PRESPECTRAL OPERATORS

E. BERKSON[1] AND H. R. DOWSON

1. Introduction

It does not seem to have attracted much attention that certain basic properties of spectral operators established in [5] are invalid or uncertain for prespectral operators. As can be seen from the proof of Theorem 5 of [5], if T is a spectral operator then a bounded operator commuting with T also commutes with any strongly countably additive resolution of the identity for T. However, Example 2.7 of [8] shows that there exist on l^∞ a prespectral operator T with a resolution of the identity $E(\cdot)$ of class l^1 and a bounded operator A which commutes with T but not with every value of $E(\cdot)$. The failure of the commutativity theorem for prespectral operators rules out direct application to such operators of the theory in [5] based on it. Thus it is not known in general if a prespectral operator of class Γ necessarily has a unique resolution of the identity of class Γ.

The purpose of this paper is to obtain results of fairly broad applicability which help to overcome the difficulties with prespectral operators arising from the failure of the commutativity theorem. Results which do not depend on special assumptions about the spectrum are presented in §3. Prespectral operators with totally disconnected spectrum are discussed in §4, and in §5 we consider scalar-type prespectral operators whose spectra are R-sets. Some new aspects and consequences of the example of Fixman, referred to above, are considered in §6. The paper concludes with a brief section on the literature which has appeared concerning prespectral operators.

2. Preliminaries

Throughout the paper, X is a complex Banach space with dual space X^*. We write $\langle x, \varphi\rangle$ for the value of the functional φ in X^* at the point x of X. For brevity the term "operator" is used to mean "bounded linear operator". The spectrum and resolvent set of an operator T are denoted by $\sigma(T)$ and $\rho(T)$ respectively. The Banach algebra of operators on X is denoted by $L(X)$. The complex plane is denoted by p and Σ_p denotes the σ-field of Borel subsets of p. Let K be a compact Hausdorff space. $C(K)$ denotes the Banach algebra of complex functions continuous on K under the supremum norm.

A family $\Gamma \subseteq X^*$ is called *total* if and only if $y \in X$, $\langle y, f\rangle = 0$ for all f in Γ imply $y = 0$. Let Σ be a σ-field of subsets of an arbitrary set Ω with $\Omega \in \Sigma$. Suppose that a mapping $E(\cdot)$ from Σ into a Boolean algebra of projections on

Received June 14, 1967.

[1] The work of this author has been partially supported by a National Science Foundation grant.

X satisfies the following conditions:

(i) $E(\delta_1) + E(\delta_2) - E(\delta_1)E(\delta_2) = E(\delta_1 \cup \delta_2)$,
(ii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$, $\delta_1, \delta_2 \in \Sigma$,
(iii) $E(\Omega \backslash \delta) = I - E(\delta)$, $\delta \in \Sigma$,
(iv) $E(\Omega) = I$,
(v) there is $M > 0$ such that $\| E(\delta) \| \leq M$ for all δ in Σ,
(vi) there is a total linear manifold $\Gamma \subseteq X^*$ such that $\langle E(\cdot)x, y\rangle$ is countably additive on Σ for each x in X and each y in Γ.

Then $E(\cdot)$ is called a *spectral measure of class* (Σ, Γ). An operator T, in $L(X)$, is called a *prespectral operator of class* Γ if and only if the following conditions (α) and (β) are satisfied.

(α) There is a spectral measure $E(\cdot)$ of class (Σ_p, Γ) with values in $L(X)$ such that

$$TE(\delta) = E(\delta)T, \quad \delta \in \Sigma_p .$$

This condition implies that the closed subspaces $E(\delta)X$, $\delta \in \Sigma_p$, are invariant under T.

(β) $\sigma(T \mid E(\delta)X) \subseteq \bar{\delta}, \delta \in \Sigma_p$,

i.e., the spectrum of the restriction of T to $E(\delta)X$ is contained in the closure of δ.

The spectral measure $E(\cdot)$ is called a *resolution of the identity* (*of class* Γ) *for* T. An operator in $L(X)$ is called a *spectral operator* if and only if it is prespectral of class X^*. It is a consequence of the Banach-Orlicz-Pettis theorem that T is a spectral operator if and only if T has a resolution of the identity which is countably additive in the strong operator topology. In this connection, it is well known that a prespectral operator on a weakly complete Banach space is automatically spectral. This can be seen for example from Lemmas 2.3 and 2.9 of [1].

Dunford initiated the study of prespectral operators in [5], but since then the majority of authors have concentrated on the case of spectral operators. The following result shows that prespectral operators arise naturally in the study of spectral operators.

2.1 THEOREM. *Let T, in $L(X)$, be a spectral operator with resolution of the identity $E(\cdot)$ of class X^*. Then T^* is prespectral on X^* with resolution of the identity $E^*(\cdot)$ of class X.*

This result was proved in [6; pp. 250–1]. We shall study this class of prespectral operators in more detail in the next section.

2.2. Now let $T \in L(X)$ and let $x \in X$. An X-valued function f_x, defined and analytic on an open subset $D(f_x)$ of p such that

$$(\zeta I - T)f_x(\zeta) = x, \quad \zeta \in D(f_x),$$

is called a *pre-imaging function for x and T.* It is easily shown that $f_x(\zeta) = (\zeta I - T)^{-1}x$ whenever $\zeta \in \rho(T) \cap D(f_x)$. If for all x in X and all pairs $f_x^{(1)}$, $f_x^{(2)}$ of pre-imaging functions for x and T we have

$$f_x^{(1)}(\zeta) = f_x^{(2)}(\zeta), \quad \zeta \in D(f_x^{(1)}) \cap D(f_x^{(2)}),$$

then T is said to have the *single-valued extension property.* In this case there is a unique pre-imaging function with maximal domain $\rho(x)$, an open set containing $\rho(T)$. The values of this function are denoted by $\{x(\xi) : \xi \in \rho(x)\}$. Let $\sigma(x) = p\backslash\rho(x)$. Clearly $\sigma(x) \subseteq \sigma(T)$. The following results concerning these concepts were proved in [5; pp. 325–9].

2.3 Theorem. (i) *A prespectral operator has the single-valued extension property.*

(ii) *Let T, in $L(X)$, be a prespectral operator with a resolution of the identity $E(\cdot)$ of class Γ. Then $E(\sigma(T)) = I$. More generally if δ is a closed subset of p then*

$$E(\delta)X = \{x \in X : \sigma(x) \subseteq \delta\}.$$

The theory of integration with respect to a spectral measure was developed in [5]. The reader is referred to pp. 330–1 and pp. 340–1 of [5] for a complete discussion. However the main consequences of the theory will be specifically recalled in §3.

3. General results

We begin this section by observing that if T is a spectral operator, and $E(\cdot)$ is a strongly countably additive resolution of the identity for T, then the commutativity theorem [5; p. 329] is valid for T and $E(\cdot)$. Thus if $F(\cdot)$ is a resolution of the identity of class Γ for T, then $\{F(\tau) : \tau \in \Sigma_p\}$ commutes with $\{E(\tau) : \tau \in \Sigma_p\}$, and the proof of Theorem 6 [5; p. 330] shows that $F(\tau) = E(\tau)$, $\tau \in \Sigma_p$. Hence all resolutions of the identity of a spectral operator, no matter what their class, are identical and countably additive in the strong operator topology. In the case of a prespectral operator T, the argument of Theorem 6 of [5] shows that if $F(\cdot)$ and $G(\cdot)$ are commuting resolutions of the identity of class Γ for T, then $F(\tau) = G(\tau)$, $\tau \in \Sigma_p$. In general it is not known that two resolutions of the identity of the same class for T commute, and so the problem of uniqueness of resolution of the identity of class Γ for a prespectral operator of class Γ is unresolved. In view of this, Lemma 6 of [5; p. 341] requires a slight change in wording. We give a preliminary definition and then state the amended version of the result, which will be required later.

Definitions. Let S be a prespectral operator with a resolution of the identity $E(\cdot)$ of class Γ such that $S = \int_{\sigma(S)} \lambda E(d\lambda)$. Then S is called a *scalar-type operator of class* Γ, and $E(\cdot)$ is called an *s-resolution of the identity of class Γ for S.*

3.1 THEOREM. *Let Σ be a σ-field of subsets of a set Ω, with $\Omega \in \Sigma$. Let $E(\cdot)$ be a spectral measure of class (Σ, Γ), and let $f \in B(\Omega)$, the set of bounded complex Σ-measurable functions defined on Ω.*

Define

$$\psi(f) = \int_\Omega f(w)E(dw).$$

Then there is a constant $v(E)$ such that

$$\| \psi(f) \| \leq v(E) \sup_{w \in \Omega} | f(w) |, \quad f \in B(\Omega).$$

For each f in $B(\Omega)$ the operator $\psi(f)$ is a prespectral operator with an s-resolution of the identity $F(\cdot)$ of class Γ where

$$F(\tau) = E(f^{-1}(\tau)), \quad \tau \in \Sigma_p .$$

Let T be a prespectral operator with a resolution of the identity $E(\cdot)$ of class Γ. Assume $\| E(\tau) \| \leq M$, $\tau \in \Sigma_p$. Note that by Lindelöf's theorem [10; p. 49], the union ρ of all open sets v in p such that $E(v) = 0$ can be expressed as a union of countably many such open sets. It follows that $E(\rho) = 0$. The complement of ρ, which we denote by K, is called the *support* of $E(\cdot)$. By 2.2, $E(\rho(T)) = 0$ and so $K \subseteq \sigma(T)$. However $E(K) = I$, and so $\sigma(T) = \sigma(T \mid E(K)X) \subseteq K$. Therefore $K = \sigma(T)$, and so $\sigma(T)$ is the intersection of all closed subsets δ such that $E(\delta) = I$. Let $f \in C(\sigma(T))$. Then by 3.1 $\psi(f) = \int_{\sigma(T)} f(\lambda)E(d\lambda)$ has a resolution of the identity $F(\cdot)$ given by

$$F(\tau) = E(f^{-1}(\tau)), \quad \tau \in \Sigma_p .$$

Therefore

$$\begin{aligned}\sigma(\psi(f)) &= \bigcap \{\delta : \delta \text{ is closed, and } F(\delta) = I\} \\ &= \bigcap \{\delta : \delta \text{ is closed, and } E(f^{-1}(\delta)) = I\}.\end{aligned}$$

However if δ is closed and $E(f^{-1}(\delta)) = I$, then since $f^{-1}(\delta)$ is a closed subset of $\sigma(T)$,

$$\sigma(T) \subseteq f^{-1}(\delta) \subseteq \sigma(T).$$

Hence

$$\sigma(\psi(f)) = \bigcap \{\delta : \delta \text{ closed, and } f^{-1}(\delta) = \sigma(T)\}.$$

There follow the familiar facts that $\sigma(\psi(f)) = f(\sigma(T))$, and that the spectral radius of $\psi(f)$ is $\sup_{\lambda \in \sigma(T)} | f(\lambda) |$.

From this and Theorem 7 of [5],

$$\sup_{\lambda \in \sigma(T)} | f(\lambda) | \leq \left\| \int_{\sigma(T)} f(\lambda)E(d\lambda) \right\| \leq 4M \sup_{\lambda \in \sigma(T)} | f(\lambda) |, \quad f \in C(\sigma(T)),$$

$$\int_{\sigma(T)} f(\lambda)g(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)E(d\lambda) \int_{\sigma(T)} g(\lambda)E(d\lambda), \quad f, g \in C(\sigma(T)).$$

Hence ψ is a bicontinuous algebra isomorphism from $C(\sigma(T))$ into $L(X)$.

Every resolution of the identity of a prespectral operator T yields an algebra of operators equivalent to $C(\sigma(T))$ in this way. It is not known in general whether different algebras of operators may be obtained from different resolutions of the identity of T. A useful criterion for uniqueness of the resolution of the identity of a particular class may be given in these terms.

3.2 LEMMA. *Let T be a prespectral operator with resolutions of the identity $E_1(\cdot)$ and $E_2(\cdot)$ of class Γ such that*

$$\int_{\sigma(T)} f(\lambda)E_1(d\lambda) = \int_{\sigma(T)} f(\lambda)E_2(d\lambda), \quad f \in C(\sigma(T)).$$

Then $E_1(\tau) = E_2(\tau)$, $\tau \in \Sigma_p$.

Proof. Let $x \in X$, $y \in \Gamma$, and $\mu_k(\delta) = \langle E_k(\delta)x, y\rangle$, $\delta \in \Sigma_p$, $k = 1, 2$. By first verifying the result for simple functions it follows in the usual way that

$$\int_{\sigma(T)} f(\lambda)\mu_k(d\lambda) = \left\langle \int_{\sigma(T)} f(\lambda)E_k(d\lambda)x, y \right\rangle, \quad f \in C(\sigma(T)), k = 1, 2.$$

Hence

$$\int_{\sigma(T)} f(\lambda)\mu_1(d\lambda) = \int_{\sigma(T)} f(\lambda)\mu_2(d\lambda), \quad f \in C(\sigma(T)).$$

μ_1 and μ_2 are finite countably additive measures with supports contained in $\sigma(T)$. Hence they are regular measures, and by the Riesz representation theorem, $\mu_1 = \mu_2$. It then follows that

$$\langle E_1(\tau)x, y\rangle = \langle E_2(\tau)x, y\rangle, \quad \tau \in \Sigma_p, x \in X, y \in \Gamma.$$

Since Γ is total, the conclusion of the lemma follows.

For our next result we make use of the notion of generalized hermiticity introduced in [12] and [16]. We shall not devote space here to a discussion of this concept, but instead refer the reader to §1 of [3].

A spectral measure is called *hermitian* if and only if all of its values are hermitian operators.

3.3 THEOREM. *Let S be a prespectral operator with Hermitian s-resolutions of the identity $E_1(\cdot)$ and $E_2(\cdot)$ of class Γ. Then $E_1(\tau) = E_2(\tau)$, $\tau \in \Sigma_p$.*

Proof. Let $R(\lambda)$ and $I(\lambda)$ denote respectively the real and imaginary parts of the complex number λ. Define

$$R_k = \int_{\sigma(S)} R(\lambda)E_k(d\lambda),$$

$$J_k = \int_{\sigma(S)} I(\lambda)E_k(d\lambda), \quad k = 1, 2.$$

Then R_k and J_k are hermitian operators, and $S = R_k + iJ_k$, $k = 1, 2$. It follows from Lemma 2(c) of [16] that $R_1 = R_2$ and $J_1 = J_2$. By virtue of the standard properties of the integral with respect to a spectral measure these equations lead to

$$\int_{\sigma(S)} p(\lambda, \bar{\lambda})E_1(d\lambda) = \int_{\sigma(S)} p(\lambda, \bar{\lambda})E_2(d\lambda),$$

where p is a polynomial in the two variables λ, $\bar{\lambda}$. Then by the Stone-Weierstrass theorem

$$\int_{\sigma(S)} f(\lambda)E_1(d\lambda) = \int_{\sigma(S)} f(\lambda)E_2(d\lambda), \quad f \in C(\sigma(S)).$$

The result now follows immediately from Lemma 3.2.

Remark. It is known that a bounded Boolean algebra of projections on X can be made into a family of hermitian operators by appropriate equivalent renorming of X. (See the proofs of Lemmas 2.2 and 2.3 of [3].) Thus if $E(\cdot)$ is an s-resolution of the identity of class Γ for the prespectral operator S, in $L(X)$, it can be made into a necessarily unique hermitian s-resolution of the identity of class Γ for S by equivalent renorming of X. We now turn to Theorem 8 of [5], the canonical decomposition theorem. For purposes of comparison we state in full the form applicable to spectral operators.

3.4 THEOREM. *An operator T is spectral if and only if it is the sum $T = S + N$ of a scalar-type spectral operator S and a quasinilpotent operator N such that $SN = NS$. Furthermore this decomposition is unique. T and S have the same spectrum and the same resolution of the identity.*

This may be proved by the arguments given in [5; pp. 333–5] or [6; pp. 226–9]. Note that in proving that the sum of a scalar-type spectral operator S and a commuting quasinilpotent N is spectral, both proofs use the commutativity theorem to show that N commutes with the resolution of the identity of S. This argument cannot be applied to the corresponding situation for prespectral operators. In fact, in §6, we will construct on l^∞ a scalar-type operator S of class l^1 and a nilpotent A with $SA = AS$ such that $S + A$ is not prespectral of any class. However the arguments of [6; pp. 226–9] do suffice to prove the following result.

3.5 THEOREM. (i) *Let T be prespectral with a resolution of the identity $E(\cdot)$ of class Γ. Define $S = \int_{\sigma(S)} \lambda E(d\lambda)$ and $N = T - S$. Then S is prespectral with an s-resolution of the identity $E(\cdot)$ of class Γ, and N is a quasinilpotent operator commuting with $\{E(\tau) : \tau \in \Sigma_p\}$. Moreover $\sigma(T) = \sigma(S)$.*

(ii) *Let S be prespectral with a resolution of the identity $E(\cdot)$ of class Γ such that $S = \int_{\sigma(S)} \lambda E(d\lambda)$. Let N be a quasinilpotent operator commuting with $\{E(\tau) : \tau \in \Sigma_p\}$. Then $S + N$ is prespectral with a resolution of the identity $E(\cdot)$ of class Γ. Moreover $\sigma(S + N) = \sigma(S)$.*

This suggests the introduction of the following terminology.

DEFINITION. A sum $T = S + N$, where S is prespectral with an s-resolution of the identity $E(\cdot)$ of class Γ, and N is a quasinilpotent commuting with $\{E(\tau) : \tau \in \Sigma_p\}$, is called a *Jordan decomposition of class* Γ *for* T. S and N are called respectively the *scalar* and *radical* parts of the decomposition. In this terminology, Theorem 3.5 states that T is prespectral if and only if T admits a Jordan decomposition. Every resolution of the identity of T then defines a Jordan decomposition of T. It is not known, in general, whether different resolutions of the identity of a prespectral operator T may yield different Jordan decompositions of T.

Let T be a spectral operator. The canonical decomposition $T = S + N$ given by Theorem 3.4 is the unique Jordan decomposition of T. This follows from Theorem 3.5 and the discussion on uniqueness of the resolution of the identity of a spectral operator at the beginning of this section. In fact we can make the following stronger assertion.

3.6 THEOREM. *Let T be a spectral operator, and let S and N be respectively the scalar and radical parts of the canonical decomposition of T. If $T = S_0 + N_0$, where S_0 is a scalar-type operator of class Γ, and N_0 is a quasinilpotent operator with $S_0N_0 = N_0S_0$, then $S = S_0$ and $N = N_0$.*

Proof. S_0 is prespectral with an s-resolution of the identity $E(\cdot)$ of class Γ. Note that N_0 commutes with S_0 and hence with T. By the commutativity theorem, N_0 commutes with the resolution of the identity of T, and hence also with S and N. Now $N - N_0$ is quasinilpotent since it is the difference of commuting quasinilpotents. By Theorem 3.4, $S_0 = S + (N - N_0)$ is spectral. Hence $E(\cdot)$ is countably additive in the strong operator topology, and so S_0 is a scalar-type spectral operator. The result now follows from Theorem 3.4.

Our next theorem, a generalization of a result of Dunford, leads to the main techniques used in this section. We denote by Σ_K the σ-field of Borel subsets of a compact Hausdorff space K.

3.7 THEOREM. *Let K be a compact Hausdorff space, and let ψ be a continuous algebra homomorphism of $C(K)$ into $L(X)$ with $\psi(1) = I$. Let N, in $L(X)$, be a quasinilpotent commuting with $\psi(f)$ for every f in $C(K)$. Then there is a spectral measure $E(\cdot)$ of class (Σ_K, X) with values in $L(X^*)$ such that*

3.7(a) $$\psi(f)^* = \int_K f(\lambda)E(d\lambda), \quad f \in C(K),$$

and

3.8 $$N^*E(\tau) = E(\tau)N^*, \quad \tau \in \Sigma_K.$$

Moreover if $S \in \psi(C(K))$, then the adjoint of $T = S + N$ is prespectral of class X, and $S^* + N^*$ is a Jordan decomposition of class X for T^*.

Proof. 3.7(a) follows by the argument used to prove Theorem 18 [5; p.

350]. Let $x \in X$ and $y \in X^*$. Define

$$\mu_1(\tau) = \langle Nx, E(\tau)y\rangle \quad \text{and} \quad \mu_2(\tau) = \langle x, E(\tau)N^*y\rangle, \quad \tau \in \Sigma_K .$$

Then as in the proof of Lemma 3.2

$$\int_K f(\lambda)\mu_1(d\lambda) = \langle Nx, \psi(f)^*y\rangle$$

$$= \langle x, \psi(f)^*N^*y\rangle = \int_K f(\lambda)\mu_2(d\lambda), \quad f \in C(K).$$

It follows that the Borel measures $\langle Nx, E(\cdot)y\rangle$ and $\langle x, E(\cdot)N^*y\rangle$ (regular by construction) are identical, and 3.8 is immediate. If $S = \psi(f)$ for some f in $C(K)$ then by 3.7(a) and 3.1, S^* has an s-resolution of the identity whose range is contained in the range of $E(\cdot)$. Hence by 3.8, $S^* + N^*$ is a Jordan decomposition of class X for T^*.

The following generalization of 3.7(a) is well known, although it does not appear explicitly in the literature.

3.9 THEOREM. *Let X be weakly complete. Let K be a compact Hausdorff space, and let ψ be a continuous algebra homomorphism of $C(K)$ into $L(X)$ with $\psi(1) = I$. Then there is a spectral measure $E(\cdot)$ of class (Σ_K, X^*) such that*

$$\psi(f) = \int_K f(\lambda)E(d\lambda), \quad f \in C(K).$$

Moreover for each f in $C(K)$, $\psi(f)$ is a scalar-type spectral operator with resolution of the identity $F(\cdot)$, where

$$F(\tau) = E(f^{-1}(\tau)), \quad \tau \in \Sigma_p .$$

Outline of proof. Consider for each x in X the map T_x which sends f in $C(K)$ into $\psi(f)x$. By Theorem VI. 7.6 of [7; p. 494], each T_x is weakly compact. Hence by Theorem VI. 7.3 of [7; p. 493], the weak completeness of X implies that for each x in X there is a vector-valued measure $\mu_x(\cdot)$ countably additive on Σ_K such that

$$T_x f = \int_K f(\lambda)\mu_x(d\lambda).$$

Define for each τ in Σ_K a map $E(\tau)$ which sends x into $\mu_x(\tau)$. Routine arguments complete the proof that $E(\cdot)$ has the properties stated. The last statement of the theorem follows from 3.1.

3.10 THEOREM. *Let T, in $L(X)$, be prespectral with a resolution of the identity $E(\cdot)$ of class Γ. Then T^* is prespectral on X^* with a resolution of the identity $F(\cdot)$ of class X such that*

$$\left(\int_{\sigma(T)} f(\lambda)E(d\lambda)\right)^* = \int_{\sigma(T)} f(\lambda)F(d\lambda), \quad f \in C(\sigma(T)).$$

Moreover if $S = \int_{\sigma(T)} \lambda E(d\lambda)$ *and* $N = T - S$, *then* $S^* + N^*$ *is the Jordan decomposition of* T^* *corresponding to* $F(\cdot)$.

Proof. From 3.1, the map ψ defined by

$$\psi(f) = \int_{\sigma(T)} f(\lambda)E(d\lambda)$$

is a continuous algebra homomorphism from $C(\sigma(T))$ into $L(X)$, and N commutes with each $\psi(f)$. Let Σ_0 denote the σ-field of Borel subsets of $\sigma(T)$. Hence by 3.7 there is a spectral measure $G(\cdot)$ of class (Σ_0, X) such that

$$\psi(f)^* = \int_{\sigma(T)} f(\lambda)G(d\lambda), \quad f \in C(\sigma(T));$$

$$N^*G(\tau) = G(\tau)N^*, \quad \tau \in \Sigma_0 .$$

Also from 3.1, S^* is prespectral with an s-resolution of the identity $F(\cdot)$ of class X, where for each δ in Σ_p $F(\delta) = G(\delta \cap \sigma(T))$. Hence

$$\psi(f)^* = \int_{\sigma(T)} f(\lambda)G(d\lambda) = \int_{\sigma(T)} f(\lambda)F(d\lambda), \quad f \in C(\sigma(T)).$$

The result now follows from 3.5.

In the case of a spectral operator more can be asserted.

3.11 THEOREM. *Let* T, *in* $L(X)$, *be a spectral operator. Then* T^*, *prespectral on* X^* *of class* X, *has a unique Jordan decomposition for resolutions of the identity of all classes. Moreover if* T^* *is also prespectral of class* Γ, *then* T^* *has a unique resolution of the identity of class* Γ.

Proof. Let K denote the compact set $\sigma(T)$. Note that $K = \sigma(T) = \sigma(T^*)$. Let $E(\cdot)$ be the resolution of the identity of T. Then by 2.1, T^* is prespectral with resolution of the identity $E^*(\cdot)$ of class X. Let $F_1(\cdot)$ and $F_2(\cdot)$ be resolutions of the identity of classes Γ_1 and Γ_2 respectively for T^*. By Theorem 3.10, T^{**} is prespectral of class X^*, and there are resolutions of the identity $G_1(\cdot)$ and $G_2(\cdot)$ of class X^* for T^{**} such that

$$\left(\int_K f(\lambda)F_1(d\lambda)\right)^* = \int_K f(\lambda)G_1(d\lambda), \quad f \in C(K),$$

$$\left(\int_K f(\lambda)F_2(d\lambda)\right)^* = \int_K f(\lambda)G_2(d\lambda), \quad f \in C(K).$$

T^{**} is a prespectral operator on X^{**}, and X is a closed subspace of X^{**} invariant under T^{**}. Moreover the restriction of T^{**} to X is a spectral operator, and so by Theorem 2.1 of [8; p. 1032]

$$G_1(\tau)x = G_2(\tau)x = E(\tau)x, \quad x \in X, \quad \tau \in \Sigma_p .$$

Therefore

$$\int_K f(\lambda)G_1(d\lambda)x = \int_K f(\lambda)G_2(d\lambda)x = \int_K f(\lambda)E(d\lambda)x, \quad f \in C(K), x \in X.$$

Let $y \in X^*$. Then for $r = 1, 2$,

$$\left\langle \int_K f(\lambda)G_r(d\lambda)x, y \right\rangle = \left\langle x, \int_K f(\lambda)F_r(d\lambda)y \right\rangle, \quad f \in C(K), x \in X, y \in X^*,$$

and so

$$\left\langle x, \int_K f(\lambda)F_1(d\lambda)y \right\rangle = \left\langle x, \int_K f(\lambda)F_2(d\lambda)y \right\rangle, \quad f \in C(K), x \in X, y \in X^*.$$

Hence

$$\int_K f(\lambda)F_1(d\lambda) = \int_K f(\lambda)F_2(d\lambda), \quad f \in C(K).$$

From this result $F_1(\cdot)$ and $F_2(\cdot)$ yield the same Jordan decomposition. Now let $\Gamma_1 = \Gamma_2 = \Gamma$. It then follows from Lemma 3.2 that T^* has a unique resolution of the identity of class Γ.

3.12 Corollary. *Let T, in $L(X)$, be a spectral operator with resolution of the identity $E(\cdot)$. Let $S = \int_{\sigma(T)} \lambda E(d\lambda)$ and $N = T - S$. Then T^* is prespectral on X^* with uniuqe resolution of the identity $E^*(\cdot)$ of class X. Moreover $S^* + N^*$ is the unique Jordan decomposition of T^* for resolutions of the identity of all classes.*

Proof. This result follows immediately from Theorems 2.1 and 3.11.

It was noted in §2 that in a weakly complete Banach space the classes of prespectral operators and spectral operators coincide. Many non-weakly complete Banach spaces have weakly complete dual spaces. (See for example the tables in [7; pp. 374–9]). We have the following results for prespectral operators on such spaces.

3.13 Theorem. *Let X^* be weakly complete and let T, in $L(X)$, be prespectral of class Γ. Then T has a unique Jordan decomposition for resolutions of the identity of all classes. Moreover T has a unique resolution of the identity of class Γ.*

Proof. By Theorem 3.10, T^* is prespectral. Since X^* is weakly complete, T^* is spectral. Let $E_1(\cdot)$ and $E_2(\cdot)$ be resolutions of the identity of classes Γ_1 and Γ_2 respectively for T. Let $F(\cdot)$ be the unique resolution of the identity for T^*. Then by Theorem 3.10,

$$\left(\int_{\sigma(T)} \lambda E_1(d\lambda)\right)^* = \int_{\sigma(T)} \lambda F(d\lambda) = \left(\int_{\sigma(T)} \lambda E_2(d\lambda)\right)^*,$$

and hence T has a unique Jordan decomposition for resolutions of the identity of all classes. Now let $\Gamma_1 = \Gamma_2$. Again by Theorem 3.10

$$\int_{\sigma(T)} f(\lambda)E_1(d\lambda) = \int_{\sigma(T)} f(\lambda)E_2(d\lambda), \quad f \in C(\sigma(T)),$$

and so by Lemma 3.2, T has a unique resolution of the identity of class Γ.

3.14 THEOREM. *Let X^* be weakly complete. Let S, in $L(X)$, be a scalar-type operator of class Γ, and let N be a quasinilpotent with $SN = NS$. Then if $T = S + N$ is prespectral, every resolution of the identity of T is an s-resolution of the identity of S, and $T = S + N$ is the unique Jordan decomposition of T. Moreover N commutes with every resolution of the identity of T.*

Proof. By Theorem 3.10, T^* is prespectral. Since X^* is weakly complete, T^* is spectral. Let S_1 and N_1 be respectively the scalar and radical parts of the canonical decomposition of T^*. Again by 3.10, S^* is prespectral with an s-resolution of the identity of class X. Hence S^* is a scalar-type spectral operator. N^* is a quasinilpotent with $S^*N^* = N^*S^*$. From the uniqueness of the canonical decomposition of T^*, we get $S_1 = S^*$ and $N_1 = N^*$. Let $E(\cdot)$ be a resolution of the identity for T. Then by 3.10

$$S^* = S_1 = \left(\int_{\sigma(T)} \lambda E(d\lambda)\right)^*.$$

The present theorem now follows immediately.

3.15 THEOREM. *Let X^* be weakly complete. Let S, in $L(X)$, be prespectral with s-resolution of the identity $E(\cdot)$ of class Γ. Let N be a quasinilpotent operator such that $SN = NS$. Then $S + N$ is prespectral of class Γ if and only if*

$$NE(\tau) = E(\tau)N, \quad \tau \in \Sigma_p.$$

Proof. The condition is certainly sufficient by 3.5. Now let $S + N$ be prespectral with resolution of the identity $F(\cdot)$ of class Γ. By the preceding theorem S is prespectral with an s-resolution of the identity $F(\cdot)$ of class Γ, and

$$NF(\tau) = F(\tau)N, \quad \tau \in \Sigma_p.$$

By 3.13, S has a unique resolution of the identity $E(\cdot)$ of class Γ. Hence $F(\cdot) = E(\cdot)$ and

$$NE(\tau) = E(\tau)N, \quad \tau \in \Sigma_p.$$

To conclude this section we state a generalization of Theorem 3.13, which may be proved in a similar way.

3.16 THEOREM. *Let T, in $L(X)$, be a prespectral operator of class Γ. Suppose that T^*, prespectral on X^* of class X, has a unique resolution of the* *identity of class X. Then T has a unique Jordan decomposition for resolutions of the identity of all classes. Also T has a unique resolution of the identity of class Γ.*

We observe that by this theorem, if it were known that every prespectral operator of class Γ had a unique resolution of the identity of class Γ, then it could be deduced that every prespectral operator had a unique Jordan decomposition for resolutions of the identity of all classes. Also if it were known that every scalar-type operator of class Γ had a unique resolution of the identity of class Γ, then it could be deduced that scalar-type operators possessed only s-resolutions of the identity. Moreover, in considering the problem of uniqueness of resolution of the identity for prespectral operators (respectively scalar-type operators) it is sufficient to consider only prespectral operators (respectively scalar-type operators) on X^* of class X.

4. Operators with totally disconnected spectra

4.1. Let $T \in L(X)$. Corresponding to each open-and-closed subset δ of $\sigma(T)$ there is a *spectral projection for* T defined by

4.2 $$A(\delta) = \frac{1}{2\pi i}\int_C (\lambda I - T)^{-1}\, d\lambda,$$

where C is a contour in $\rho(T)$ which encloses δ but excludes $\sigma(T)\backslash\delta$. Moreover the operator $A(\delta)$ does not depend on the particular contour C chosen. The map $\delta \leftrightarrow A(\delta)$ is an isomorphism from the Boolean algebra of open-and-closed subsets of $\sigma(T)$ onto a Boolean algebra of projections in $L(X)$. For each open-and-closed subset δ of $\sigma(T)$ we have $TA(\delta) = A(\delta)T$ and

4.3 $$\sigma(T \mid A(\delta)X) = \delta.$$

The reader is referred to [15; pp. 298–302] for a complete discussion and proofs of these properties. In order to prove the first theorem of this section a preliminary result is required. The concepts introduced in 2.2 will be used in the proof.

4.4 Lemma. *Let T, in $L(X)$, be a prespectral operator with a resolution of the identity $E(\cdot)$ of class Γ. Then for each open-and-closed subset δ of $\sigma(T)$, $E(\delta)$ is equal to the spectral projection $A(\delta)$ for T.*

Proof. Let $x \in E(\delta)X$. By 2.3, T has the single-valued extension property and $\sigma(x) \subseteq \delta$. Therefore by 4.2,

$$A(\sigma(T)\backslash\delta)\, x = 0.$$

Hence $A(\delta)x = x$, and we have

4.5 $$A(\delta)E(\delta) = E(\delta).$$

Now let $y \in E(\sigma(T)\backslash\delta)X$. By 2.3, $\sigma(y) \subseteq \sigma(T)\backslash\delta$, and so by 4.2, $A(\delta)y = 0$.

Hence

4.6 $$A(\delta)E(\sigma(T)\backslash\delta) = 0.$$

Addition of 4.5 and 4.6 gives the required result.

DEFINITION. A subset of p is called *totally disconnected* if and only if the connected component of each point is the set consisting of the point itself.

A compact set in p is totally disconnected if and only if its topology has a base of open-and-closed subsets [9; p. 247]. Hence by Lindelöf's theorem [10; p. 49] a compact totally disconnected subset of p has a countable base of open-and-closed subsets.

4.7 THEOREM. *Let T, in $L(X)$, be prespectral of class Γ*, and let $\sigma(T)$ *be totally disconnected. Then T has a unique resolution of the identity of class Γ. Also T has a unique Jordan decomposition for resolutions of the identity of all classes.*

Proof. Let $E(\cdot)$ and $F(\cdot)$ be resolutions of the identity of class Γ for T. By Lemma 4.4, $E(\delta) = F(\delta)$ for each open-and-closed subset δ of $\sigma(T)$. The topology of $\sigma(T)$ has a countable base of such subsets. Hence for each relatively open subset τ of $\sigma(T)$

$$\langle E(\tau)x, y\rangle = \langle F(\tau)x, y\rangle, \qquad x \in X, \quad y \in \Gamma.$$

Since $E(\sigma(T)) = F(\sigma(T)) = I$ and Γ is total, it follows that

$$E(\tau) = F(\tau), \quad \tau \in \Sigma_p .$$

Hence T has a unique resolution of the identity of class Γ. By 3.10, T^* is prespectral of class X. Since $\sigma(T^*)$ is totally disconnected, T^* has a unique resolution of the identity of class X. Application of 3.16 completes the proof.

4.8 THEOREM. *Let S be a scalar-type operator of class Γ, and let $\sigma(S)$ be totally disconnected. Let N be a quasi-nilpotent operator with $SN = NS$. Then if $T = S + N$ is prespectral, every resolution of the identity of T is an s-resolution of the identity of S, and $T = S + N$ is the unique Jordan decomposition for T. Moreover N commutes with every resolution of the identity for T.*

Proof. The argument of [6; pp. 227–8] shows that $\sigma(T) = \sigma(S)$. Hence $\sigma(T)$ is totally disconnected. Let $F(\cdot)$ be a resolution of the identity for T, and let $E(\cdot)$ be the s-resolution of the identity of class Γ for S. Let δ be an open-and-closed subset of $\sigma(T)$. There exist disjoint open sets G_1 and G_2 with $\delta \subseteq G_1$ and $\sigma(T)\backslash\delta \subseteq G_2$. Define a function f by $f(z) = 1$, $z \in G_1$, and $f(z) = 0$, $z \in G_2$. Note that f is analytic on a neighborhood of $\sigma(T) = \sigma(S)$. It follows from Corollary VII. 6. 12 of [7; p. 592] that

$$f(T) = f(S + N) = \sum_{n=0}^{\infty} \frac{f^{(n)}(S)N^n}{n!} = f(S),$$

in the usual notation. Now by 4.4, $f(T) = F(\delta)$ and $f(S) = E(\delta)$. Hence $F(\delta) = E(\delta)$. Let $\varepsilon > 0$ be given. Since the compact set $\sigma(T)$ has a base of open-and-closed subsets, it is easily shown that there exist a partition of $\sigma(T)$ into open-and-closed subsets $\{\delta_i : i = 1, 2, \cdots n\}$ and points λ_i in δ_i, $i = 1, 2, \cdots n$, such that

$$|\lambda - \lambda_i| < \varepsilon, \quad \lambda \in \delta_i, \quad i = 1, 2, \cdots n.$$

Now $\sum_{i=1}^{n} \lambda_i E(\delta_i) = \sum_{i=1}^{n} \lambda_i F(\delta_i)$. Since ε is an arbitrary positive number, it follows from 3.1 that

$$S = \int_{\sigma(T)} \lambda E(d\lambda) = \int_{\sigma(T)} \lambda F(d\lambda).$$

The conclusions of the theorem now follow immediately.

4.9 THEOREM. *Let S be a prespectral operator with an s-resolution of the identity $E(\cdot)$ of class Γ, and let $\sigma(S)$ be totally disconnected. Let N be a quasi-nilpotent operator with $SN = NS$. Then $S + N$ is prespectral of class Γ if and only if*

$$NE(\tau) = E(\tau)N, \quad \tau \in \Sigma_p.$$

Proof. The condition is clearly sufficient by 3.5. Now let $S + N$ be prespectral with resolution of the identity $F(\cdot)$ of class Γ. By the previous theorem, S is prespectral with an s-resolution of the identity $F(\cdot)$ of class Γ, and

$$NF(\tau) = F(\tau)N, \quad \tau \in \Sigma_p.$$

By 4.7, S has a unique resolution of the identity $E(\cdot)$ of class Γ. Hence $F(\cdot) = E(\cdot)$ and

$$NE(\tau) = E(\tau)N, \quad \tau \in \Sigma_p.$$

In order to prove the next theorem we require the following well-known elementary result.

4.10 LEMMA. *Let $T \in L(X)$. Let E, F be projections in $L(X)$ such that $EF = F$ and T, E, F commute. Then*

$$\sigma(T \mid FX) \subseteq \sigma(T \mid EX) \subseteq \sigma(T).$$

Proof. Let $\lambda \in \rho(T)$. Now E commutes with T and hence also with $(\lambda I - T)^{-1}$. Therefore $(\lambda I - T)^{-1}$ leaves EX invariant, and its restriction to that subspace is a bounded operator, clearly inverse to $(\lambda I - T) \mid EX$. Hence $\lambda \in \rho(T \mid EX)$, and $\sigma(T \mid EX) \subseteq \sigma(T)$. Similarly $T \mid EX$ commutes with $F \mid EX$, and $\sigma(T \mid FX) \subseteq \sigma(T \mid EX)$.

4.11 THEOREM. *Let T, in $L(X)$, have totally disconnected spectrum. In order that T^* be prespectral of class X it is necessary and sufficient that the set $\{A(\delta) : \delta$ open-and-closed in $\sigma(T)\}$ of spectral projections for T be uniformly* *bounded in norm. If this is the case,* T^* *has a unique resolution of the identity* $E(\cdot)$ *of class* X, *where for each open-and-closed subset* δ *of* $\sigma(T)$, $E(\delta) = A(\delta)^*$.

Proof. If T^* is prespectral with resolution of the identity $E(\cdot)$ of class X, then by 4.4, $E(\delta) = A(\delta)^*$, and so the condition is necessary. In this case, $E(\cdot)$ is unique by 4.7. Conversely suppose $\| A(\delta)\| \leq M$, δ open-and-closed. Let $f \in C(\sigma(T))$, and let $\varepsilon > 0$ be given. Since the topology of the compact set $\sigma(T)$ has a base of open-and-closed subsets, it is easily shown that there exist a partition of $\sigma(T)$ into open-and-closed subsets $\{\delta_i : i = 1, 2, \cdots n\}$ and points λ_i in δ_i, $i = 1, 2, \cdots n$, such that

$$|f(\lambda_i) - f(\lambda)| < \varepsilon, \quad \lambda \in \delta_i, \quad i = 1, 2, \cdots n.$$

Hence the algebra $\tilde{A}$ of finite linear combinations of characteristic functions of disjoint open-and-closed subsets of $\sigma(T)$ is dense in $C(\sigma(T))$. Define a map ψ from $\tilde{A}$ into $L(X)$ by

$$\psi(\textstyle\sum_{i=1}^n \alpha_i \chi(\tau_i)) = \sum_{i=1}^n \alpha_i A(\tau_i), \quad \tau_i \cap \tau_j = \emptyset \quad \text{if} \quad i \neq j,$$

where $\chi(\tau_i)$ denotes the characteristic function of τ_i. It is easy to see that ψ is well defined and that

$$\| \psi(f)\| \leq 4M \sup_{\lambda \in \sigma(T)} |f(\lambda)|, \quad f \in \tilde{A}.$$

Therefore ψ can be extended to a continuous homomorphism from $C(\sigma(T))$ into $L(X)$. Let Σ_0 denote the σ-field of Borel subsets of $\sigma(T)$. By 3.7(a) there is a spectral measure $E_0(\cdot)$ of class (Σ_0, X) such that

4.12 $$\psi(f)^* = \int_{\sigma(T)} f(\lambda)E_0(d\lambda), \quad f \in C(\sigma(T)).$$

Define $E(\cdot)$ on Σ_p by

$$E(\delta) = E_0(\delta \cap \sigma(T)), \quad \delta \in \Sigma_p.$$

Then $E(\cdot)$ is a spectral measure of class (Σ_p, X). Let τ be an open-and-closed subset of $\sigma(T)$. By 4.12, $A(\tau)^* = E(\tau)$. Also

$$\langle Tx, E(\tau)y\rangle = \langle Tx, A(\tau)^*y\rangle = \langle x, E(\tau)T^*y\rangle, \quad x \in X, \quad y \in X^*.$$

The topology of $\sigma(T)$ has a countable base of open-and-closed subsets. Since $E(\sigma(T)) = I$, it follows that the regular measures $\langle Tx, E(\cdot)y\rangle$ and $\langle x, E(\cdot)T^*y\rangle$ are identical for all x in X, y in X^*. Hence

$$T^*E(\delta) = E(\delta)T^*, \quad \delta \in \Sigma_p.$$

Finally let $\delta \in \Sigma_p$, and let τ be an open-and-closed subset of $\sigma(T)$ with $\bar{\delta} \cap \sigma(T) \subseteq \tau$. Then, since $A(\tau)^* = E(\tau)$, it follows from 4.10 and 4.3 that

$$\sigma(T^* \mid E(\delta)X^*) \subseteq \sigma(T^* \mid E(\bar{\delta} \cap \sigma(T))X^*) \subseteq \sigma(T^* \mid E(\tau)X^*) = \tau.$$

Now since $\sigma(T)$ is totally disconnected, $\bar{\delta} \cap \sigma(T)$ is equal to the intersection of all open-and-closed subsets τ of $\sigma(T)$ with $\bar{\delta} \cap \sigma(T) \subseteq \tau$. Hence

$$\sigma(T^* \mid E(\delta)X^*) \subseteq \bar{\delta} \cap \sigma(T) \subseteq \bar{\delta}, \quad \delta \in \Sigma_p .$$

By using Theorem 3.9 instead of 3.7 (a) at the appropriate stage of the proof, we can obtain the following result by similar arguments.

4.13 THEOREM. *Let X be weakly complete and let T, in $L(X)$, have totally disconnected spectrum. In order that T be a spectral operator it is necessary and sufficient that the set $\{A(\delta) : \delta$ open-and-closed in $\sigma(T)\}$ of spectral projections for T be uniformly bounded in norm.*

Dunford [6; p. 252] proved this theorem by a different method.

5. Scalar-type operators whose spectra are R-sets

In order to prove the main result of this section a preliminary lemma, due to Lumer, is required.

5.1 LEMMA. *Let A and B be commuting hermitian operators. Let N be a quasinilpotent operator with $N = A + iB$. Then $A = B = 0$.*

Proof. By Lemma 15 of [13; p. 82], $B = 0$. Since iN is quasinilpotent, similar reasoning applied to the equation

$$iN = -B + iA$$

yields $A = 0$.

DEFINITION. A compact subset K of p is called an *R-set* if and only if the rational functions with poles in $p\backslash K$ are uniformly dense in $C(K)$.

We observe that every R-set is nowhere dense, but that there exist nowhere dense compact subsets of p which are not R-sets. If a compact subset of p has plane Lebesgue measure 0, or if it is nowhere dense and its complement has a finite number of components, then it is an R-set.

5.2 THEOREM. *Let S, in $L(X)$, be a scalar-type operator of class Γ, and let $\sigma(S)$ be an R-set. Then every resolution of the identity for S is an s-resolution of the identity. Also S has a unique resolution of the identity of class Γ.*

Proof. Let $E(\cdot)$ be an s-resolution of the identity of class Γ for S, and let $F(\cdot)$ be a resolution of the identity of class Γ_0 for S. Let $S_0 = \int_{\sigma(S)} \lambda F(d\lambda)$, and let $S = S_0 + N$ be the Jordan decomposition of S corresponding to $F(\cdot)$. Define

$$R = \int_{\sigma(S)} R(\lambda)E(d\lambda), \quad J = \int_{\sigma(S)} I(\lambda)E(d\lambda),$$

$$R_0 = \int_{\sigma(S)} R(\lambda)F(d\lambda), \qquad J_0 = \int_{\sigma(S)} I(\lambda)F(d\lambda).$$

Since $\sigma(S)$ is an R-set, there are sequences $\{r_n\}$ and $\{j_n\}$ of rational functions with poles outside $\sigma(S)$ converging uniformly on $\sigma(S)$ to $R(\cdot)$ and $I(\cdot)$ respectively. It follows from 3.1 that in the uniform operator topology

$$r_n(S) \to R, \quad r_n(S_0) \to R_0, \quad j_n(S) \to J \quad \text{and} \quad j_n(S_0) \to J_0 .$$

From this and the relation $SS_0 = S_0S$ it is clear that the operators R, R_0, J and J_0 commute. Since each of these four operators can be made hermitian by equivalent renorming of X [3; Theorem 2.5], and since these operators commute, it follows from Corollary 7 of [13; p. 78] that after some appropriate equivalent renorming of X they are simultaneously hermitian. We assume that this renorming has been carried out. By applying Lemma 5.1 to the equation

$$N = (R - R_0) + i(J - J_0)$$

we obtain $N = 0$. Hence $S = \int_{\sigma(S)} \lambda F(d\lambda)$. For every rational function g with poles outside $\sigma(S)$ we have

$$g(S) = \int_{\sigma(S)} g(\lambda)E(d\lambda) = \int_{\sigma(S)} g(\lambda)F(d\lambda).$$

By hypothesis such rational functions are uniformly dense in $C(\sigma(S))$, and so by 3.1

$$\int_{\sigma(S)} f(\lambda)E(d\lambda) = \int_{\sigma(S)} f(\lambda)F(d\lambda), \quad f \in C(\sigma(S)).$$

If $F(\cdot)$ is also of class Γ then, by 3.2, $F(\cdot) = E(\cdot)$. This completes the proof.

5.3 Theorem. *Let $S \in L(X)$, and let $\sigma(S)$ be an R-set. In order that S^* be a scalar-type operator of class X it is necessary and sufficient that there exist a constant $M > 0$ such that for each rational function g with poles outside $\sigma(S)$*

$$\| g(S)\| \leq M \sup_{\lambda\in\sigma(S)} | g(\lambda)|.$$

Proof. Necessity is obvious, since if S^* is prespectral with an s-resolution of the identity $E(\cdot)$ of class X, then

$$g(S)^* = \int_{\sigma(S)} g(\lambda)E(d\lambda).$$

Conversely suppose the condition is satisfied. It follows that the map $r \to r(S)$, which sends a rational function in $C(\sigma(S))$ into an element of $L(X)$, is well defined. Since $\sigma(S)$ is an R-set, this map can be extended to a continuous algebra homomorphism from $C(\sigma(S))$ into a subalgebra of $L(X)$ containing S. The conclusion now follows at once from Theorem 3.7.

The next result, the last in this section, may be deduced in a similar manner from Theorem 3.9.

5.4 THEOREM. *Let X be weakly complete, and let $S \in L(X)$. Let $\sigma(S)$ be an R-set. Then S is a scalar-type spectral operator if and only if there is a constant $M > 0$ such that for each rational function g with poles outside $\sigma(S)$*

$$\| g(S)\| \leq M \sup_{\lambda \in \sigma(S)} | g(\lambda)|.$$

6. The Fixman example

This section is devoted to simplifying Fixman's example 2.7 [8; pp. 1035–6] and developing some further consequences. On the subspace of l^∞ consisting of convergent sequences, the map which assigns to each such sequence its limit is a linear functional of norm 1. Throughout this section L denotes a fixed linear functional on l^∞ with $\| L \| = 1$ such that for each convergent sequence $\{\xi_n\}$

$$L(\{\xi_n)\}) = \lim_n \{\xi_n\}.$$

Define operators S and A on l^∞ by

$$\begin{aligned} S\{\xi\} = \{\eta_n\}, \quad \text{where } \eta_n &= \xi_n, \text{ if } n = 1 \text{ or } 2, \\ &= \frac{n-2}{n-1}\xi_n, \text{ if } n = 3, 4, 5, \cdots; \end{aligned}$$

$$A\{\xi_n\} = \{L(\{\xi_n\}), L(\{\xi_n\}), 0, 0, 0, \cdots\}.$$

The operators S and A here defined are modifications of those employed in [8] and are more convenient to our purposes. Clearly $\| A \| = 1$ and $A^2 = 0$. Also

$$\begin{aligned} S\{\xi_n\} = \{\xi_n\} - \{\gamma_n\}, \quad \text{where } \gamma_n &= 0, \text{ if } n = 1, 2, \\ &= \frac{1}{n-1}\xi_n, \text{ if } n = 3, 4, 5, \cdots \end{aligned}$$

Since $L(\{\gamma_n\}) = 0$, then $AS\{\xi_n\} = A\{\xi_n\}$. It is easy to see that $SA\{\xi_n\} = A\{\xi_n\}$, and hence

$$AS = SA.$$

$\sigma(S)$ is the totally disconnected set consisting of 1 and the numbers $(n-2)/(n-1)$ for $n = 3, 4, 5, \cdots$. By regarding S as the adjoint of an operator on l^1 it follows from Theorem 4.11 that S is prespectral with a unique resolution of the identity $E(\cdot)$ of class l^1 satisfying

6.1
$$\begin{aligned} E(\{1\})\{\xi_k\} &= \{\xi_1, \xi_2, 0, 0, 0, \cdots\}, \\ E\left(\left\{\frac{n-2}{n-1}\right\}\right)\{\xi_k\} &= \{\delta_{kn}\,\xi_k\} \text{ for } n = 3, 4, \cdots, \end{aligned}$$

where $\delta_{kn} = 1$ if $n = k$, and $\delta_{kn} = 0$ if $n \neq k$. Define the sequence $\{\lambda_n\}$ by setting

$$\lambda_n = 1 \quad \text{if} \quad n = 1, 2; \quad \lambda_n = (n-2)/(n-1) \quad \text{if} \quad n = 3, 4, 5, \cdots$$

Then it is easy to see from 6.1 that for τ in Σ_p, $E(\tau)$ is the operator which multiplies the n^{th} term of a sequence by 1 if $\lambda_n \in \tau$ and by 0 if $\lambda_n \notin \tau$. The sequence $\{f_n\}$ of functions on $\sigma(S)$, given by

$$\begin{aligned} f_n(\lambda) &= \lambda \quad \text{if} \quad \lambda < (n-2)/(n-1), \\ &= 1 \quad \text{if} \quad \lambda \geq (n-2)/(n-1), \end{aligned}$$

for $n = 3, 4, 5, \cdots$, converges uniformly to the function identically equal to λ on $\sigma(S)$. One sees directly that $\int_{\sigma(S)} f_n(\lambda)E(d\lambda)$ converges to S in the uniform operator topology, and hence

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

Observe that:

(i) S is a scalar-type operator on l^∞ of class l^1, and $\sigma(S)$ is an R-set;
(ii) S is the adjoint of a scalar-type spectral operator by 5.3 and 5.4;
(iii) $(l^\infty)^*$ is weakly complete by IV. 8.16 and IV. 9.9 of [7];
(iv) $\sigma(S)$ is totally disconnected.

It follows from any one of the Theorems 3.11, 3.13, 4.7 and 5.2 that every resolution of the identity for S is an s-resolution of the identity, and any two resolutions of the identity of the same class for S are identical. Also by 2.1, $E(\cdot)$ arises from the resolution of the identity of a scalar-type spectral operator on l^1 by taking adjoints. Since

$$AE(\{1\})\{1, 1, 1, \cdots\} = \{0, 0, 0, \cdots\}$$

and

$$E(\{1\})A\{1, 1, 1, \cdots\} = \{1, 1, 0, \cdots\}$$

we have

6.2 *A commutes with S but not with the resolution of the identity of class l^1 for S.*

Next we define $T_1 = S + A$. Thus T_1 is the sum of S and a nilpotent commuting with S. It is clear from 6.2 and Theorem 4.9 that T_1 is not prespectral of class l^1. In fact we shall show

6.3. *T_1 is not prespectral of any class.*

Suppose to the contrary that $G(\cdot)$ is a resolution of the identity of class Γ for T_1. By either 3.14 or 4.8, $G(\cdot)$ is an s-resolution of the identity of class Γ for S, and A commutes with every value of $G(\cdot)$. Now by 2.3 the projections $G(\{1\})$ and $E(\{1\})$ have the same range. Also

$$G(\{1\})\{1, 1, 1, \cdots\} \in E(\{1\})l^\infty$$

and

$$AG(\{1\})\{1, 1, 1, \cdots\} = \{0, 0, 0, \cdots\}.$$

However

$$A\{1, 1, 1, \cdots\} = \{1, 1, 0, \cdots\} \in E(\{1\})l^\infty = G(\{1\})l^\infty$$

and

$$G(\{1\})A\{1, 1, 1, \cdots\} = \{1, 1, 0, \cdots\}.$$

This gives a contradiction, and so 6.3 is established.

If A were to commute with some resolutionof the identity for S, then Theorem 3.5 would give a contradiction to 6.3. Thus

6.4. *A does not commute with any resolution of the identity for S.*

Resolutions of the identity other than $E(\cdot)$ can be constructed for S by the method of Fixman. Define

6.5 $$F(\tau) = E(\tau) + AE(\tau) - E(\tau)A, \quad \tau \in \Sigma_p .$$

Using the relations $A^2 = 0$ and $AE(\tau)A = 0$, $\tau \in \Sigma_p$, it is easily verified that $F(\cdot)$ is a homomorphism from Σ_p into a Boolean algebra of projections on l^∞ with $F(\sigma(S)) = I$. Vlearly $\| F(\tau)\| \leq 3$, $\tau \in \Sigma_p$.

For each positive integer n let e_n in l^1 be given by $e_n = \{\delta_{nk}\}_{k=1}^\infty$, and let e_n^* be the corresponding linear functional on l^∞. Let Γ_1 be the total linear manifold in $(l^\infty)^*$ generated by $e_1^* - L$, $e_2^* - L$, and $\{e_n^* : n = 3, 4, 5, \cdots\}$. Since for each τ in Σ_p and x in l^∞

$$\langle F(\tau)x, e_n^*\rangle = \langle E(\tau)x, e_n^*\rangle, \quad n = 3, 4, 5, \cdots ,$$

$$\langle F(\tau)x, e_n^* - L\rangle = \chi_\tau(1)\langle x, e_n^* - L\rangle, \quad n = 1, 2,$$

where χ_τ is the characteristic function of τ, it follows that $F(\cdot)$ is Γ_1-countably additive. Since $E(\cdot)$ and A commute with S, elementary algebra shows that $F(\tau)S = SF(\tau)$, $\tau \in \Sigma_p$. In order to prove that $F(\cdot)$ is a resolution of the identity for S it remains only to show that $\sigma(S \mid F(\tau)l^\infty) \subseteq \bar{\tau}$, $\tau \in \Sigma_p$. By virtue of Lemma 4.10 it suffices to prove this inclusion when τ is a closed subset of $\sigma(S)$. Again by Lemma 4.10, and the fact that $\sigma(S)$ is totally disconnected, it is sufficient to prove the inclusion for an open-and-closed subset τ of $\sigma(S)$. It is easy to see from the definition of $F(\cdot)$ that $E(\cdot)$ and $F(\cdot)$ agree on finite subsets of $\sigma(S)\backslash\{1\}$. Since every open-and-closed subset of $\sigma(S)$ is such a set or the complement in $\sigma(S)$ of such a set, $F(\cdot)$ and $E(\cdot)$ agree on open-and-closed subsets of $\sigma(S)$. Therefore

$$\sigma(S \mid F(\tau)l^\infty) = \sigma(S \mid E(\tau)l^\infty) \subseteq \tau$$

for τ open-and-closed in $\sigma(S)$. (This is simpler than the proof given in [8]). In establishing 6.2 it was shown that A and $E(\{1\})$ do not commute. Hence from 6.5, $F(\{1\}) \neq E(\{1\})$. Therefore $F(\cdot)$ and $E(\cdot)$ are distinct.

In contrast to the property of A stated in 6.4 we show

6.6. *There is a nilpotent N commuting with $E(\cdot)$ but not with $F(\{1\})$.*

We define N on l^∞ by setting

$$N(\xi_n\} = \{\xi_2, 0, 0, 0, \cdots\}.$$

Then $\| N \| = 1$, $N^2 = 0$ and N commutes with $E(\cdot)$. Moreover

$$F(\{1\})\{1, \tfrac{1}{2}, 1, 1, 1, \cdots\} = \{0, -\tfrac{1}{2}, 0, 0, 0, \cdots\};$$

$$NF(\{1\})\{1, \tfrac{1}{2}, 1, 1, 1, \cdots\} = \{-\tfrac{1}{2}, 0, 0, 0, 0, \cdots\}.$$

However

$$N\{1, \tfrac{1}{2}, 1, 1, 1, \cdots\} = \{\tfrac{1}{2}, 0, 0, 0, 0, \cdots\};$$

$$F(\{1\})N\{1, \tfrac{1}{2}, 1, 1, 1, \cdots\} = \{\tfrac{1}{2}, 0, 0, 0, 0, \cdots\}.$$

Therefore 6.6 is demonstrated.

Define $T_2 = S + N$. Since N commutes with $E(\cdot)$, T_2 is prespectral of class l^1 by Theorem 3.5. By either 3.14 or 4.8, every resolution of the identity of T_2 is an s-resolution of the identity of S. Now T_2 does not commute with $F(\cdot)$, and so S has an s-resolution of the identity $F(\cdot)$ which is not a resolution of the identity of T_2. Moreover, if in the statements of Theorems 3.15 and 4.9 the words "of class Γ" are deleted in both places, then the theorems fail.

To round off the considerations in 6.4 and 6.6 we show:

6.7. *If Q is a quasinilpotent commuting with every resolution of the identity for S, then $Q = 0$.*

Since for $n = 3, 4, 5, \cdots$,

$$Q \mid E\left(\left\{\frac{n-2}{n-1}\right\}\right) l^\infty$$

is a quasinilpotent on a 1-dimensional space, it is 0. Therefore

$$QE\left(\left\{\frac{n-2}{n-1}\right\}\right) = 0,$$

and so

6.8 $$E\left(\left\{\frac{n-2}{n-1}\right\}\right) Q = 0, \quad n = 3, 4, 5, \cdots$$

Let $\{\xi_k\} \in l^\infty$, and let $Q\{\xi_k\} = \{\eta_k\}$. Then from 6.8 it follows that

$$0 = E\left(\left\{\frac{n-2}{n-1}\right\}\right) Q\{\xi_k\} = \{\delta_{nk}\eta_k\}_{k=1}^{\infty}.$$

Hence $\eta_n = 0$ for $n \geq 3$. If further $\xi_1 = \xi_2 = 0$ then clearly

$$QE(\{1\})\{\xi_k\} = Q\{\xi_1, \xi_2, 0, 0, 0, \cdots\} = 0,$$

and so in this case

$$0 = E(\{1\})Q\{\xi_k\} = \{\eta_1, \eta_2, 0, 0, 0, \cdots\},$$

which gives

(6.9) $$Q\{\xi_k\} = 0 \quad \text{if} \quad \xi_1 = \xi_2 = 0.$$

Now we consider $Q \mid E(\{1\})l^\infty$. Representing this operator by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

relative to the basis $\{1, 0, 0, \cdots\}$ and $\{0, 1, 0, 0, \cdots\}$, we observe that for any $\{\xi_k\}$ in l^∞,

$$Q\{\xi_1, \xi_2, 0, 0, 0, \cdots\} = \{a\xi_1 + b\xi_2, c\xi_1 + d\xi_2, 0, 0, 0, \cdots\}.$$

Hence by 6.9,

$$\begin{aligned} Q\{\xi_k\} &= Q\{\xi_1, \xi_2, 0, 0, 0, \cdots\} + Q\{0, 0, \xi_3, \xi_4, \xi_5, \cdots\} \\ &= Q\{\xi_1, \xi_2, 0, 0, 0, \cdots\} \\ &= \{a\xi_1 + b\xi_2, c\xi_1 + d\xi_2, 0, 0, 0, \cdots\}. \end{aligned} \tag{6.10}$$

Direct computation with 6.5 shows that

$$F(\{1\})\{1, 0, 1, 1, 1, \cdots\} = \{0, -1, 0, 0, 0, \cdots\},$$

and we see with the aid of (6.10) that

$$QF(\{1\})\{1, 0, 1, 1, 1, \cdots\} = \{-b, -d, 0, 0, 0, \cdots\}. \tag{6.11}$$

However

$$Q\{1, 0, 1, 1, 1, \cdots\} = \{a, c, 0, 0, \cdots\}.$$

The right-hand member of this last equation belongs to the range of $F(\{1\})$. Therefore

$$F(\{1\})Q\{1, 0, 1, 1, 1, \cdots\} = \{a, c, 0, 0, 0, \cdots\}.$$

Since Q commutes with $F(\cdot)$ it follows from this equation and 6.11 that

$$a + b = 0, \qquad c + d = 0. \tag{6.12}$$

Define an operator A_1 on l^∞ by

$$A_1\{\xi_k\} = \{L(\{\xi_k\}), 0, 0, 0, \cdots\}.$$

As in the proof of the corresponding results for A we have $A_1^2 = 0, A_1 S = SA_1$, and $A_1 E(\{1\}) \neq E(\{1\})A_1$. Denote by Γ_2 the total linear manifold in $(l^\infty)^*$ generated by $e_1^* - L, \{e_n^* : n = 2, 3, 4, \cdots\}$; the set function $H(\cdot)$ defined by

$$H(\tau) = E(\tau) + A_1 E(\tau) - E(\tau)A_1, \quad \tau \in \Sigma_p,$$

is a resolution of the identity of class Γ_2 for S. From the definition of $H(\cdot)$ we obtain

$$H(\{1\})\{1, 1, 1, 1, 1, \cdots\} = \{0, 1, 0, 0, 0, \cdots\}.$$

Therefore

$$QH(\{1\})\{1, 1, 1, 1, 1, \cdots\} = \{b, d, 0, 0, 0, \cdots\}. \tag{6.13}$$

However by 6.10 and 6.12,

$$Q\{1, 1, 1, 1, 1, \cdots\} = \{a + b, c + d, 0, 0, 0, \cdots\} = 0.$$

Using this last fact and the relation $QH(\{1\}) = H(\{1\})Q$, we deduce from 6.13 that $b = d = 0$. Therefore by 6.12 and 6.10, $Q = 0$.

7. Comments on the literature

The problems with prespectral operators (as opposed to spectral operators) mentioned in §1 have not always been taken into account in the literature, with the result that errors have occurred in various places. Without attempting a general analysis of this situation, we shall take up some of these errors in this section.

In the work of E. Berkson, one non-trivial error occurs. This concerns Theorem 3.3 of [3; p. 370–1]. Although this theorem was intended to apply to a scalar-type operator S on X of arbitrary class Γ, its proof depends on knowing that S^* has a unique resolution of the identity of class X. With Γ arbitrary it is not known if S^* has a unique s-resolution of the identity of class X (which would be enough for the proof). However, by Corollary 3.12 above, the proof of Theorem 3.3 of [3] is valid if S is of class X^*. Thus the statement of Theorem 3.3 of [3] must be revised to include the additional hypothesis that S is of class X^*. It should be mentioned that this theorem is subsequently applied in [3] and [4] only to scalar-type spectral operators, and so no further difficulty arises from it. For another proof of Theorem 3.3 of [3] when S is of class X^*, see also Proposition 9 of [13].

In conclusion we consider the following proposition from [2; p. 858].

7.1 THEOREM. *Let A be a commutative Banach algebra with radical R such that for some compact Hausdorff space Ω, the algebra A/R is isomorphic to $C(\Omega)$. If A is the direct sum of a closed subalgebra B and the radical R, then the closed subalgebra B is uniquely determined.*

There are two difficulties with the proof of 7.1 as presented in [2]. These are:

7.2. The sum of a scalar-type operator and a commuting quasinilpotent need not be prespectral of any class by 6.3;

7.3. The decomposition of a prespectral operator into the sum of a scalar-type operator of the same class and a commuting quasinilpotent is not known to be unique.

However 7.1 is valid by virtue of the following proof based on the notion of generalized hermiticity. We do not assume in this proof that the algebra A has an identity.

Proof of 7.1. Let B_1 and B_2 be closed subalgebras of A complementary to

R. We shall denote by T_a the image of a in A under the extended left regular representation of A in A_1. (See [14; p. 4] for this terminology and notation.) If x is an element of an arbitrary Banach algebra, we use the symbol $E_0(x)$ to denote the element $\sum_{n=1}^{\infty} x^n/n!$. We shall show that $B_1 \subseteq B_2$; similar reasoning gives the reverse inclusion. Since each of the algebras B_1, B_2 is algebraically equivalent to $C(\Omega)$, each is topologically isomorphic to $C(\Omega)$ by the Corollary [11; p. 77].

Let $b_1 \in B_1$. Then b_1 can be written $b_1 = b_2 + r$ where $b_2 \in B_2$, $r \in R$, and we wish to show that $r = 0$. It is known that the hermitian elements (in the Vidav sense) of a C^*-algebra with identity are precisely those elements which are self-adjoint with respect to the given involution. (See the proof of Theorem 21 of [12; p. 41].) Further, it is known that if u is a hermitian element of an arbitrary Banach algebra (possessing an identity of norm 1), then the set $\{E_0(itu) : t \text{ is real}\}$ is uniformly bounded in norm [16; Hilfssatz 1]. Applying these facts about generalized hermiticity to $C(\Omega)$ and using the Banach algebra equivalence of each of B_1 and B_2 with $C(\Omega)$, we find that there are elements u_j, v_j, $j = 1, 2$, such that

7.4(a) $b_j = u_j + iv_j, \qquad u_j, v_j \in B_j$;
7.4(b) the set $\{\|E_0(itu_j)\| + \|E_0(itv_j)\| : t \text{ is real}\}$ is bounded.

From the equation $b_1 = b_2 + r$ we obtain

$$T_{u_1} + iT_{v_1} = T_{u_2} + iT_{v_2} + T_r .$$

From the boundedness condition 7.4(b) and Theorem 6 of [13; p. 77], it follows that the underlying space A_1 can be renormed with an equivalent Banach space norm which makes the operators T_{u_1}, T_{u_2}, T_{v_1}, T_{v_2} simultaneously hermitian. Let such an equivalent renorming be carried out. Then

$$(T_{u_1} - T_{u_2}) + i(T_{v_1} - T_{v_2}) = T_r .$$

Since T_r is quasinilpotent it follows from Lemma 5.1 that $T_r = 0$. Hence $r = 0$.

We observe that in the above proof of 7.1, it is clear from the reasoning in [2] that $T_{b_1}^*$ is a scalar-type operator of class A_1. Since this operator is equal to $T_{b_2}^* + T_r^*$, the latter is trivially prespectral of class A_1. Moreover by 3.7, $T_{b_2}^* + T_r^*$ is a Jordan decomposition of class A_1 for $T_{b_1}^*$. Thus the difficulty 7.2 can be overcome in the proof given in [2]. By reducing the problem to the case in which the spectrum of $T_{b_1}^*$ is real, the other difficulty 7.3 can be circumvented by appealing to Theorem 5.2. This reduction can be effected, because B_1 is equivalent to $C(\Omega)$. We omit the details.

REFERENCES

1. W. G. BADE, *On Boolean algebras of projections and algebras of operators*, Trans. Amer. Math. Soc., vol. 80(1955), pp. 345–360.
2. W. G. BADE AND P. C. CURTIS, JR., *The Wedderburn decomposition of commutative Banach algebras*, Amer. J. Math., vol. 82(1960), pp. 851–866.

3. E. BERKSON, *A characterization of scalar-type operators on reflexive Banach spaces*, Pacific J. Math., vol. 13(1963), pp. 365–373.
4. ———, *Some characterizations of C^*-algebras*, Illinois J. Math., vol. 10(1966), pp. 1–8
5. N. DUNFORD, *Spectral operators*, Pacific J. Math., vol. 4 (1954), pp. 321–54.
6. ———, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., vol. 64 (1958), pp. 217–74.
7. N. DUNFORD AND J. T. SCHWARTZ, *Linear operators—Part 1: General theory*, Interscience, New York, 1958.
8. U. FIXMAN, *Problems in spectral operators*, Pacific J. Math., vol. 9(1959), pp. 1029–1051.
9. L. GILLMAN AND M. JERISON, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.
10. J. L. KELLEY, *General topology*, Van Nostrand, Princeton, 1955.
11. L. H. LOOMIS, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.
12. G. LUMER, *Semi-inner product spaces*, Trans. Amer. Math. Soc., vol. 100(1961), pp. 29–43.
13. ———, *Spectral operators, Hermitian operators and bounded groups*, Acta. Sci. Math., vol. 25(1964), pp. 75–85.
14. C. E. RICKART, *General theory of Banach algebras*, Van Nostrand, New York, 1960.
15. A. E. TAYLOR, *Introduction to functional analysis*, Wiley, New York, 1958.
16. I. VIDAV, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Zeitschr., vol. 66(1956), pp. 121–128.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS