A TAUBERIAN THEOREM FOR DIRICHLET CONVOLUTIONS[1]

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In the development of the proof of the prime number theorem, the relation

$$\text{(A)} \qquad \sum_{d \leqq x} f(x/d) = ax \log x + bx + E(x)$$

where a and b are constants has been frequently discussed. In particular should be mentioned Landau [6, pp. 597–604] and Ingham [4]. The theorem that if $f(x)$ is positive non-decreasing and $E(x) = o(x)$, then $f(x) \sim ax$ as $x \to \infty$, has been attributed by Karamata [5] to Jakimovski, using the fact (deeper than a mere asymptotic form of the prime number theorem) that $\sum_{n \leqq x} \mu(n) = O(x/\log^2 x)$ where $\mu(n)$ is the Möbius function. However, in [4] Ingham had shown how (using Wiener's Tauberian theory), one could deduce from (A), and $f(x)$ positive nondecreasing, that $f(x) \sim ax$, by appealing only to the fact that the Riemann zeta-function has no zeros on the line $\sigma = 1$; thus providing an independent proof of the prime number theorem.

In this note we develop Ingham's procedure to consider the more general convolution

$$\text{(B)} \qquad \sum_{d \leqq x} k(d) f(x/d) = ax \sum_{d \leqq x} (k(d))/d + bx + o(x)$$

where $k(d)$ is subject to certain restrictions. Recently Erdös and Ingham [2] have considered the convolution

$$f(x) + \sum f(x/a_n) = (1 + \sum (1/a_n) x + o(x)$$

where the a_n are real numbers $1 < a_1 \leqq a_2 \leqq \cdots$ subject to the condition $\sum (1/a_n)$ converges. If the a_n are integers this reduces to the form (B) where $k(d)$ takes only the values 0 and 1; however, there is no overlap between the results of [2] and those discussed here. Although the proof of the theorem below follows the method introduced by Ingham in [4], there is perhaps some interest in elucidating those properties of $[x] = \sum_{d \leqq x} 1$ which play a role in Tauberian deductions from (A).

Throughout this paper, $k(d)$ is an arithmetic function with $k(1) = 1$ and $k^*(d)$ is the "Dirichlet inverse" of $k(d)$ defined by

$$\sum_{d \mid n} k(d) k^*(n/d) = 1, \quad n = 1$$
$$= 0, \quad \text{otherwise}.$$

Empty sums are interpreted as $= 0$. s is a complex variable and $\sigma = \mathrm{Re}\,(s)$. x is a real variable and all functions of x are real-valued. All error terms are as the variable $\to \infty$. All unexplained terminology or notation is as in [3].

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THEOREM. *Suppose $k(d)$ is a non-negative arithmetic function with $k(1) = 1$ which enjoys the following properties:*

(i) $\sum_{d \leqq x} k(d) = Ax + E(x)$

where $E(x) = o(x)$, and $\int_1^x (|E(t)|/t^2)dt$ converges as $x \to \infty$, and A is a positive constant;

(ii) $\lim_{\delta \to 0+} \sum_{d=1}^{\infty} k(d) d^{-1-ix-\delta} \neq 0$ *for all real $x \neq 0$.*

Then, if $f(x)$ is a positive, non-decreasing function for $x \geqq 1$, which satisfies (B), *and $f(x) = O(x)$, then*

$$f(x) = ax + o(x) \quad \text{as} \quad x \to \infty.$$

Proof. For convenience, define $f(x) = 0$ for $0 \leqq x < 1$. Then

$$(1) \quad \begin{aligned} \int_1^x \frac{1}{u} \sum_{d \leqq u} k(d) f(u/d)\, du &= \sum_{d \leqq x} k(d) \int_d^x \frac{1}{u} f\left(\frac{u}{d}\right) du \\ &= \sum_{d \leqq x} k(d) \int_1^{x/d} \frac{f(t)}{t}\, dt = \int_1^x \frac{f(t)}{t} \sum_{d \leqq x/t} k(d)\, dt, \end{aligned}$$

and so extending the range of integration formally to $(0, \infty)$ on the right and substituting (B) on the left in (1) gives

$$(2) \quad \int_0^\infty \frac{f(t)}{t} \sum_{d \leqq x/t} k(d)\, dt = a \int_1^x \sum_{d \leqq t} \frac{k(d)}{d}\, dt + bx + o(x)$$

as $x \to \infty$.

Replacing x by x/α, α a fixed positive constant, in (2) gives

$$(3) \quad \int_0^\infty \frac{f(t)}{t} \sum_{d \leqq x/(\alpha t)} k(d)\, dt = a \int_1^{x/\alpha} \sum_{d \leqq t} \frac{k(d)}{d}\, dt + \frac{bx}{\alpha} + o(x)$$

For $\alpha \neq \beta$, α, β fixed constants >1 to be determined more precisely later, define

$$(4) \quad \begin{aligned} L(y) &=_{\text{def}} 2 \sum_{d \leqq y} k(d) - \alpha \sum_{d \leqq y/\alpha} k(d) - \beta \sum_{d \leqq y/\beta} k(d) \\ &= 2\, E(y) - \alpha E(y/\alpha) - \beta E(y/\beta) \end{aligned}$$

by (i). Then from (3), after some appropriate changes of variable, we have

$$(5) \quad \begin{aligned} &\int_0^\infty \frac{f(t)}{t} L\left(\frac{x}{t}\right) dt \\ &\quad = a \int_1^x \left\{ 2 \sum_{d \leqq t} \frac{k(d)}{d} - \sum_{d \leqq t/\alpha} \frac{k(d)}{d} - \sum_{d \leqq t/\beta} \frac{k(d)}{d} \right\} dt + o(x). \end{aligned}$$

On the other hand by (i) and partial summation,

$$(6) \quad \begin{aligned} \sum_{d \leqq t} \frac{k(d)}{d} &= A \log t + A + \int_1^t \frac{E(u)}{u^2}\, du + o(1) \\ &= A \log t + A + K + o(1) \end{aligned}$$

since the integral converges by hypothesis (i) as $t \to \infty$. Substitution of (6) in (5) yields on letting $L(1/x) = G(x)$,

$$\frac{1}{x}\int_0^\infty \frac{f(t)}{t}\, G\left(\frac{t}{x}\right) dt = aA \log(\alpha\beta) + o(1). \tag{7}$$

Now, for $0 \leqq t < 1$, by definition, $E(t) = \sum_{d \leqq t} k(d) - At = -At$, and so by (4), for $0 \leqq t < 1$, $L(t) = 0$. Hence by the definition of $G(x)$ and (4),

$$\int_0^\infty |G(t)|\, dt = \int_0^\infty \frac{|L(u)|}{u^2}\, du = \int_1^\infty \frac{|2E(u) - \alpha E(u/\alpha) - \beta E(u/\beta)|}{u^2}\, du,$$

which is convergent by hypothesis (i). Hence $|G(x)|$ is integrable in $(0, \infty)$; if furthermore

$$\int_0^\infty G(u)u^{ix}\, du \neq 0 \quad \text{for all real } x \tag{8}$$

then the Wiener-Pitt Tauberian theory may be applied to (7).

It is easily seen, however, writing $s = \sigma + ix$, that for $\sigma > 0$,

$$\int_1^\infty L(t)t^{-s-2}\, dt = (2 - \alpha^{-s} - \beta^{-s})\left(\int_1^\infty E(t)t^{-s-2}\, dt + A/s\right)$$

by computing $\int E(t/\alpha)t^{-s-2}\, dt$ and $\int E(t/\beta)t^{-s-2}$ dt in terms of $\int E(t)t^{-s-2}dt$ while, as above, $L(t) = 0$ for $0 \leqq t < 1$. Hence for $\sigma > 0$,

$$\begin{aligned} \int_0^\infty G(u)u^s\, du &= \int_0^\infty L(t)t^{-s-2}\, dt \\ &= (2 - \alpha^{-s} - \beta^{-s})\left(\int_1^\infty E(t)t^{-s-2}\, dt + A/s.\right) \end{aligned} \tag{9}$$

But by hypothesis (i) and the arguments above, $\int_1^\infty G(u)u^s\, du$ and $\int_{-1}^\infty E(t)t^{-s-2}\, dt$ are convergent for $\sigma = 0$ also. Hence, by a well-known continuity theorem, on taking limits of both sides of (9) as $\sigma \to 0^+$ we may interchange the limit with the integration and obtain,

$$\int_0^\infty G(u)u^{ix}\, du = (2 - \alpha^{-ix} - \beta^{-ix})\left(\int_1^\infty E(t)t^{-ix-2} + \frac{A}{ix}\right) \tag{10}$$

for all real $x \neq 0$, and

$$\int_0^\infty G(u)\, du = A \log(\alpha\beta). \tag{11}$$

If α and β are chosen so that $(\log \alpha)/(\log \beta)$ is irrational, then the first factor on the right in (10) $\neq 0$, and the right side of (11) $\neq 0$. For the second factor on the right in (10) we have by (i), partial summation, and the above quoted continuity result, for $\delta > 0$,

$$\int_1^\infty E(t)t^{-ix-2}\,dt = \int_1^\infty (\sum_{d\leqq t} k(d) - At)t^{-ix-2}\,dt$$

$$= -\frac{A}{ix} + \lim_{\delta\to 0^+} \frac{1}{1+ix+\delta}\sum_{d=1}^{\infty} k(d)\, d^{-1-ix-\delta}.$$

Hence, by hypothesis (ii), the second factor on the right side of (10) also $\neq 0$.

So (8) is true and, taking sight of (11), the theorem will follow from (7) and a well-known result of Pitt [3, Theorem 233] provided we can prove that $f(x)/x$ is slowly decreasing in $(0, \infty)$ in the sense of Schmidt.

To show this, it suffices to note that for all $p > 1$ and all $x > 0$,

$$f(px)/px - f(x)/x \geqq (f(x)/x)(1 - 1/p) = O(1)(1 - 1/p)$$

since $f(x)$ is positive and non-decreasing and $f(x) = O(x)$.

Hence the theorem follows.

Remarks. (a) It would be desirable to eliminate if possible the necessity of hypothesizing $f(x) = O(x)$. In the classical case $k(d) = 1$ considered by Ingham, an argument going back to Tschebyscheff allows the deduction of $f(x) = O(x)$ from (B). An attempt to imitate this argument for a more general nonconstant $k(d)$ leads to the condition:

There exists an integer $m \geqq 2$ such that for all integers $d \geqq 1$,

$$\sum_{v=1}^{m} k(md - m + v) \geqq mk(d).$$

While, with the assumption of this condition, one can indeed deduce $f(x) = O(x)$ from (B), unfortunately it appears likely, though no proof is known, that the only functions $k(d)$ satisfying this condition and (i) are constants, and so it represents no advance over Ingham's case.

It would be in particular useful to eliminate the hypothesis $f(x) = O(x)$ or replace it by a weaker one in the kind of situation considered in [1]. Here, it seems almost as difficult to prove the relevant function is $O(x)$ as it does to prove that actually it is $\sim x$ as $x \to \infty$.

(b) Writing $k(n) = nh(n)$, and if $f(x) = \sum_{n\leqq x} n\, a_n$ the theorem can also be formulated in terms of the $(\mathfrak{D}, h(n))$-summability methods introduced in [7].

(c) The condition that $f(x)$ be positive may be ameliorated to $f(x) \geqq -M$ by considering $f(x) + M$ in place of $f(x)$.

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27 (1967), no. 6 (December), p. 2019B. An abstract of the present paper occurs in the *Notices of the American Mathematical Society*, vol. 15 (1968), no. 1 (January), pp. 145–146 .

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