) \circ b$, $a \in \Delta_1$, $b \in \Delta_2$, and $0 < \lambda < 1$. We define $s(x) = \lambda \circ s'(a) + (1 - \lambda) \circ b$.

It is easily verified that s is well defined. By construction, s has the required properties.

3. Proof of Theorems 1 and 2

For convenience of notation, we state the proof for topological manifolds only. Clearly, the arguments hold also in the piecewise linear and in the differentiable case.

We may assume in the following that dimension $n \geq 2$.

Let $C_\rho^n = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n \text{ with } |x_i| \leq \rho,\ i = 1, \cdots, n\}$ be the cube in $\mathbf{R}^n$ with side length 2ρ and with the origin as centre, and let

$$\text{int } C_\rho^n = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n \quad \text{with} \quad |x_i| < \rho, \quad i = 1, \cdots, n\}.$$

(A) *Let M^n be a k-connected n-dimensional topological manifold without boundary and let $k \leq n - 3$ if $k > 0$. Suppose that $g_i : \mathbf{R}^n \to M^n$, $i = 1, \cdots, m$, where $m = \langle (n + 1)/(k + 1) \rangle$, and $g : \mathbf{R}^n \to M^n$ are homeo-*

morphisms into the manifold M^n. *Let* α, β, and γ *be such that* $\alpha > 0$ *and* $\gamma > \beta > 0$. *Then there are homeomorphisms* h_i *of* M^n *onto itself*, $i = 1, \cdots, m$, *such that*

(1) $g(C^n_\alpha) \subset \bigcup_{i=1}^m h_i \circ g_i(C^n_\gamma)$, *and*

(2) h_i *is the identity on* $g_i(C^n_\beta)$, $i = 1, \cdots, m$.

Proof. We choose a λ with $\beta < \lambda < \gamma$. Let H be the simplicial complex determined by a simplicial subdivision of $\mathbf{R}^n$ such that

(a) C^n_α forms the set of points of a subcomplex K of H, and

(b) if $g(\Delta) \cap g_i(C^n_\beta)$ is not empty for a simplex Δ of H, then $g(\Delta) \subset g_i(C^n_\lambda)$, $i = 1, \cdots, m$.

Let

$$K_0 = \{\Delta \in K : g(\Delta) \cap g_i(C^n_\beta) \text{ is empty for each } i = 1, \cdots, m\}.$$

If K_0 is not n-dimensional, then $g(C^n_\alpha) \subset \bigcup_{i=1}^m g_i(C^n_\gamma)$ by property (b) of the simplicial complex H, and we define h_i to be the identity on M^n, $i = 1, \cdots, m$, in this case. We assume now that K_0 is an n-dimensional simplicial complex. We construct inductively sequences $K_1, \cdots, K_{m-1}$ and $L_1, \cdots, L_{m-1}$ of simplicial complexes as follows. Suppose that K_{i-1}, $i \geq 1$, is defined. Let L_i be the first barycentric subdivision of the k-skeleton of K_{i-1}, and let K_i be the complementary complex of L_i in the first barycentric subdivision of K_{i-1}, $i = 1, \cdots, m-1$. It follows that

$$\dim (K_i) = n - i \circ (k+1).$$

Hence $\dim(K_{i-1}) \leq k$ if $n - (i-1) \circ (k+1) \leq k$, or if $i \geq (n+1)/(k+1)$. Thus $\dim (K_{m-1}) \leq k$.

Define $L_m = K_{m-1}$. We consider the simplicial complexes $L_1, \cdots, L_m$ in $\mathbf{R}^n$. Let $M^n_i = M^n - g_i(C^n_\lambda)$. The n-dimensional topological manifold M^n_i without boundary is again k-connected. (Namely, if $k > 0$ it is simply connected by van Kampen's Theorem for example, the singular homology groups are trivial in dimensions $j = 1, \cdots, k$, and therefore $\pi_j(M^n) = 0$ for $j = 1, \cdots, k$ by the Hurewicz Isomorphism Theorem. Alternatively, a direct proof can be given by "pushing out" homotopies from the cell $g_i(C^n_\lambda)$.) The open subset $V_i = g_i\,(\text{int } C^n_\gamma - C^n_\lambda)$ of M^n_i is obviously $(k-1)$-connected. Let $P_i = |L_i| - g^{-1}(g_i(C^n_\lambda))$. Then P_i is an open subset of $|L_i|$ and therefore a k-dimensional polyhedron in the open subset $U_i = \mathbf{R}^n - g^{-1}(g_i(C^n_\lambda))$ of $\mathbf{R}^n$, $g(P_i)$ is closed in M^n_i, and $g(P_i) - V_i$ is compact. Lemma 2 (or 1 respectively) can be applied. There is a homeomorphism h''_i of M^n_i onto itself with $g(P_i) \subset h''_i(V_i)$ and such that h''_i is the identity on $M^n - C_i$ for some compact subset $C_i \subset M^n_i$. The homeomorphism h''_i can be extended to a homeomorphism h'_i of M^n onto itself by defining h'_i to be the identity on $g_i(C^n_\lambda)$. We conclude that

$$g(|L_i|) \subset h'_i \circ g_i\,(\text{int } C^n_\gamma), \qquad i = 1, \cdots, m.$$

Now let $W_i = g^{-1}(h'_i \circ g_i\ (\text{int}\ C^n_\gamma))$, $i = 1, \cdots, m$. Then W_i is an open subset of $\mathbf{R}^n$ with $|L_i| \subset W_i$, $i = 1, \cdots, m$. Applying Lemma 3, we construct inductively a sequence of stretching homeomorphisms s_{m-i} of $\mathbf{R}^n$ onto itself, $i = 1, \cdots, m-1$, such that

(1) $|K_{m-2}| \subset s_{m-1}(W_{m-1}) \cup W_m$,
$|K_{m-3}| \subset s_{m-2}(W_{m-2}) \cup s_{m-1}(W_{m-1}) \cup W_m$,
$\vdots$
$|K_0| \subset s_1(W_1) \cup \cdots \cup s_{m-1}(W_{m-1}) \cup W_m$,

(2) s_{m-i} moves each point of $\mathbf{R}^n$ only within a simplex of H, and
(3) s_{m-i} is the identity on $(\mathbf{R}^n - \text{int}\ C^n_{\alpha+1}) \cup g^{-1}(g_i(C^n_\beta))$.

We lift the homeomorphisms s_i onto the manifold M^n defining $s'_i : M^n \to M^n$ by $s'_i(p) = g \circ s_i(g^{-1}(p))$ for $p \in g(\mathbf{R}^n)$ and $s'_i(p) = p$ for $p \in M^n - g(\mathbf{R}^n)$, $i = 1, \cdots, m-1$. It follows that

$$g(|K_0|) \subset s'_1 \circ h'_1 \circ g_1(C^n_\gamma) \cup \cdots \cup s'_{m-1} \circ h'_{m-1} \circ g_{m-1}(C^n_\gamma) \cup h'_m \circ g_m(C^n_\gamma).$$

We claim further that

$$g(C^n_\alpha) \subset s'_1 \circ h'_1 \circ g_1(C^n_\gamma) \cup \cdots \cup s'_{m-1} \circ h'_{m-1} \circ g_{m-1}(C^n_\gamma) \cup h'_m \circ g_m(C^n_\gamma).$$

Namely, consider $x \in C^n_\alpha = |K|$. If $x \in |K_0|$, then $g(x)$ is a point of the set at the right side by the preceding. If $x \notin |K_0|$, then $x \in \Delta$, $\Delta \in H$, and $g(\Delta) \cap g_{i_0}(C^n_\beta)$ is not empty for some i_0 by definition of K_0. Consequently, $g(\Delta) \subset g_{i_0}(C^n_\lambda)$ by property (b) of the simplicial complex H. It follows from property (2) of s_{i_0} that $s_{i_0}(\Delta) = \Delta$. Hence $g(x) \in s'_{i_0} \circ h'_{i_0} \circ g_{i_0}(C^n_\gamma)$.

Finally, let $h_i = s'_i \circ h'_i$, $i = 1, \cdots, m-1$, and let $h_m = h'_m$. Properties (1) and (2) are satisfied.

(B) If M^n is compact, Theorems 1 and 2 follow immediately from (A); one simply chooses finitely many homeomorphisms $g_j : \mathbf{R}^n \to M^n$ such that $\{g_j(C^n_1)\}$ covers M^n, and applies (A) a finite number of times to make the first m of these cubes engulf the remaining ones. If M^n is not compact, the same method still works.

Consider homeomorphisms $g_j : \mathbf{R}^n \to M^n$, $j = 1, 2, \cdots$, such that $\{g_j(C^n_1)\}_{j=1}^\infty$ covers M^n. We construct inductively m sequences $\{f_{i,j}\}_{j=0}^\infty$, $i = 1, \cdots, m$, of homeomorphisms $f_{i,j} : \mathbf{R}^n \to M^n$ into M^n with $f_{i,0} = g_i$, $i = 1, \cdots, m$, such that

(1) $\bigcup_{l=1}^{m+j} g_l(C^n_1) \subset \bigcup_{i=1}^m f_{i,j}(C^n_{j+1})$, and
(2) $f_{i,j} \mid C^n_j = f_{i,j-1} \mid C^n_j$, $i = 1, \cdots, m, j \geq 1$.

Suppose that $f_{i,j-1}$, $i = 1, \cdots, m$, $j \geq 1$, are constructed. We apply (A). There are homeomorphisms h_i of M^n onto itself, $i = 1, \cdots, m$, such that

$$g_{m+j}(C^n_1) \subset \bigcup_{i=1}^m h_i \circ f_{i,j-1}(C^n_{j+1})$$

and

$$h_i \text{ is the identity on } f_{i,j-1}(C_j^n), \qquad i = 1, \cdots, m.$$

Then $f_{i,j} = h_i \circ f_{i,j-1}$, $i = 1, \cdots, m$, satisfy the required properties.

Finally, we pass to the limit and define maps $f_i : \mathbf{R}^n \to M^n$, $i = 1, \cdots, m$, by $f_i(x) = \lim_{j\to\infty} f_{i,j}(x)$. It follows from property (2) that each map f_i is well defined and a homeomorphism onto an open subset of M^n. The images of $f_1(\mathbf{R}^n), \cdots, f_m(\mathbf{R}^n)$ are open n-cells in M^n and cover M^n by construction.

4. Some consequences of Theorem 2

If x is a real number, let $[x]$ denote the greatest integer less than or equal to x.

COROLLARY 1. *If $n \geq 5$ every $[n/2]$-connected n-dimensional topological (piecewise linear, differentiable) manifold without boundary can be covered with two open topological (piecewise linear, differentiable) cells.*

COROLLARY 2. *A simply connected 4-dimensional topological (piecewise linear, differentiable) manifold without boundary can be covered with three open topological (piecewise linear, differentiable) cells.*

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