Illinois Journal of Mathematics
Volume 54, Number 2, Summer 2010, Pages 753–770
S 0019-2082

BILIPSCHITZ HOMOGENEOUS JORDAN CURVES, MÖBIUS MAPS, AND DIMENSION

DAVID M. FREEMAN

ABSTRACT. We characterize fractal chordarc curves in Euclidean space by the fact that they remain bilipschitz homogeneous under inversion. We illustrate this result by constructing two examples. The techniques used in these constructions provide a means of calculating various dimensions of bilipschitz homogeneous Jordan curves.

1. Introduction

A homeomorphism $f:(X,d_X)\to(Y,d_Y)$ is *L-bilipschitz* provided that for all $x,y\in X$,

$$L^{-1}d_X(x,y)\le d_Y(f(x),f(y))\le Ld_X(x,y).$$

A metric space X is *L-bilipschitz homogeneous* provided that for any two points $x,y\in X$, there exists an L-bilipschitz self map of X sending x to y.

We focus our attention on bilipschitz homogeneous Jordan curves. There is a large subcollection of such curves known as fractal chordarc curves (see definitions and references in Section 2). Ghamsari and Herron proved that the fractal chordarc property for Jordan curves is preserved by Möbius maps [GH98, Theorems B, C], and discovered that bilipschitz homogeneity is equivalent to the chordarc condition for a rectifiable curve [GH99, Theorem B]. In the present paper, we utilize bilipschitz homogeneity and Möbius maps to provide a new characterization of fractal chordarc curves.

THEOREM 1.1. *Let $\hat{\Gamma}\subset\hat{\mathsf{R}}^n$ be a Jordan curve of B-bounded turning. The following three statements are quantitatively equivalent*:

(i) *$\hat{\Gamma}$ is a (C,α)-fractal chordarc curve.*
(ii) *For every $M\in\text{Möb}(\hat{\mathsf{R}}^n)$, $M(\hat{\Gamma})\cap\mathsf{R}^n$ is L-bilipschitz homogeneous.*

Received September 9, 2009; received in final form November 18, 2009.
The author gratefully acknowledges support from the Henry Laws Fellowship.
2010 *Mathematics Subject Classification.* Primary 30C35. Secondary 51F99.

(iii) *Both* $\hat{\Gamma} \cap \mathsf{R}^n$ *and* $\hat{\Gamma}^* \cap \mathsf{R}^n$ *are* L*-bilipschitz homogeneous, where* $\hat{\Gamma}^*$ *denotes an inversion of* $\hat{\Gamma}$*. If* $\infty \in \hat{\Gamma}$*, we require that* $\infty \notin \hat{\Gamma}^*$*, and conversely.*

We remark that by results from [Fre] we need not assume the bounded turning condition in the case that $\hat{\Gamma} \subset \hat{\mathsf{R}}^2$ and $\hat{\Gamma} \cap \mathsf{R}^2$ is unbounded (see also [Bis01, Theorem 1.1]). In Section 5, we prove Theorem 1.1 by use of the following, which is proved in Section 4.

THEOREM 1.2. *Let* $\hat{\Gamma} \subset \hat{\mathsf{R}}^n$ *be a Jordan curve such that* $\{0, \infty\} \subset \hat{\Gamma}$*. Then* $\hat{\Gamma}$ *is* (C, α)*-fractal chordarc if and only if* $\hat{\Gamma}$ *is of* B*-bounded turning and both* $\hat{\Gamma}$ *and* $\hat{\Gamma} \cap \mathsf{R}^n$ *are* L*-bilipschitz homogeneous. This equivalence is quantitative.*

We remark that previous work of Bishop, Ghamsari, Herron, Mayer, and Rohde is crucial to the above results. In Section 7, we illustrate Theorem 1.2 by constructing planar curves that are bilipschitz homogeneous with respect to Euclidean distance but not chordal distance, and vice-versa. These construction methods lead to a means of calculating the dimensions of planar bilipschitz homogeneous Jordan curves, described in Section 8.

2. Preliminaries

Given a constant C, we write $C = C(A, B, \dots)$ to indicate that C is determined solely by the numbers $A, B, \dots$. Given positive numbers A and B, we write $A \simeq B$ to indicate the existence of a constant $C \in [1, +\infty)$ such that $C^{-1}A \le B \le CA$. Here we require that C be independent of A and B. We write $A \lesssim B$ to indicate that $A \le CB$. We say two conditions are *quantitatively equivalent* if the constants for each condition depend only on the constants for the other.

We use N to denote the natural numbers and N_0 to denote $\mathsf{N} \cup \{0\}$. The integers are referred to by Z. The real line is R, Euclidean space is R^n, and the upper half space is $\mathsf{H}^n \subset \mathsf{R}^n$. The unit circle is S, and the unit sphere is $\mathsf{S}^n \subset \mathsf{R}^{n+1}$. We denote an open ball of radius $r > 0$ centered at a point x by $B(x; r)$. Then $S(x; r) := \partial B(x; r)$. We define $\mathsf{B}^n := B(0; 1) \subset \mathsf{R}^n$; so $\mathsf{S}^n = \partial \mathsf{B}^{n+1}$. For $r < s$, we write $A(x; r, s) := \{y \in X : r < d(x, y) < s\}$. When working in two dimensions, we write $D(x; r)$ to denote the disk of radius r centered at x, and $C(x; r) := \partial D(x; r)$.

We write $\hat{\mathsf{R}}^n$ to denote $\mathsf{R}^n \cup \{\infty\}$ under the chordal distance χ, where

$$\forall z, w \in \mathsf{R}^n, \quad \chi(z, w) := \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \qquad \chi(z, \infty) := \frac{2}{\sqrt{1 + |z|^2}}.$$

Stereographic projection $\Phi : \mathsf{R}^n \to \hat{\mathsf{R}}^n \setminus \{\infty\}$ is an isometry (by definition). We also have the following (cf. [GH98, Lemma 2.4]).

FACT 2.1. The following statements are true:

(a) Stereographic projection is 2-bilipschitz on the unit ball.

(b) Let $R \in [2,+\infty)$ be given. For $z, w \in A(0; R/2, 2R) \subset \mathsf{R}^n$,

$$|z-w|/8R^2 \leq \chi(\Phi(z),\Phi(w)) \leq 8|z-w|/R^2.$$

(c) Given $R \in (0,+\infty)$, there is a 4-bilipschitz map $\psi : A(0; R/2, 2R) \to A(0; R/2, 2R)$ such that for all $z \in A(0; R/2, 2R)$, $\psi(z)/R^2 = z/|z|^2$.

We write Möb($\hat{\mathsf{R}}^n$) to denote the collection of Möbius maps $M : \hat{\mathsf{R}}^n \to \hat{\mathsf{R}}^n$ (see [Bea95, Chapter 3]). Given a point $x \in \mathsf{R}^n$, we write $\varphi_x : \hat{\mathsf{R}}^n \to \hat{\mathsf{R}}^n$ to denote inversion in the sphere $S(x;1) \subset \mathsf{R}^n$; thus, $\varphi_x \in$ Möb($\hat{\mathsf{R}}^n$). For example, $\varphi_0(z) = z/|z|^2$ (where $0 \mapsto \infty$ and $\infty \mapsto 0$).

A *Jordan curve*, generally denoted by Λ, is a proper homeomorphic image of S or R; here *proper* means that closed and bounded sets are compact. We reserve $\hat{\Gamma}$ to denote a Jordan curve in $\hat{\mathsf{R}}^n$, and write $\Gamma := \hat{\Gamma} \cap \mathsf{R}^n$.

Given Λ in (X,d), and two points $x, y \in \Lambda$, we write $\Lambda[x,y]$ to denote the smaller (with respect to diameter) closed subarc of Λ joining x to y. We say that Λ is of *B-bounded turning* provided that for all $x, y \in \Lambda$, $\operatorname{diam}(\Lambda[x,y]) \leq Bd(x,y)$. A curve Λ is (C,α)-*fractal chordarc* if there exist constants $C \in [1,+\infty)$ and $\alpha \in (0,+\infty)$ such that, for $x, y \in \Lambda$, we have

$$C^{-1} d(x,y)^\alpha \leq \mathcal{H}^\alpha(\Lambda[x,y]) \leq C d(x,y)^\alpha.$$

Here $\mathcal{H}^\alpha$ denotes α-dimensional Hausdorff measure. For a detailed study of fractal chordarc curves, we direct the reader to [GH98] and [GH99]. In R^n, such curves are of bounded turning and bilipschitz homogeneous, quantitatively (see [GH98, Theorem 4.5] and the proof of Theorem 1.2).

Given subset E of a metric space X and a scale $r > 0$, we define a *covering number* as

$$N(r;E) := \inf\left\{ k \in \mathsf{N} : \exists \{x_i\}_{i=1}^k \subset X \text{ such that } E \subset \bigcup_{i=1}^k B(x_i;r) \right\}.$$

A metric space X is D-*doubling* if there exists $D \in [1,+\infty)$ such that for any $x \in X$, we have $N(r; B(x;2r)) \leq D$. When X is doubling, $E \subset X$ is compact, and $A \in [1,+\infty)$, one can verify that

$$N(Ar;E) \leq N(r;E) \leq DA^{\log_2(D)} N(Ar;E).$$

Note that both R^n and $\hat{\mathsf{R}}^n$ are D-doubling, with $D = D(n)$.

We say that a Jordan curve Λ possesses a *bounded covering property* provided there exists a constant $C \in [1,+\infty)$ with the following property: If $x, y, z, w \in \Lambda$ satisfy $d(x,y) \leq d(z,w)$, then for any $r > 0$ we have

$$N(r; \Lambda[x,y]) \leq C N(r; \Lambda[z,w]).$$

A set $S \subset X$ is r-*separated* provided that for every pair of distinct points $x, y \in S$, we have $d(x,y) \geq r$. Given $r > 0$ and a subset E of X, the *packing number* $P(r;E)$ is defined as the supremal cardinality of r-separated sets in E. It is straightforward to check that $N(r;E) \leq P(r;E) \leq N(r/2;E)$. A set E

is (H,α)-*homogeneous* provided that for every $x \in E$ and all $0 < r \le s < \operatorname{diam}(E)$, we have $P(r; B(x;s)) \le H(s/r)^\alpha$. It is well known (and not hard to check) that this is equivalent to the doubling condition.

We say that sets E and F are L-*bilipschitz equivalent* if there exists an L-bilipschitz homeomorphism from E to F.

LEMMA 2.2. *Let E and F be L-bilipschitz equivalent subsets of a D-doubling metric space. Then for any $r > 0$, $N(r;E) \simeq N(r;F)$, up to the constant $D^3 L^{\log_2(D)}$.*

Proof. Assume E and F are bounded. Let $\{B_i\}_{i=1}^k$ be a minimal (with respect to cardinality) cover of F by balls $B_i := B(x_i, r)$, where $x_i \in F$. Given an L-bilipschitz map $f : E \to F$, we know that $\{B(f^{-1}(x_i); Lr)\}$ covers E. Therefore, $N(Lr;E) \le k \le D^2 N(r;F)$, where the factor of D^2 comes from the requirement that each $x_i \in F$. Using f^{-1} we obtain $N(Lr;F) \le D^2 N(r;E)$. Via the doubling condition, we are done. □

3. Bilipschitz homogeneous curves in doubling spaces

We make frequent use of the following facts from [May95, Lemma 2.3] and [Bis01, Corollary 1.2(6) ⇒ (10)], respectively.

FACT 3.1. Let Λ be an L-bilipschitz homogeneous B-bounded turning Jordan curve in a D-doubling metric space:

(a) There exists a constant $C = C(B,D,L)$ with the following property: if $x,y,z,w \in \Lambda$ and $0 < r \le s < +\infty$ are such that $s = d(x,y) \le d(z,w)$, then

$$C^{-1} N(r;\Lambda[z,w]) \le N(r;\Lambda[x,y]) N(s;\Lambda[z,w]) \le C N(r;\Lambda[z,w]).$$

(b) For any constant $C \in [1,+\infty)$ and any two subarcs $I_1, I_2 \subset \Lambda$ with $C^{-1}\operatorname{diam}(I_1) \le \operatorname{diam}(I_2) \le C \operatorname{diam}(I_1)$, there exists an M-bilipschitz map $f : \Lambda \to \Lambda$ with $f(I_1) = f(I_2)$. Here, $M = M(B,C,D,L)$.

A *dimension gauge* is a nondecreasing function $\delta : (0,+\infty) \to (0,+\infty)$ with $\delta(t) \to 0$ as $t \searrow 0$. The *generalized Hausdorff measure* of a set E is $\mathcal{G}^\delta(E) := \lim_{r\to 0^+} \mathcal{G}^\delta_r(E)$, where

$$\mathcal{G}^\delta_r(E) := \inf\left\{ \sum_i \delta(\operatorname{diam}(E_i)) : E \subset \bigcup_i E_i, \operatorname{diam}(E_i) \le r \right\}.$$

When $\delta(t) = t^\alpha$ for all sufficiently small t, we have $\mathcal{G}^\delta = \mathcal{H}^\alpha$. Given a Jordan curve Λ and a dimension gauge δ, we say that Λ satisfies a (C,δ)-*generalized chordarc condition* for $C \in [1,+\infty)$ provided that for every $x,y \in \Lambda$,

$$C^{-1}\delta(d(x,y)) \le \mathcal{G}^\delta(\Lambda[x,y]) \le C\delta(d(x,y)).$$

We define canonical dimension gauges for bounded turning bilipschitz homogeneous Jordan curves in doubling metric spaces. When Λ is bounded, for $t \in (0,+\infty)$, we define $\delta_\Lambda(t) := N(t;\Lambda)^{-1}$. When Λ is unbounded, choose some basepoint x_0 and an orientation on Λ. Given $t \in (0, +\infty)$, we move in the positive direction along Λ until we reach the first point x_t such that $|x_t - x_0| = t$. Then write Λ_t to denote the subarc $\Lambda[x_0, x_t]$, and define

$$\delta_\Lambda(t) := \begin{cases} N(t;\Lambda_1)^{-1}, & \text{if } t \leq 1, \\ N(1;\Lambda_t), & \text{if } t \geq 1. \end{cases}$$

Clearly, this definition depends on the choice of x_0. However, by Fact 3.1(a), different choices of x_0 change δ_Λ only up to a multiplicative constant. Moreover, this constant depends only on the bilipschitz homogeneity, bounded turning and doubling constants. We have the following behavior for such dimension gauges (see [FH, Proposition 3.20]).

FACT 3.2. Let Λ be an L-bilipschitz homogeneous Jordan curve of B-bounded turning in a D-doubling metric space, with canonical dimension gauge $\delta := \delta_\Lambda$. Then there exist constants $C \in [1, +\infty)$ and $\alpha \in [1, +\infty)$ depending only on B, D, and L, such that for all $0 < r \leq s < \operatorname{diam}(\Lambda)$ we have

$$C^{-1}\frac{s}{r} \leq \frac{\delta(s)}{\delta(r)} \leq C\left(\frac{s}{r}\right)^{\alpha}. \tag{3.1}$$

The following fact from [HM99, Theorem E] ties together several of the concepts we have discussed thus far.

FACT 3.3. For a Jordan curve Λ in a D-doubling metric space, the following are quantitatively equivalent:

(1) Λ is L-bilipschitz homogeneous and of B-bounded turning.

(2) Λ enjoys an A-bounded covering property.

(3) Λ satisfies a (C, δ)-generalized chordarc condition for some dimension gauge δ.

4. Proof of Theorem 1.2

We first prove necessity. Suppose that $\hat{\Gamma} \subset \hat{\mathsf{R}}^n$ is (C, α)-fractal chordarc. Since $\hat{\Gamma}$ is of B-bounded turning, with $B := 2C^{1/\alpha}$ ([GH98, Theorem 4.5]), $\hat{\Gamma}$ is P-porous in $\hat{\mathsf{R}}^n$, with $P = P(B)$. Therefore, there exists $r = r(B) \in (0, 2)$ and some point $x \in \hat{\mathsf{R}}^n$ such that $B(x; r) \subset \hat{\mathsf{R}}^n \setminus \hat{\Gamma}$. We 'rotate' $\hat{\mathsf{R}}^n$ so that $x \mapsto \infty$, obtaining $\hat{\Gamma}'$. Then $\hat{\Gamma}'$ is bilipschitz equivalent to $\hat{\Gamma}' \cap \mathsf{R}^n$, with bilipschitz constant depending only on r. Therefore, $\hat{\Gamma}' \cap \mathsf{R}^n$ is fractal chordarc, with constants depending only on C and α. Since $\hat{\Gamma}' \cap \mathsf{R}^n = M(\hat{\Gamma}) \cap \mathsf{R}^n$ for some Möbius map M, by [GH98, Theorems B and C], $\hat{\Gamma} \cap \mathsf{R}^n$ is also fractal chordarc, with constants depending only on C and α. It follows from [GH99, Proposition 4.1] and [GH98, Theorem B] that fractal chordarc curves in R^n and $\hat{\mathsf{R}}^n$ are L-bilipschitz homogeneous, with $L = L(C, \alpha)$.

We now prove sufficiency. It follows from [HM99, Theorem E] (cf. [May95, Lemme 4.2]) that an L-bilipschitz homogeneous D-doubling Jordan curve of

B-bounded turning satisfies a generalized chordarc condition with respect to its canonical dimension gauge. Moreover, the comparability constant depends only on B, D, and L. Therefore, if the canonical dimension gauge for Γ is comparable to a power function, then we conclude that Γ is fractal chordarc.

We write $\hat{\delta}$ and δ to denote the canonical dimension gauges for $\hat{\Gamma}$ and $\Gamma = \hat{\Gamma} \cap \mathsf{R}^n$, respectively. The proof breaks down into two main parts: We first demonstrate that for any positive numbers s, t we have $\delta(st) \simeq \delta(s)\delta(t)$. This property is then used to obtain the desired conclusion that $\delta(t) \simeq t^\alpha$, where $\alpha \in [1, n)$ is the Hausdorff dimension of Γ.

Part 1. For any positive numbers s, t, we have $\delta(st) \simeq \delta(s)\delta(t)$, up to a constant depending only on B, L, and n.

Step 1. Let $0 < r \leq 1$. We prove that $\delta(r) \simeq \hat{\delta}(r)$, where the comparability depends only on B, L, n. Let Γ_1 be as in the definition of δ, choosing $\Gamma_1 \subset \mathsf{B}^n$. By Fact 2.1(a), Fact 3.1(b), and Lemma 2.2, we have

$$\delta(r) = N(r; \Gamma_1)^{-1} \simeq N(r; \Phi(\Gamma_1))^{-1} \simeq N(r; \hat{\Gamma})^{-1} = \hat{\delta}(r).$$

Step 2. Let $0 < s \leq 1$ and $0 < t \leq 1$. We verify that $\delta(st) \simeq \delta(s)\delta(t)$. Again the comparability depends only on B, L, n. Begin by assuming that Γ_1 has basepoint z_0 with $|z_0| = 4s^{-1/2} \geq 4$. Therefore, any ball B of radius t intersecting Γ_1 must lie in the annulus $A(0; |z_0|/2, 2|z_0|)$. We assert that

$$N(t; \Gamma_1) \simeq N(st; \Phi(\Gamma_1)). \tag{4.1}$$

Indeed, let $\{B_i\}$ be a finite cover of Γ_1 by balls of radius t. Then by Fact 2.1(b), Φ maps each B_i to a ball of radius comparable to $|z_0|^{-2}t \simeq st$. The assertion then follows from the metric doubling property as in the proof of Lemma 2.2.

Again using Fact 2.1(b), we have

$$\frac{s}{128} = \frac{1}{8|z_0|^2} \leq \operatorname{diam}(\Phi(\Gamma_1)) \leq \frac{8}{|z_0|^2} = \frac{s}{2}. \tag{4.2}$$

Therefore,

$$\frac{1}{\delta(t)} \simeq N(t; \Gamma_1) \simeq N(st; \Phi(\Gamma_1)) \simeq N(st; \hat{\Gamma}_s) \simeq \frac{N(st; \hat{\Gamma})}{N(s; \hat{\Gamma})} \simeq \frac{\hat{\delta}(s)}{\hat{\delta}(st)}.$$

The first equality follows from the definition of δ. The second is (4.1). The third follows from (4.2), Fact 3.1(b), and Lemma 2.2. The fourth is a consequence of Fact 3.1(a). The final follows from the definition of $\hat{\delta}$.

Using these calculations along with Step 1, we conclude that

$$\delta(st) \simeq \hat{\delta}(st) \simeq \delta(t)\hat{\delta}(s) \simeq \delta(t)\delta(s).$$

All comparability statements depend only on B, L, and n.

Step 3. Let $1 \leq s \leq t$. We show that $\delta(s/t) \simeq \delta(s)/\delta(t)$, with comparability constant depending only on B, L, n. Choose Γ_t so that it has an endpoint z_0

with $|z_0| = 4t$. By the definitions of δ and $\hat{\delta}$, and by Fact 3.1(a), we have

$$\delta(t) = N(1;\Gamma_t) \simeq N(1;\Gamma_s)N(s;\Gamma_t) = \delta(s)N(s;\Gamma_t).$$

The comparability depends only on B, L, and n.

Suppose $B \cap \Gamma_t \neq \varnothing$, where B is a ball of radius s. Since $s \leq t$ and $|z_0| = 4t$, we have $B \subset A(0;|z_0|/2, 2|z_0|)$. As in the verification of (4.1), we have $N(s;\Gamma_t) \simeq N(s/|z_0|^2;\Phi(\Gamma_t))$. Using the metric doubling property, we then have $N(s/|z_0|^2;\Phi(\Gamma_t)) \simeq N(s/t^2;\Phi(\Gamma_t))$. By Fact 2.1(b), $\operatorname{diam}(\Phi(\Gamma_t)) \simeq 1/t$, up to the constant 128. Then

$$N\big(s/t^2;\Phi(\Gamma_t)\big) \simeq N(s/t^2;\hat{\Gamma}_{1/t}) \simeq \frac{N(s/t^2;\hat{\Gamma})}{N(1/t;\hat{\Gamma})} \simeq \frac{\hat{\delta}(1/t)}{\hat{\delta}(s/t^2)}.$$

The first equality follows from Fact 3.1(b) and Lemma 2.2. The second follows from Fact 3.1(a), and the final follows from the definition of $\hat{\delta}$. Using Steps 1 and 2,

$$\frac{\hat{\delta}(1/t)}{\hat{\delta}(s/t^2)} \simeq \frac{\delta(1/t)}{\delta(1/t)\delta(s/t)} = \frac{1}{\delta(s/t)}.$$

Stringing together the above observations yields $\delta(s/t) \simeq \delta(s)/\delta(t)$. The comparability depends only on B, L, and n.

Step 4. Let $s, t > 0$. We confirm that $\delta(st) \simeq \delta(s)\delta(t)$, up to a constant depending only on B, L, n. We perform a case analysis in order to prove the equivalent conclusion that for every $s, t > 0$ we have $\delta(s/t) \simeq \delta(s)/\delta(t)$.

Case 1: $s \leq 1$. Suppose first that $t \geq 1$. Then

$$\delta(s/t) \simeq \delta(s)\delta(1/t) \simeq \delta(s)\delta(1)/\delta(t) \simeq \delta(s)/\delta(t).$$

The first equality follows from Step 2 and the second from Step 3. The final follows from the definition of δ.

Suppose now that $t < 1$. If $s/t \leq 1$, then by Step 2 we have

$$\delta(s) = \delta\big((s/t)t\big) \simeq \delta(s/t)\delta(t).$$

If $s/t > 1$, then by Step 3,

$$\delta(1/s)/\delta(1/t) \simeq \delta(t/s) = \delta\big(1/(s/t)\big) \simeq \delta(1)/\delta(s/t) \simeq 1/\delta(s/t). \tag{4.3}$$

Furthermore, since $s \leq 1$, by Step 3 we have

$$\delta(s) = \delta\big(1/(1/s)\big) \simeq \delta(1)/\delta(1/s) \simeq 1/\delta(1/s).$$

Similarly, $\delta(t) \simeq 1/\delta(1/t)$. Putting this together, we get $\delta(s/t) \simeq \delta(s)/\delta(t)$, where the comparability constant depends only on B, L, and n.

Case 2: $s > 1$. Suppose first that $t \geq 1$. If $s/t \leq 1$, then by Step 3

$$\delta(s/t) \simeq \delta(s)/\delta(t).$$

If $s/t > 1$, then again by Step 3 we have

$$\delta(s/t) \simeq 1/\delta(t/s) \simeq \delta(s)/\delta(t).$$

Now suppose that $t < 1$ (so $s/t > 1$). By the calculations in (4.3), $\delta(s/t) \simeq 1/\delta(t/s)$. By Step 2, $\delta(t/s) \simeq \delta(t)\delta(1/s)$. By Step 3, $\delta(1/s) \simeq \delta(1)/\delta(s)$. Putting this together yields $\delta(s/t) \simeq \delta(s)/\delta(t)$. The comparability depends only on B, L, and n.

Part 2. There exists $\alpha \in [1, n)$ such that for any $t \in (0, +\infty)$ we have $\delta(t) \simeq t^\alpha$, up to a constant depending only on B, L, and n.

To this end, let $\alpha(t) := \log(\delta(t))/\log(t)$. Define

$$\underline{\alpha} := \liminf_{t\to+\infty} \alpha(t), \qquad \overline{\alpha} := \limsup_{t\to+\infty} \alpha(t).$$

Using the definition of δ and the fact that $\Gamma \subset \mathsf{R}^n$, one can check that $1 \le \underline{\alpha} \le \overline{\alpha} \le n$. Furthermore, given any $\varepsilon > 0$ and $s \ge 1$, there exists $r_0 \in [1, +\infty)$ such that for any $r \ge r_0$ and $r \le t \le sr$ we have $\alpha(r) - \varepsilon \le \alpha(t) \le \alpha(r) + \varepsilon$. We refer to this property as *asymptotic continuity at* $+\infty$.

To verify the asymptotic continuity property, fix $\varepsilon > 0$ and $s \ge 1$. Let $D \in [1, +\infty)$ and $\beta \in [1, n)$ denote the constants from (3.1). Then for any $1 \le r \le t \le sr$, we have

$$\begin{aligned}\alpha(t) = \frac{\log(\delta(t))}{\log(t)} &\le \frac{\log(D(t/r)^\beta\delta(r))}{\log(t)} \le \frac{\log(Ds^\beta\delta(r))}{\log(t)} \\ &\le \frac{\log(Ds^\beta)}{\log(r)} + \frac{\log(\delta(r))}{\log(r)} < \alpha(r) + \varepsilon\end{aligned}$$

for $r > (Ds^\beta)^{1/\varepsilon}$. Similarly, for $r \ge \max\{s^{6/\varepsilon}s^{-1}, D^{2/\varepsilon}s^{-1}, \delta(1)D\}$ we have

$$\begin{aligned}\alpha(t) &\ge \frac{\log(D^{-1}(t/r)\delta(r))}{\log(t)} \ge \frac{\log(D^{-1}\delta(r))}{\log(t)} \ge \frac{\log(D^{-1})}{\log(sr)} + \frac{\log(\delta(r))}{\log(sr)} \\ &= \frac{\log(D^{-1})}{\log(sr)} + \frac{\log(\delta(r))}{\log(r)}\frac{\log(r)}{\log(sr)} \ge \alpha(r)\left(\frac{\log(r)}{\log(sr)}\right) - \varepsilon/2 \\ &= \alpha(r) - \alpha(r)\left(\frac{\log(s)}{\log(sr)}\right) - \varepsilon/2 \ge \alpha(r) - \varepsilon.\end{aligned}$$

The final inequality is true because $\alpha(r) \le 3$ when $r \ge \delta(1)D$ (by (3.1)).

Step 1: We begin by demonstrating that $\overline{\alpha} = \underline{\alpha}$. We accomplish this by way of contradiction; thus we assume $\overline{\alpha} > \underline{\alpha}$. Let $0 < \varepsilon < (\overline{\alpha} - \underline{\alpha})/4$. Fix $s \ge 2$ for which $\alpha(s) > \overline{\alpha} - \varepsilon/2$. Due to the asymptotic continuity of α, there exists $r_0 \ge s$ such that for any $r \ge r_0$, $r \le t \le sr$ implies that $\alpha(r) - \varepsilon/2 \le \alpha(t) \le \alpha(r) + \varepsilon/2$. We may also choose r_0 such that $\alpha(r_0) > \overline{\alpha} - \varepsilon/2$.

Choose $r_1 > r_0$ such that $\alpha(r_1) < \underline{\alpha} + \varepsilon/2$. Let $k \ge 0$ be the largest integer such that $\alpha(s^k r_0) > \underline{\alpha} + \varepsilon$ and $s^k r_0 \le r_1$. Note that $\alpha(s^0 r_0) = \alpha(r_0) > \underline{\alpha} + \varepsilon$, so such a k does exist. If $\alpha(s^{k+1}r_0) > \alpha(s^k r_0)$, then by the choice of k we must have $s^{k+1}r_0 > r_1$, and then by asymptotic continuity,

$$\underline{\alpha} + \varepsilon/2 < \alpha(s^k r_0) - \varepsilon/2 \le \alpha(r_1) < \underline{\alpha} + \varepsilon/2.$$

This contradiction reveals that $\alpha(s^{k+1}r_0) \le \alpha(s^k r_0)$. In fact, the same reasoning implies that $\alpha(s^{k+1}r_0) \le \underline{\alpha} + \varepsilon$. Again using asymptotic continuity, it follows that $\alpha(s^k r_0) < \underline{\alpha} + 2\varepsilon$.

By the above paragraph, we may assume that we have an $r \ge r_0$ such that $\underline{\alpha} + \varepsilon < \alpha(r) < \underline{\alpha} + 2\varepsilon$ while $\alpha(sr) \le \alpha(r)$. By Part 1,

$$(sr)^{\alpha(sr)} \simeq s^{\alpha(s)} r^{\alpha(r)}.$$

Here the comparability depends only on B, L, n. Notice that $\alpha(s) - \alpha(sr) > \varepsilon$ while $\alpha(sr) - \alpha(r) \le 0$. Therefore,

$$s^{\varepsilon} < s^{\alpha(s)-\alpha(sr)} \simeq r^{\alpha(sr)-\alpha(r)} \le 1.$$

Since ε and the comparability constant are independent of s, we reach a contradiction for a large enough choice of s. We conclude that $\underline{\alpha} = \overline{\alpha} =: \omega$.

Step 2: Now we prove that $\delta(s) \simeq s^{\omega}$, up to a constant depending only on B, L, n. To this end, suppose $s > 1$ is such that $\alpha(s) > \omega$.

Case 1: There exist arbitrarily large numbers r such that $\alpha(r) > \omega$. Since $\alpha(r) \to \omega$ as $r \nearrow +\infty$, we must be able to find arbitrarily large r such that $\alpha(r) \ge \alpha(sr)$. To see this, suppose that there existed $r_0 \in (1, +\infty)$ with the property that $r \ge r_0 \Rightarrow \alpha(r) < \alpha(sr)$. Since $s > 1$, for every $r \ge r_0$ and for every $k \in \mathsf{N}$, we would have $\alpha(r) < \alpha(s^k r)$. Sending $k \nearrow +\infty$, we would have $\alpha(r) \le \omega$ for all $r \ge r_0$. This would contradict our assumption that there exist arbitrarily large r for which $\alpha(r) > \omega$, so our claim is verified. Furthermore, when r is large enough we also have $\alpha(s) \ge \alpha(sr)$. Thus, we choose $r \ge s$ such that $\alpha(s) \ge \alpha(sr)$ and $\alpha(r) \ge \alpha(sr)$. For such $r \ge s$ we use Part 1 to conclude that

$$r^{\alpha(sr)-\alpha(r)} \simeq s^{\alpha(s)-\alpha(sr)} \ge 1 \ge r^{\alpha(sr)-\alpha(r)}.$$

Therefore, $s^{\alpha(s)-\alpha(sr)} \simeq 1$, or equivalently $s^{\alpha(s)} \simeq s^{\alpha(sr)}$, up to a constant that depends only on B, L, n. Since there exist arbitrarily large r with the above properties, $s^{\alpha(s)} \simeq s^{\omega}$.

Case 2: There exists $r_0 > 1$ such that for all $r \ge r_0$, $\alpha(r) \le \omega$. If there exist arbitrarily large r for which $\alpha(r) = \omega$, then the existence of arbitrarily large r for which $\alpha(sr) \le \alpha(r)$ and $\alpha(sr) \le \alpha(s)$ is clear; as in Case 1, $s^{\alpha(s)} \simeq s^{\omega}$. Therefore, we may assume that for all $r \ge r_0$, $\alpha(r) < \omega$. Since we have arbitrarily large r for which $\alpha(r) < \omega$, the tactics used in Case 1 can be used to verify that $r^{\alpha(r)} \simeq r^{\omega}$ for any $r > 1$ such that $\alpha(r) < \omega$. Thus, for any $r \ge r_0$, we have $r^{\alpha(r)} \simeq r^{\omega}$ and $(sr)^{\alpha(sr)} \simeq (sr)^{\omega}$. Using Part 1, we find that $s^{\alpha(s)} \simeq s^{\omega}$. The comparability constant depends only on B, L, n.

The same strategy can be used to verify that for any $s > 1$ with $\alpha(s) < \omega$ we have $s^{\alpha(s)} \simeq s^{\omega}$. With straightforward adjustments, this strategy will also yield such comparability for all $0 < s < 1$. In conclusion, $\delta(s) \simeq s^{\omega}$ for all $s \in (0, +\infty)$. The comparability constant depends only on B, L, n.

5. Proof of Theorem 1.1

(i)⇒(ii). This follows from [GH98, Theorems B and C] as in the proof of necessity in Theorem 1.2.

(ii)⇒(iii). This is immediate, as inversions are Möbius maps.

(iii)⇒(i). First, we assume that $\Gamma = \hat{\Gamma} \cap \mathsf{R}^n$ is bounded, L-bilipschitz homogeneous, and there exists a point $x \in \Gamma$ such that $\varphi_x(\hat{\Gamma}) \cap \mathsf{R}^n$ is also L-bilipschitz homogeneous (recall the definition of φ_x in Section 2). Note that if we apply the translation $z \mapsto z - x$ to Γ and then apply φ_0, the resulting curve is a translation of $\varphi_x(\hat{\Gamma}) \cap \mathsf{R}^n$. Therefore, we may assume $x = 0 \in \Gamma$ and $\varphi_0(\hat{\Gamma}) \cap \mathsf{R}^n$ is L-bilipschitz homogeneous. Furthermore, for a set $E \subset \mathsf{R}^n$ and $c > 0$, $\varphi_0(cE) = c^{-1}\varphi_0(E)$; here cE denotes the image of E under the map $z \mapsto cz$. Since bilipschitz homogeneity in R^n is invariant under such rescalings (and under rotations), we may also assume that $\Gamma \subset \overline{\mathsf{B}}$ and $1 \in \Gamma$ without affecting our assumption that $\varphi_0(\hat{\Gamma}) \cap \mathsf{R}^n$ is L-bilipschitz homogeneous.

Write $\hat{\Gamma}^*$ to denote $\varphi_0(\hat{\Gamma})$, then write $\hat{\Gamma}^*_0$ to denote the image of $\hat{\Gamma}^*$ under the translation $z \mapsto z-1$; so $0 \in \hat{\Gamma}^*_0$. Our goal is to show that $\hat{\Gamma}^*_0$ is $100L$-bilipschitz homogeneous. By Fact 2.1(a), $\hat{\Gamma}$ is $4L$-bilipschitz homogeneous. Note that $\hat{\Gamma}$ is isometric to $\hat{\Gamma}^*$ via φ_0. Furthermore, $\hat{\Gamma}^*$ is 5-bilipschitz equivalent to $\hat{\Gamma}^*_0$. Therefore, $\hat{\Gamma}^*_0$ is $100L$-bilipschitz homogeneous.

Since both $\hat{\Gamma}^*_0$ and $\Gamma^*_0 = \hat{\Gamma}^*_0 \cap \mathsf{R}^n$ are $100L$-bilipschitz homogeneous, by Theorem 1.2, we conclude that $\hat{\Gamma}^*_0$ is (C, α)-chordarc, with $C = C(B, L, n)$. Since $\hat{\Gamma}$ is 5-bilipschitz equivalent to $\hat{\Gamma}^*_0$, it is (C', α)-chordarc; $C' = C'(C)$.

Next, we assume that Γ is unbounded, L-bilipschitz homogeneous, and there exists a point $x \notin \Gamma$ such that $\varphi_x(\hat{\Gamma}) \cap \mathsf{R}^n$ is L-bilipschitz homogeneous. As in the bounded case, we may translate, rotate, and rescale so that $x = 0$, $1 \in \Gamma$, and $\Gamma \cap \mathsf{B} = \varnothing$ while maintaining the assumption that $\varphi_0(\hat{\Gamma}) \cap \mathsf{R}^n$ is L-bilipschitz homogeneous. Let Γ_0 denote a translation of Γ so that $0 \in \Gamma_0$.

We demonstrate that $\hat{\Gamma}_0$ is $100L$-bilipschitz homogeneous. Since $\Gamma^* = \varphi_0(\hat{\Gamma}) \cap \mathsf{R}^n \subset \mathsf{B}$, by Fact 2.1(a) we conclude that $\hat{\Gamma}^*$ is $4L$-bilipschitz homogeneous. Furthermore, $\hat{\Gamma}^*$ is isometric to $\hat{\Gamma}$ via φ_0. Because $\hat{\Gamma}$ is 5-bilipschitz equivalent to $\hat{\Gamma}_0$, it follows that $\hat{\Gamma}_0$ is $100L$-bilipschitz homogeneous.

Since both Γ_0 and $\hat{\Gamma}_0$ are $100L$-bilipschitz homogeneous, Theorem 1.2 tells us that $\hat{\Gamma}_0$ (and therefore $\hat{\Gamma}$) is (C, α)-chordarc, with $C = C(B, L, n)$.

6. Snowflake functions

The curves we construct in Section 7 involve elements of the catalogue $\mathcal{HS}$ constructed in [Roh01]. We implement additional notation to precisely analyze such curves. Let $\sigma : \mathsf{N}_0 \to \mathsf{N}_0$ be any function with the following properties: $\sigma(0) = 0$, and for all k, $\sigma(k+1) = \sigma(k)$ or $\sigma(k+1) = \sigma(k) + 1$. We call such a function a *snowflake function*.

FIGURE 1. Subarcs I and J_p.

We let I and J_p be as pictured in Figure 1. Suppose $\sigma(1)=0$. Then we replace each side of S_0 (the square centered at the origin with sides of unit length parallel to the coordinate axes) with a similarity (scaled, rotated, translated) copy of I to obtain S_1. The linear subarcs of the similarity copies of I form the $4\cdot 4^1$ 1-*edges* in S_1, each of length 1/4. Suppose instead that $\sigma(1)=1$. Then we replace each side of S_0 with a similarity copy of J_p. Each similarity copy of J_p contains 4^1 1-edges, each of length p. In all such replacements, we require the copies of J_p to 'point out' of the closed curve S_1.

We continue inductively. For $k\geq 2$, let S_k be given, consisting of $4\cdot 4^k$ k-edges, each of length $a_k := p^{\sigma(k)}4^{\sigma(k)-k}$. If $\sigma(k+1)=\sigma(k)$, then we replace each k-edge in S_k with a similarity copy of I to obtain S_{k+1}; each copy of I contains four of the $4\cdot 4^{k+1}$ $(k+1)$-edges in S_{k+1}, each of length $a_{k+1} = p^{\sigma(k+1)}4^{\sigma(k+1)-(k+1)}$. If $\sigma(k+1)=\sigma(k)+1$, then we replace each k-edge in S_k with a similarity copy of J_p; S_{k+1} consists of $4\cdot 4^{k+1}$ $(k+1)$-edges, each of length $a_{k+1} := p^{\sigma(k+1)}4^{\sigma(k+1)-(k+1)}$.

Therefore, given a parameter $p\in[1/4,1/2)$ and a snowflake function σ, we obtain a sequence (S_k) of Jordan curves that converge in the Hausdorff distance to some Jordan curve $S\subset \mathsf{R}^2$. We call such a curve a (σ,p)-*snowflake curve*. Note that $\mathcal{HS}$ is precisely the collection of all (σ,p)-snowflake curves as such σ and $p\in[1/4,1/2)$ assume all possible configurations. Indeed, given any curve $S\in\mathcal{HS}$, there exists a unique pair (σ,p) with its associated sequence (S_k) converging to S. We also obtain a (σ,p)-*sequence* (a_k) consisting of the lengths (=diameters) of the k-edges in each S_k; so $a_k = p^{\sigma(k)}4^{\sigma(k)-k}$. We use the term k-*arc* to describe the smaller subarc of S whose endpoints coincide with the corresponding k-edge in S_k. As a point of reference, note that if $p=1/3$ and $\sigma=\mathrm{id}$, then $(a_k)=(3^{-k})$ and the sequence (S_k) converges to a middle-third von Koch snowflake curve.

We can also apply this 'snowflaking process' to curves and arcs other than the unit square. In particular, given any finite line segment I and a pair (σ,p) as above, we can apply the same process to obtain a sequence of piecewise linear arcs I_k that converge to a limit arc J. We call such a limit arc a (σ,p)-*snowflake arc*.

The following items will be useful both in constructing examples and in performing dimension calculations in Section 8.

Lemma 6.1. *Let $S \in \mathcal{HS}$ be given. Let (a_k) denote the corresponding (σ,p)-sequence. For any $j \le k$ and any $x \in S$, $P(a_k; D(x;a_j) \cap S) \simeq 4^{k-j}$, up to the constant $64D^2$, where D is the doubling constant for R^2.*

Proof. Let $k \ge j$ and $x \in S$. Given an n-arc $J \subset S$, let I denote the n-edge in S_n whose endpoints coincide with those of J. We write $\triangle(I)$ to denote the right isosceles triangle with base I. Thus, $\mathcal{H}^2(\triangle(I)) = \mathrm{diam}(I)^2/4$, and $\triangle(I) \cap S = J$. Let $z \in J$, and let $\{J_i\}$ denote the n-arcs of S. For every J_i such that $\triangle(J_i) \cap D(z;a_n) \ne \varnothing$, we must have $\triangle(J_i) \subset D(z;2a_n)$. Therefore, by area considerations, at most $16\pi < 64$ n-arcs can intersect $D(z;a_n)$.

Since every j-arc in S has diameter a_j, $\overline{D}(x,a_j) \cap S$ contains at least one j-arc of S, denoted by J. Note that J contains precisely 4^{k-j} k-arcs. Since every disk of radius a_k centered in J can intersect at most 64 k-arcs from S, we conclude that

$$N\big(a_k; D(x;a_j) \cap S\big) \ge N(a_k; J) \ge 4^{k-j}/64.$$

On the other hand, since at most 64 j-arcs of S can intersect $D(x;a_j)$, each containing precisely 4^{k-j} k-arcs, we have

$$N\big(a_k; D(x;a_j) \cap S\big) \le 64 \cdot 4^{k-j}.$$

The statement concerning packing numbers follows from the comparability of packing and covering numbers in doubling spaces. □

We need a bit more notation. Let $S \in \mathcal{HS}$ be given, with corresponding (σ,p)-sequence (a_k). Given $j < k$, define

$$\alpha_{j,k} := \left[1 - \frac{\sigma(k) - \sigma(j)}{k-j} \log_4(4p)\right]^{-1}.$$

A straightforward calculation verifies that $4^{k-j} = (a_j/a_k)^{\alpha_{j,k}}$. We thus refer to the numbers $\alpha_{j,k}$ as (σ,p)-*exponents.*

Corollary 6.2. *Let $S \in \mathcal{HS}$ be a (σ,p)-snowflake with corresponding sequence (a_k) and exponents $\{\alpha_{j,k}\}$. For any $0 < r \le s \le 1$ and $x \in S$, choose $j \le k$ such that $a_k \le r \le a_{k-1}$ and $a_j \le s \le a_{j-1}$. Then $P(r; D(x;s) \cap S) \simeq (s/r)^{\alpha_{j,k}}$, up to the constant $256D^2$, where D is the doubling constant for R^2.*

7. Curve constructions

Here, we construct two curves that illustrate Theorem 1.2.

Example 7.1. *There exists a Jordan curve $\hat{\Gamma} \subset \hat{\mathsf{R}}^2$ that is not bilipschitz homogeneous while $\hat{\Gamma} \cap \mathsf{R}^2$ is bilipschitz homogeneous.*

Proof. Note that the identity function $\mathrm{id} : \mathsf{N}_0 \to \mathsf{N}_0$ is a snowflake function. For each $n \in \mathsf{Z}$, we use the identity function to construct a snowflake arc K_n from the line segment $[n, n+1]$; so K_n is a middle-third von Koch arc. Set $\Gamma = \bigcup_{n \in \mathsf{Z}} K_n$. Thus, Γ is an unbounded Jordan curve in R^2. Since each subarc

K_n is 4-bounded turning (see [GH99, Fact 5.2]), it is straightforward to see that Γ itself is 9-bounded turning. In addition, we note that each subarc K_n satisfies an α-dimensional fractal chordarc condition, with $\alpha := \log_3(4)$ and $\mathcal{H}^\alpha(K_n) = 1$.

We begin by demonstrating that Γ is bilipschitz homogeneous. By Fact 3.3, it suffices to show that Γ satisfies a generalized chordarc condition with respect to some doubling dimension gauge. Given $r \in \mathsf{R}_+$, define

$$\delta(r) := \begin{cases} r^\alpha, & \text{if } r \le 1, \\ r, & \text{if } r \ge 1. \end{cases}$$

It is easy to verify that this dimension gauge is doubling, with constant 2^α, and that $\mathcal{G}^\delta = \mathcal{H}^\alpha$.

Now let $x, y \in \Gamma$. Suppose first that $|x-y| \le 1$. Then we note that $\Gamma[x,y]$ is contained in at most 2 adjacent subarcs K_{n-1}, K_n with common endpoint z_n. If both points x and y are contained in the same subarc K_{n-1}, the fractal chordarc property yields

$$\mathcal{H}^\alpha(\Gamma[x,y]) \simeq |x-y|^\alpha.$$

Assume that $x \in K_{n-1}$ and $y \in K_n$. By additivity,

$$\mathcal{H}^\alpha(\Gamma[x,y]) = \mathcal{H}^\alpha(\Gamma[x,z_n]) + \mathcal{H}^\alpha(\Gamma[z_n,y]) \simeq |x-z_n|^\alpha + |z_n - y|^\alpha.$$

We note that $\max\{|x-z_n|, |z_n-y|\} \le |x-y|$, and so $\mathcal{H}^\alpha(\Gamma[x,y]) \lesssim 2|x-y|^\alpha$. Without loss of generality, $|x-z_n| \ge |x-y|/2$. Therefore, $\mathcal{H}^\alpha(\Gamma[x,y]) \gtrsim 2^{-\alpha}|x-y|^\alpha$.

Now suppose that $|x-y| > 1$. Then let $K_n, \ldots, K_{n+m}$ denote the minimal collection of consecutive subarcs such that $\bigcup_{i=1}^m K_{n+i}$ joins x to y. Thus, $x \in K_n$ and $y \in K_{n+m}$, where $m > 0$. Let z_{n+1} denote the right endpoint of K_n, and let z_{n+m} denote the left endpoint of K_{n+m}. Thus, $x \le z_{n+1} \le z_{n+m} \le y$ in Γ. If $m = 1$, then we are in the situation described in the above paragraph, so we assume $m \ge 2$. Since $\mathcal{H}^\alpha(K_i) = 1$ for each i, we have

$$|x-y|/3 \le m-1 \le \mathcal{H}^\alpha(\Gamma[x,y]) \le m+1 \le 3|x-y|.$$

In conclusion, we find that for every $x, y \in \Gamma$, we have $\mathcal{H}^\alpha(\Gamma[x,y]) \simeq \delta(|x-y|)$, up to an absolute constant. Therefore, Γ satisfies a generalized chordarc condition, and by Fact 3.3 we know that Γ is bilipschitz homogeneous.

Now we turn our attention to $\hat{\Gamma} := \Phi(\Gamma) \cup \{\infty\}$ to show that it is not bilipschitz homogeneous. Since $\hat{\Gamma}$ is of bounded turning, by Fact 3.3 it suffices to demonstrate that $\hat{\Gamma}$ does not possess the bounded covering property.

Note that for $R \ge 1$, $\varphi_0(\Gamma \setminus D(0;R)) \subset D(0;1/R) \cap (\mathsf{H} \setminus D(i/2;1/2))$. Recall that φ_0 is inversion through S. Furthermore, $D(0;1/R) \cap (\mathsf{H} \setminus D(i/2;1/2))$ is contained in a rectangle of base $2/R$ and height $2/R^2$. Therefore,

$$N\big(2/R^2; D(0;1/R) \cap \varphi_0(\Gamma)\big) \le 2R + 1 \le 3R.$$

For the remainder of this example, $\hat{D}(x;r)$ and $\hat{C}(x;r)$ denote disks and circles in $\hat{\mathsf{R}}^2$. Fix $0 < r \le 1/2$. By Fact 2.1(a), Lemma 2.2, the doubling property, and the above paragraph,

$$N\big(r^2;\hat{D}(\infty;r)\cap\hat{\Gamma}\big)=N\big(r^2;\hat{D}(0;r)\cap\varphi_0(\hat{\Gamma})\big)\simeq N\big(r^2;D(0;r)\cap\varphi_0(\Gamma)\big)\lesssim 1/r.$$

We turn our attention to the point $0\in\Gamma$. Note that the sequence (a_k) derived from the construction of each K_n is given by $(a_k)=(3^{-k})$. By Fact 2.1(a), Lemma 2.2, Lemma 6.1, and the metric doubling property, for $k\in\mathsf{N}$ we have

$$\begin{aligned}N\big(3^{-2k};\hat{D}(0;a_k)\cap\hat{\Gamma}\big)&\simeq N\big(3^{-2k};D(0;a_k)\cap\Gamma\big)=N\big(a_k^2;D(0;a_k)\cap\Gamma\big)\\&=N\big(a_{2k};D(0;a_k)\cap\Gamma\big)\simeq 4^k=(4/3)^k a_k^{-1}.\end{aligned}$$

Putting the above estimates together tells us that for $k\in\mathsf{N}$,

$$N\big(a_{2k};\hat{D}(0;a_k)\cap\hat{\Gamma}\big)\gtrsim(4/3)^k N\big(a_{2k};\hat{D}(\infty;a_k)\cap\hat{\Gamma}\big).$$

Now let $\hat{I}_k$ denote the minimal subarc (with respect to inclusion) of $\hat{\Gamma}$ that contains $\hat{D}(0;a_k)\cap\hat{\Gamma}$. Since $\hat{\Gamma}$ is B-bounded turning (for some absolute constant B), $\operatorname{diam}(\hat{I}_k)\le 2Ba_k$. Let $\hat{J}_k\subset\hat{D}(0;a_k)$ denote a subarc of $\hat{I}_k$ with left endpoint 0 and right endpoint on $\hat{C}(0;a_k)$. Thus, $\operatorname{diam}(\hat{J}_k)\ge a_k$. By Lemma 2.2 and Fact 3.1(b), $N(a_{2k};\hat{I}_k)\simeq N(a_{2k};\hat{J}_k)$ up to a constant depending only on B and L. We set x_k,y_k to be the endpoints of $\hat{J}_k$; so $\hat{J}_k=\hat{\Gamma}[x_k,y_k]$.

Finally, for each $k\in\mathsf{N}$ set $z_k=\infty$ and choose w_k to be the first point in $\hat{C}(\infty,a_k)\cap\hat{\Gamma}$ as we move from z_k in the positive direction along $\hat{\Gamma}$. Then $\chi(x_k,y_k)=\chi(z_k,w_k)$. These points and the scales a_{2k} suffice to demonstrate that $\hat{\Gamma}$ does not possess a bounded covering property, and therefore is not bilipschitz homogeneous. Indeed, for each $k\in\mathsf{N}$ we have $\chi(x_k,y_k)=\chi(z_k,w_k)$ while $N(a_{2k};\hat{\Gamma}[x_k,y_k])\gtrsim(4/3)^k N(a_{2k};\hat{\Gamma}[z_k,w_k])$. ☐

EXAMPLE 7.2. *There exists a Jordan curve $\hat{\Gamma}\subset\hat{\mathsf{R}}^2$ that is bilipschitz homogeneous while $\hat{\Gamma}\cap\mathsf{R}^2$ is not bilipschitz homogeneous.*

Proof. Similar to Example 7.1, we use a certain (σ,p)-snowflake curve as the starting point for our construction of Γ. Set $p:=1/3$ and define $q:=\log_4(4p)=\log_4(4/3)<1$. We define a snowflake function $\sigma:\mathsf{N}_0\to\mathsf{N}_0$, beginning with $\sigma(1):=1$. Given $\sigma(k)$, let j_k denote the smallest number such that $\sigma(j_k)=\sigma(k)$. Note that j_k may equal k; in particular, $j_1=1$. Given an interval $[n,m]\subset\mathsf{N}_0$, we say that σ is constant on $[n,m]\subset\mathsf{N}_0$ if $\sigma(m)=\sigma(n)$. We say that σ is strictly increasing on $[n,m]$ if $\sigma(m)=\sigma(n)+(m-n)$.

We use j_k to define $g(k):=\lceil j_k-q\sigma(j_k)\rceil\ge 1$; here $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. We also define the number

$$h_k:=\min\{n\in\mathsf{N}:n\le k,\ \sigma\text{ is strictly increasing on }[n,k]\}.$$

We now give a recursive definition for $\sigma(k+1)$. Suppose first that $j_k<k$. If $k-j_k<g(k)$, then set $\sigma(k+1):=\sigma(k)$. If $k-j_k\ge g(k)$, then set $\sigma(k+1):=$

$\sigma(k)+1$. Suppose now that $j_k = k$. If $k < 2h_k$, then set $\sigma(k+1) := \sigma(k)+1$. If $k \geq 2h_k$ then set $\sigma(k+1) := \sigma(k)$.

Using this recursion rule, we find that

$$\begin{array}{lllll} \sigma(2)=1, & \sigma(3)=2, & \sigma(4)=3, & \sigma(5)=3, & \sigma(6)=3, \\ \sigma(7)=3, & \sigma(8)=3, & \sigma(9)=4, & \sigma(10)=5, & \ldots. \end{array}$$

Let (k_n) denote the subsequence of numbers k for which $\sigma(k-1) < \sigma(k) = \sigma(k+1)$. Thus, $k_1 = 1$, $k_2 = 4, \ldots$. Moreover, $k_n \nearrow +\infty$ as $n \nearrow +\infty$. It follows from these definitions that σ is constant precisely on intervals of the form $[k_n, k_n + g(k_n)]$. In addition, σ is strictly increasing on the intervals of the form $[k_n + g(k_n), k_{n+1}]$, each of length $k_n + g(k_n)$.

While this recursive definition may seem cumbersome, it yields the following useful behavior for the $(\sigma, 1/3)$-sequence (a_k): for every $n \in \mathsf{N}$ it is straightforward to verify that,

$$a_{k_n+g(k_n)} \leq a_{k_n}^2 \leq 4a_{k_n+g(k_n)}.$$

Let S denote the $(\sigma, 1/3)$-snowflake curve obtained from the unit square in $S_0 \subset \mathsf{R}^2$. We define $\hat{\Gamma}' := \Phi(2S)$. Then we rotate $\hat{\Gamma}'$ to obtain $\hat{\Gamma}$, rotating so that $\Phi(i) \mapsto 0$ and $\Phi(-i) \mapsto \infty$. Thus, $\varphi_0(\hat{\Gamma}) = \hat{\Gamma}$. Since $\hat{\Gamma}$ is 6-bilipschitz equivalent to $S \in \mathcal{HS}$, it is bilipschitz homogeneous. However, we claim that $\Gamma := \hat{\Gamma} \cap \mathsf{R}^2$ is not bilipschitz homogeneous.

To verify this claim, we first note that $\varphi_0(\hat{\Gamma}) \cap \mathsf{R}^2 = \hat{\Gamma} \cap \mathsf{R}^2$, and that Γ is B-bounded turning for some finite constant B. For $n \geq 2$, we choose points $x_n \in \Gamma \cap C(0; a_{k_n}^{-1/2})$, where k_n is defined as above. Let $\Gamma_n \subset D(x_n; 1)$ denote the minimal subarc of Γ with left endpoint x_n and right endpoint $y_n \in C(x_n; 1)$. Thus, $|x_n - y_n| = 1$.

We now estimate $N(a_{k_n}; \Gamma[x_n, y_n])$. To do this, we look at $\varphi_0(\Gamma[x_n, y_n]) \subset \mathsf{D}$. Note that any disk of radius a_{k_n} which intersects $\Gamma[x_n, y_n]$ must lie in the annulus $A(0; a_{k_n}^{-1/2}/2, 2a_{k_n}^{-1/2})$. Therefore, using Fact 2.1(c) [cf. (4.1) in the proof of Theorem 1.2], we have

$$\begin{aligned} N(a_{k_n}; \Gamma[x_n, y_n]) &\simeq N(a_{k_n}^2; \varphi_0(\Gamma[x_n, y_n])) \\ &\simeq N(a_{k_n+g(k_n)}; \varphi_0(\Gamma[x_n, y_n])) \\ &\leq N\big(a_{k_n+g(k_n)}; \varphi_0(D(x_n; 1)) \cap \Gamma\big). \end{aligned}$$

Again by Fact 2.1(c), $\varphi_0(D(x_n; 1))$ is a disk centered at some point $x_n' \in \mathsf{D}$ of radius no greater than $8a_{k_n}$. Since $\Gamma \cap \mathsf{D}$ is bilipschitz equivalent to $S \cap \mathsf{H}$, we can use Lemma 2.2 and Lemma 6.1 to verify that

$$N\big(a_{k_n+g(k_n)}; D(x_n'; 8a_{k_n}) \cap \Gamma\big) \lesssim 4^{g(k_n)}.$$

The comparability depends only on B and L.

Now we construct points $x_0, y_0 \in \Gamma$. Let $x_0 := 0$ and let y_0 be the first point on Γ (as we move in the positive direction along Γ) such that $|x_0 - y_0| = 1$.

Using Fact 3.1(b) and Lemma 2.2,

$$N(a_{k_n};\Gamma[x_0,y_0]) \simeq N(a_{k_n};\mathsf{D}\cap\Gamma) \simeq 4^{k_n}.$$

Therefore, for $n \geq 2$, $|x_0 - y_0| = |x_n - y_n|$ and

$$N(a_{k_n};\Gamma[x_0,y_0]) \gtrsim 4^{k_n - g(k_n)} N(a_{k_n};\Gamma[x_n,y_n]).$$

Here the comparability is independent of n. Note that $k_n - g(k_n) = k_n - \lceil k_n - q\sigma(k_n)\rceil \geq q\sigma(k_n) - 1 \to +\infty$. Therefore, Γ does not possess a bounded covering property, and so is not bilipschitz homogeneous. □

8. Dimension calculations

In this section, we calculate the lower and upper Minkowski and Assouad dimensions of compact bilipschitz homogeneous Jordan curves in R^2. These dimensions are denoted by $\underline{\dim}_{\mathcal{M}}$, $\overline{\dim}_{\mathcal{M}}$, and $\dim_{\mathcal{A}}$, respectively. Recall that, for a (bounded) metric space X,

$$\underline{\dim}_{\mathcal{M}}(X) := \liminf_{r\to 0} \frac{\log(N(r;X))}{\log(1/r)} \leq \limsup_{r\to 0} \frac{\log(N(r;X))}{\log(1/r)} =: \overline{\dim}_{\mathcal{M}}(X),$$
$$\dim_{\mathcal{A}}(X) := \inf\{\alpha : \exists H \in [1,+\infty) \text{ such that } X \text{ is } (H,\alpha)\text{-homogeneous}\}.$$

By [May95, Proposition 4.3], the Hausdorff dimension of a bilipschitz homogeneous Jordan curve in R^2 equals its lower Minkowski dimension.

Given a (compact) bilipschitz homogeneous Jordan curve $\Gamma \subset \mathsf{R}^2$, there exists a curve $S \in \mathcal{HS}$ that is the image of Γ under a bilipschitz map of R^2 (see [Roh01, Theorem 1.3]). Given S, there exists a unique snowflake function $\sigma : \mathsf{N}_0 \to \mathsf{N}_0$ and parameter $p \in [1/4, 1/2)$ such that S is a (σ,p)-snowflake curve. We now proceed to calculate the various dimensions of S in terms of σ and p. Since the dimensions under consideration are preserved by bilipschitz maps, we obtain the dimensions of Γ.

Given a function $f : \mathsf{N}_0 \times \mathsf{N}_0 \to \mathsf{R}$, we write

$$\limsup_{(k-j)\to+\infty} f(j,k) := \lim_{n\to+\infty}[\sup\{f(j,k) : k - j \geq n\}].$$

THEOREM 8.1. *Let $S \in \mathcal{HS}$ be a (σ,p)-snowflake curve with exponents $\alpha_{j,k}$.*

$$\dim_{\mathcal{A}}(S) = \limsup_{(k-j)\to+\infty} \alpha_{j,k},$$
$$\underline{\dim}_{\mathcal{M}}(S) = \liminf_{k\to+\infty} \alpha_{0,k} \leq \limsup_{k\to+\infty} \alpha_{0,k} = \overline{\dim}_{\mathcal{M}}(S).$$

Proof. Write $\overline{\alpha} := \limsup_{(k-j)\to+\infty} \alpha_{j,k}$. Since $(1/p)^{k-j} \leq a_j/a_k \leq 4^{k-j}$, $(k-j) \to +\infty$ if and only if $a_j/a_k \to +\infty$. Therefore, for any $\alpha > \overline{\alpha}$, there exists some $M > 0$ such that $\alpha_{j,k} < \alpha$ whenever $a_j/a_k \geq M$.

Let $\overline{\alpha} < \alpha < \overline{\alpha}+1$ and M be given as above. Let $0 < r \le s \le \operatorname{diam}(S)$. There exist $j \le k$ with $a_k \le r \le 2a_{k-1}$ and $a_j \le s \le 2a_{j-1}$. In the case that $a_j/a_k \ge M$, the above paragraph and Corollary 6.2 yield

$$P\big(r; D(x,s)\cap S\big) \simeq \left(\frac{s}{r}\right)^{\alpha_{j,k}} < \left(\frac{s}{r}\right)^{\alpha}.$$

If $a_j/a_k < M$, then again using Corollary 6.2, we have

$$P\big(r; D(x,s)\cap S\big) \simeq \left(\frac{s}{r}\right)^{\alpha_{j,k}} = \left(\frac{s}{r}\right)^{\alpha_{j,k}-\alpha}\left(\frac{s}{r}\right)^{\alpha} \lesssim M^2\left(\frac{s}{r}\right)^{\alpha}.$$

Therefore, for any $\alpha > \overline{\alpha}$, there exists $H \in [1,+\infty)$ for which S is (H,α)-homogeneous. It follows that $\dim_{\mathcal{A}}(S) \le \overline{\alpha}$.

Now let $0 < \alpha < \overline{\alpha}$. For any $N \in \mathsf{N}$, there exists some $a_j/a_k \ge N$ such that $\alpha_{j,k} > \alpha + (\overline{\alpha}-\alpha)/2$. Then by Corollary 6.2,

$$P\big(a_k; D(x,a_j)\cap S\big) \simeq \left(\frac{a_j}{a_k}\right)^{\alpha_{j,k}} = \left(\frac{a_j}{a_k}\right)^{\alpha_{j,k}-\alpha}\left(\frac{a_j}{a_k}\right)^{\alpha} \ge N^{(\overline{\alpha}-\alpha)/2}\left(\frac{a_j}{a_k}\right)^{\alpha}.$$

Since $N \in \mathsf{N}$ was arbitrary, we see that S is not (H,α)-homogeneous for any finite H if $\alpha < \overline{\alpha}$. Thus, $\dim_{\mathcal{A}}(S) \ge \overline{\alpha}$, and so $\dim_{\mathcal{A}}(S) = \overline{\alpha}$.

Continuing, for $k \in \mathsf{N}$, we have $N(a_k; S) \simeq 4^k = (1/a_k)^{\alpha_{0,k}}$, up to the constant 64 (cf. Lemma 6.1). For any $0 < r \le \operatorname{diam}(S)$, there exists k such that $a_k \le r \le 2a_{k-1}$. Therefore,

$$\begin{aligned}
\overline{\dim}_{\mathcal{M}}(S) &= \limsup_{r\to 0} \frac{\log(N(r;S))}{\log(1/r)} = \lim_{j\to+\infty}\left[\sup\left\{\frac{\log(N(r;S))}{\log(1/r)} : r \le a_j\right\}\right] \\
&\le \lim_{j\to+\infty}\left[\sup\left\{\frac{\log(N(a_k;S))}{\log(1/2a_{k-1})} : k \ge j\right\}\right] \\
&\le \lim_{j\to+\infty}\left[\sup\left\{\frac{\alpha_{0,k}\log(1/a_k)}{\log(1/2a_{k-1})} + \frac{\log(64)}{\log(1/2a_{k-1})} : k \ge j\right\}\right] \\
&= \lim_{j\to+\infty}\left[\sup\{\alpha_{0,k} : k \ge j\}\right] = \limsup_{j\to+\infty} \alpha_{0,j} =: \overline{\alpha}.
\end{aligned}$$

By a similar argument one obtains $\overline{\dim}_{\mathcal{M}}(S) \ge \overline{\alpha}$, and so $\overline{\dim}_{\mathcal{M}}(S) = \overline{\alpha}$. By parallel methods, $\underline{\dim}_{\mathcal{M}}(S) = \liminf_{k\to+\infty} \alpha_{0,k}$. □

As an application, these dimension calculations can be used to verify the existence of bilipschitz homogeneous Jordan curves $\Gamma \subset \mathsf{R}^2$ for which

$$\begin{aligned}
\underline{\dim}_{\mathcal{M}}(\Gamma) &< \overline{\dim}_{\mathcal{M}}(\Gamma) < \dim_{\mathcal{A}}(\Gamma), \\
\underline{\dim}_{\mathcal{M}}(\Gamma) &= \overline{\dim}_{\mathcal{M}}(\Gamma) < \dim_{\mathcal{A}}(\Gamma), \\
\underline{\dim}_{\mathcal{M}}(\Gamma) &< \overline{\dim}_{\mathcal{M}}(\Gamma) = \dim_{\mathcal{A}}(\Gamma).
\end{aligned}$$

Similar techniques yield a bilipschitz homogeneous Jordan curve that has Assouad dimension 1 yet fails to satisfy a chordarc condition. The interested reader may consult [Fre09] for the verification of these claims.

References

[Bea95] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1995, Corrected reprint of the 1983 original. MR 1393195

[Bis01] C. J. Bishop, *Bi-Lipschitz homogeneous curves in* $\mathbb{R}^2$ *are quasicircles*, Trans. Amer. Math. Soc. **353** (2001), 2655–2663 (electronic). MR 1828465

[FH] D. M. Freeman and D. A. Herron, *Bilipschitz homogeneity and inner diameter distance*, to appear in J. Anal. Math.

[Fre] D. M. Freeman, *Unbounded bilipschitz homogeneous Jordan curves*, submitted.

[Fre09] D. M. Freeman, Ph.D. thesis, Dept. of Mathematical Sciences, University of Cincinnati, 2009; available at http://etd.ohiolink.edu/view.cgi?acc_num=ucin1251229498.

[GH98] M. Ghamsari and D. A. Herron, *Higher dimensional Ahlfors regular sets and chordarc curves in* $\mathbf{R}^n$, Rocky Mountain J. Math. **28** (1998), 191–222. MR 1639853

[GH99] M. Ghamsari and D. A. Herron, *Bi-Lipschitz homogeneous Jordan curves*, Trans. Amer. Math. Soc. **351** (1999), 3197–3216. MR 1608313

[HM99] D. A. Herron and V. Mayer, *Bi-Lipschitz group actions and homogeneous Jordan curves*, Illinois J. Math. **43** (1999), 770–792. MR 1712522

[May95] V. Mayer, *Trajectoires de groupes à 1-paramètre de quasi-isométries*, Rev. Mat. Iberoamericana **11** (1995), 143–164. MR 1321776

[Roh01] S. Rohde, *Quasicircles modulo bilipschitz maps*, Rev. Mat. Iberoamericana **17** (2001), 643–659. MR 1900898

David M. Freeman, University of Cincinnati, Raymond Walters College, 9555 Plainfield Road, Cincinnati, OH 45236, USA

E-mail address: freemadd@ucrwcu.rwc.uc.edu