Illinois Journal of Mathematics
Volume 54, Number 2, Summer 2010, Pages 501–508
S 0019-2082

ON THE REGULARITY OF CERTAIN PROJECTIVE MONOMIAL CURVES

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ABSTRACT. In this paper we present a method to find the regularity of projective monomial curves in terms of an ordering of monoids associated to them. We use this result to find the regularity of certain monomial curves and investigate where regularity is attained in their minimal graded free resolutions.

1. Introduction

In this paper we discuss the regularity of projective monomial curves. Our main result is Theorem 3.4, which describes the regularity of a projective monomial curve in terms of combinatorial data associated to the curve. Our proof of Theorem 3.4 is elementary and self contained, using only the definition of a curve, basic commutative algebra, and elementary facts about regularity found in [7]. In Section 2 we give background on projective monomial curves, in Section 3 we prove Theorem 3.4, and in Section 4 we apply it to obtain the exact value of the regularity for a certain infinite class of almost arithmetic progression curves (Theorem 4.4) and an algorithm for computing regularity (Theorem 4.1). Regularity of a projective variety has been studied by many authors. In particular, the papers [2], [3] and [12] also give descriptions of the regularity of projective varieties. Also note [10] which uses projective monomial curves to study more general projective curves.

2. Projective monomial curves

This section reviews background (from [11], [14], [15]) needed for our proof of Theorem 3.4. The stable basis is a new idea, which we introduce because it arises naturally in our proof. Let $R = \mathbb{K}[x_0, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, graded by $\deg(x_i) = 1$. Suppose that $\mathscr{S} = \{a_1, \dots, a_n\}$ is a set

Received July 6, 2008; received in final form April 27, 2010.
The second author was partially supported by an NSERC grant.
2010 *Mathematics Subject Classification.* Primary 14H99, 13D02. Secondary 14Q05.

of relatively prime positive integers such that $a_1 < \cdots < a_n$. Let $a_0 = 0, d = a_n$, and S be the submonoid of $\mathbb{N}^2$ generated by $\{(d,0),(d-a_1,a_1),\ldots,(d-a_{n-1},a_{n-1}),(0,d)\}$. Define a $\mathbb{K}$-algebra homomorphism $\varphi : R \to \mathbb{K}[s,t]$ by $\varphi(x_i) = s^{d-a_i}t^{a_i}, 0 \le i \le d$. Let $I_{\mathscr{S}}$ be the kernel of φ. Then $I_{\mathscr{S}}$ is the defining ideal of a projective monomial curve $\mathfrak{C}$ of degree d in $\mathbb{P}^n_{\mathbb{K}}$ which we will refer to as the curve defined by $\mathscr{S}$. The homogeneous coordinate ring of $\mathfrak{C}$ is $R_{\mathscr{S}} = R/I_{\mathscr{S}} \cong \mathrm{Im}(\varphi) = \mathbb{K}[s^d, s^{d-a_1}t^{a_1},\ldots,s^{d-a_{n-1}}t^{a_{n-1}},t^d]$. It is sometimes convenient to represent $R_{\mathscr{S}}$ in the form $\mathbb{K}[s^d, s^{d-a_1}t^{a_1},\ldots,s^{d-a_{n-1}}t^{a_{n-1}},t^d]$, which we will denote $\mathbb{K}[S]$. We make $\mathbb{K}[S]$ into a graded $\mathbb{K}$-algebra by setting $\deg(s^{d-a_i}t^{a_i}) = 1$.

Let Γ be the monoid generated by $\mathscr{S}$, that is, $\Gamma := \{\sum_{i=1}^n m_i a_i \mid m_i \in \mathbb{N}\}$. Let Θ_i be the set of all elements of Γ which can be written as a sum of i elements of $\mathscr{S}$ (repetition is allowed) and set $\mathfrak{M}_i := \Theta_i \setminus (\bigcup_{j<i}\Theta_j)$. The elements of $\mathfrak{M}_i$ are those elements of Γ which can be minimally expressed as the sum of i elements of $\mathscr{S}$. For every $n \in \Gamma$ there is a unique i such that $n \in \mathfrak{M}_i$. In this case we set $\mathrm{ord}_{\mathscr{S}}(n) = i$.

An element $a \in \Gamma$ is called *unstable* if there exists a $k \in \mathbb{N}$ such that $\mathrm{ord}_{\mathscr{S}}(ka_n + a) < k + \mathrm{ord}_{\mathscr{S}}(a)$ and it is called *stable* if no such k exists. By $\mathrm{UStb}(\mathscr{S})$ we mean the set of all unstable elements of Γ and by $\mathrm{Stb}(\mathscr{S})$ we mean $\Gamma \setminus \mathrm{UStb}(\mathscr{S})$.

The following lemma is easy to prove.

LEMMA 2.1. *Let $b \in \mathrm{UStb}(\mathscr{S})$, then for any $i \in \{1,\ldots,n\}$ at least one of the following holds*:

(1) $\mathrm{ord}_{\mathscr{S}}(b + a_i) \le \mathrm{ord}_{\mathscr{S}}(b)$,
(2) $b + a_i \in \mathrm{UStb}(\mathscr{S})$.

We set $\mathrm{StdB}(\mathscr{S}) := \{a \in \Gamma \mid a - d \notin \Gamma\}$, and call it the *standard basis* of Γ and set $\mathrm{StbB}(\mathscr{S}) := \{a \in \mathrm{Stb}(\mathscr{S}) \mid a - d \notin \mathrm{Stb}(\mathscr{S})\}$ and call it the *stable basis* of Γ.

$\mathrm{StbB}(\mathscr{S})$ and $\mathrm{StdB}(\mathscr{S})$ both contain exactly d elements of different congruence classes modulo d. The smallest element of Γ in a given congruence class mod d is the standard basis element in that class, and the smallest stable element in the class is the stable basis element in that class. Furthermore, $\#\,\mathrm{UStb}(\mathscr{S})$ is finite.

CONSTRUCTION 2.2. For every $i \in \mathbb{N}$, we take $\mathrm{gr}(\mathscr{S})_i$ to be the $\mathbb{K}$-vector space with basis $\{\tau^a \mid a \in \Gamma, \mathrm{ord}_{\mathscr{S}}(a) = i\}$ and let $\mathrm{gr}(\mathscr{S}) = \bigoplus_{i\ge 0}\mathrm{gr}(\mathscr{S})_i$. Define a multiplication on $\mathrm{gr}(\mathscr{S})$ by

$$\tau^a.\tau^b = \begin{cases} \tau^{a+b}; & \text{if } \mathrm{ord}_{\mathscr{S}}(a+b) = \mathrm{ord}_{\mathscr{S}}(a) + \mathrm{ord}_{\mathscr{S}}(b), \\ 0; & \text{otherwise.} \end{cases}$$

This makes $\mathrm{gr}(\mathscr{S})$ into a positively graded $\mathbb{K}$-algebra with degree i part equal to $\mathrm{gr}(\mathscr{S})_i$. It is easy to see that $\mathbb{K}[S]/s^d\mathbb{K}[S] \cong \mathrm{gr}(\mathscr{S})$, as graded $\mathbb{K}$-algebras, with the class of $s^{md-x}t^x$ in $\mathbb{K}[S]/s^d\mathbb{K}[S]$ corresponding to $\tau^x \in \operatorname{gr}(\mathscr{S})$ (where $\operatorname{ord}_{\mathscr{S}}(x) = m$). Equivalently, we may identify $\operatorname{gr}(\mathscr{S})$ with $R/(x_0, I_{\mathscr{S}})$, in which case the canonical image of x_i in $R/(x_0, I_{\mathscr{S}})$ corresponds to $\tau^{a_i} \in \operatorname{gr}(\mathscr{S})$. Since $\mathbb{K}[S]$ is an integral domain, s^d is a nonzero-divisor on $\mathbb{K}[S]$. The existence of an unstable element is equivalent to t^d being a zero-divisor on $\mathbb{K}[S]$. Since s^d, t^d is a system of parameters for $\mathbb{K}[S]$, we thus have that $\mathbb{K}[S]$ is Cohen–Macaulay if and only if $\operatorname{UStb}(\mathscr{S}) = \emptyset$. If $(a,b) \in S$, we define $\deg(a,b) = (a+b)/d$ so that $\deg(a,b)$ equals the degree of $s^a t^b$ as an element of $\mathbb{K}[S]$.

Notation 2.3. By $\mathbb{Z}$ and $\mathbb{N}$ we mean the set of all integers and nonnegative integers respectively. If X is a set, we show its cardinality by $\#X$. Other notation introduced in this section, such as $R, \operatorname{gr}(\mathscr{S}), \operatorname{StdB}(\mathscr{S}), \operatorname{StbB}(\mathscr{S})$, will be used throughout the paper. We may refer to $\mathscr{S}$ as a curve, and write simply ord instead of $\operatorname{ord}_{\mathscr{S}}$.

3. The regularity of a projective monomial curve

In this section we first recall the definition of regularity, and then some facts about Hilbert functions. Then we prove our main theorem.

Let $R = \mathbb{K}[x_0, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, graded by $\deg(x_i) = 1$. Every finitely generated graded R-module M has a minimal graded free resolution of the form

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{p,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}} \longrightarrow M \longrightarrow 0, \tag{3.1}$$

where p is the projective dimension of M. We set $c_i = \max\{j \mid \beta_{i,j} \neq 0\}$ for $i \in \{0, \ldots, p\}$. The *Castelnuovo–Mumford regularity*, or simply the *regularity*, of M is defined as $\operatorname{reg}(M) = \max\{c_i - i \mid i = 0, \ldots, p\}$.

Let M be a finitely generated positively graded R-module with $\dim(M) = r$ and $\operatorname{pd}(M) = p$. The *Hilbert function* $H(M, -) : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by $H(M, i) := \dim_{\mathbb{K}}(M_i)$. There is a polynomial $P_M(i)$ of degree $r-1$, called the *Hilbert polynomial* of M, such that $H(M,i) = P_M(i)$ for $i \gg 0$. The *Hilbert series* $H_M(t)$ of M is defined by $H_M(t) := \sum_{i \geq 0} H(M,i)t^i$. There exists a unique $Q_M(t) \in \mathbb{Z}[t]$ with $Q_M(1) \neq 0$ such that $H_M(t) = Q_M(t)/(1-t)^r$ [6, Corollary 4.1.8]. If $Q_M(t) = h_0 + h_1 t + \cdots + h_b t^b$ with $h_b \neq 0$, then $(h_0, h_1, \ldots, h_b)$ is called the *h-vector* of M. The total degree $\deg(H_M(t)) = b - r$ is called the a-*invariant* of M and is denoted by $\mathrm{a}(M)$.

In particular, for the algebra $\operatorname{gr}(\mathscr{S})$ introduced in Section 2, we have $H(\operatorname{gr}(\mathscr{S}), i) = \#\mathfrak{M}_i$ and $P_{\operatorname{gr}(\mathscr{S})}(i) = d$ [15, Theorem 3(a)].

Lemma 3.1 ([6, Proposition 4.1.12]). *With notation as above* $H(M, \mathrm{a}(M)) \neq P_M(\mathrm{a}(M))$ *and* $H(M,i) = P_M(i)$ *for all* $i \geq \mathrm{a}(M) + 1$.

Lemma 3.2. *Let M be a finitely generated graded R-module. Suppose that y is a homogeneous nonzero divisor of M. Then* $\mathrm{a}(M/yM) = \mathrm{a}(M) + \deg(y)$.

Proof. Suppose that $\deg(y) = s$. From the short exact sequence

$$0 \longrightarrow M(-s) \xrightarrow{y.} M \longrightarrow M/yM \longrightarrow 0$$

it follows that $H_{M/yM}(t) = (1 - t^s)H_M(t)$. Therefore $\mathrm{a}(M/yM) = \deg(H_{M/yM}(t)) = \deg(H_M(t)) + s = \mathrm{a}(M) + \deg(y)$. □

REMARK 3.3. Let M be a graded R-module, with resolution and notation as in (3.1). Then by [1, Proposition 1.1] (and the graded Auslander–Buchsbaum theorem) we have

(1) If M is Cohen–Macaulay, then $p := \mathrm{pd}(M) = n+1-\dim(M)$. Furthermore, $c_0 < c_1 < \cdots < c_p$ so that $\mathrm{reg}(M) = c_p - p$ (i.e., regularity is attained at the last step of the resolution). From the resolution (noting that $\mathrm{a}(R) = -(n+1)$), we have $\mathrm{a}(M) = c_p - (n+1)$, so that $\mathrm{reg}(M) = \mathrm{a}(M) + \dim(M)$. (In particular, $p = n-1$ if $M = R/I_{\mathscr{S}}$, $R/I_{\mathscr{S}}$ Cohen–Macaulay.)

(2) If $M = R/I_{\mathscr{S}}$ is not Cohen–Macaulay, then $p = n$ and we still have $c_0 < c_1 < \cdots < c_{n-1}$, so that the regularity of $R/I_{\mathscr{S}}$ is attained in the last or second last step of the resolution. Furthermore, tensoring the Koszul complex of the R-module $\mathbb{K}$ with $\mathrm{gr}(\mathscr{S})$, we see that $c_n - n$ is the largest degree of an unstable element of $\mathscr{S}$.

Now we are ready to state and prove our main result.

THEOREM 3.4. *Let R, $\mathscr{S}$ and $R_{\mathscr{S}}$ be as introduced in Section 2, and let $\mathrm{reg}(R_{\mathscr{S}})$ be the regularity of $R_{\mathscr{S}}$, regarded as an R-module. Then we have*

$$\begin{aligned}\mathrm{reg}(R_{\mathscr{S}}) &= \max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S}) \cup \mathrm{StbB}(\mathscr{S})\}\\ &= \max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S}) \cup \mathrm{StdB}(\mathscr{S})\}.\end{aligned}$$

Proof. Consider the short exact sequence of graded R-modules

$$0 \to R_{\mathscr{S}}(-1) \xrightarrow{x_0.} R_{\mathscr{S}} \to \mathrm{gr}(\mathscr{S}) \to 0.$$

By [7, Proposition 20.20], $\mathrm{reg}(R_{\mathscr{S}}) = \mathrm{reg}(R_{\mathscr{S}}/x_0R_{\mathscr{S}}) = \mathrm{reg}(\mathrm{gr}(\mathscr{S}))$. So we compute the regularity of $\mathrm{gr}(\mathscr{S})$.

Let $\mathfrak{a}$ be the ideal of $\mathrm{gr}(\mathscr{S})$ generated by all elements τ^a, where a runs over the set of all unstable elements of Γ. If $a \in \mathrm{UStb}(\mathscr{S})$, then either $a + a_i \in \mathrm{UStb}(\mathscr{S})$ or $\tau^a\tau^{a_i} = 0$, for any $i \in \{1, \ldots, n\}$, by Lemma 2.1, and therefore $\mathfrak{a} = \bigoplus_{a\in\mathrm{UStb}(\mathscr{S})} \mathbb{K}\tau^a$. Now we have $\mathrm{gr}(\mathscr{S})/\mathfrak{a}$ is a one dimensional Cohen–Macaulay ring and $H(\mathrm{gr}(\mathscr{S})/\mathfrak{a}, i) = \#\,\mathrm{StbB}(\mathscr{S})_{\leq i}$. By Lemma 3.1, $\mathrm{a}(\mathrm{gr}(\mathscr{S})/\mathfrak{a}) = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StbB}(\mathscr{S})\} - 1$. By Remark 3.3(1), $\mathrm{reg}(\mathrm{gr}(\mathscr{S})/\mathfrak{a}) = \mathrm{a}(\mathrm{gr}(\mathscr{S})/\mathfrak{a}) + 1 = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StbB}(\mathscr{S})\}$. Also $\mathfrak{a}$ is a zero dimensional Cohen–Macaulay module with $H(\mathfrak{a}, i) = \#\,\mathrm{UStb}(\mathscr{S})_i$. Thus $\mathrm{a}(\mathfrak{a}) = \max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\}$ and again by Remark 3.3(1) we have $\mathrm{reg}(\mathfrak{a}) = \mathrm{a}(\mathfrak{a}) + \dim(\mathfrak{a}) = \max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\}$. Now from the short exact sequence

$$0 \to \mathfrak{a} \to \mathrm{gr}(\mathscr{S}) \to \frac{\mathrm{gr}(\mathscr{S})}{\mathfrak{a}} \to 0$$

and [7, Corollary 20.19(d)] we have $\operatorname{reg}(\operatorname{gr}(\mathscr{S})) = \max\{\operatorname{reg}(\mathfrak{a}), \operatorname{reg}(\operatorname{gr}(\mathscr{S})/\mathfrak{a})\} = \max\{\operatorname{ord}(a) \mid a \in \operatorname{UStb}(\mathscr{S}) \cup \operatorname{StbB}(\mathscr{S})\}$. This proves the first equality in the theorem. If $a \in \operatorname{StdB}(\mathscr{S}) \setminus \operatorname{StbB}(\mathscr{S})$ then $a \in \operatorname{UStb}(\mathscr{S})$ and from this we deduce that $\max\{\operatorname{ord}(a) \mid a \in \operatorname{UStb}(\mathscr{S}) \cup \operatorname{StdB}(\mathscr{S})\} \leq \operatorname{reg}(\operatorname{gr}(\mathscr{S}))$. Conversely, let $a \in \operatorname{StbB}(\mathscr{S}) \setminus \operatorname{StdB}(\mathscr{S})$, then $a - d \in \operatorname{UStb}(\mathscr{S})$ and $\operatorname{ord}(a) \leq \operatorname{ord}(a-d)$ by Lemma 2.1. Therefore $\operatorname{reg}(\operatorname{gr}(\mathscr{S})) \leq \max\{\operatorname{ord}(a) \mid a \in \operatorname{UStb}(\mathscr{S}) \cup \operatorname{StdB}(\mathscr{S})\}$. □

Our description of regularity in Theorem 3.4 seems quite different from that in [2] or [12] and is, we feel, more elementary. Unlike [2], we do not use initial ideals, and, as we see in the next section, computation with Theorem 3.4 does not require the use of Gröbner bases (cf. [2, Remark 2.6] or [12, Remark 2.3]). However, [12, Corollary 2.2] does apply more generally.

4. Applications

In this section, we describe several applications of Theorem 3.4. First of all, we have the following theorem.

THEOREM 4.1. *Let i_0 be the smallest integer such that $\#\mathfrak{M}_{i_0} = d$ and $\mathfrak{M}_{i_0} \subseteq d + \mathfrak{M}_{i_0-1}$ (equivalently $\dim_{\mathbb{K}}(\operatorname{gr}(\mathscr{S})_{i_0}) = d$ and $t^d : \operatorname{gr}(\mathscr{S})_{i_0-1} \to \operatorname{gr}(\mathscr{S})_{i_0}$ is onto). Then $\operatorname{reg}(R_{\mathscr{S}}) = i_0 - 1$.*

Proof. By [11, Lemma 1.4], $\mathfrak{M}_{j+1} = d + \mathfrak{M}_j$ for all $j \geq i_0$. From this, it follows that there are no elements of $\operatorname{UStb}(\mathscr{S})$ or $\operatorname{StdB}(\mathscr{S})$ in degrees greater than or equal to i_0. By assumption we have $\#\mathfrak{M}_{i_0-1} \geq \#\mathfrak{M}_{i_0}$. If $\#\mathfrak{M}_{i_0-1} > \#\mathfrak{M}_{i_0}$, then $\operatorname{UStb}(\mathscr{S}) \cap \mathfrak{M}_{i_0-1} \neq \emptyset$. If $\#\mathfrak{M}_{i_0-1} = \#\mathfrak{M}_{i_0}$, then $\mathfrak{M}_{i_0} = d + \mathfrak{M}_{i_0-1}$ and $\mathfrak{M}_{i_0-1} \nsubseteq d + \mathfrak{M}_{i_0-2}$ (by definition of i_0) so $\operatorname{StbB}(\mathscr{S}) \cap \mathfrak{M}_{i_0-1} \neq \emptyset$. By Theorem 3.4 we now have $\operatorname{reg}(R_{\mathscr{S}}) = i_0 - 1$. □

Theorem 4.1 can be turned into an algorithm as follows. By the method described in the second paragraph of Section 2, one can recursively compute the sets $\mathfrak{M}_i$. One eventually obtains $\#\mathfrak{M}_i = d$ and $\mathfrak{M}_i \subseteq d + \mathfrak{M}_{i-1}$. The regularity of $R_{\mathscr{S}}$ then follows from Theorem 4.1.

EXAMPLE 4.2. Let $\mathscr{S} = \{2, 5, 7\}$. Then $\mathfrak{M}_0 = \{0\}$, $\mathfrak{M}_1 = \{2, 5, 7\}$, $\mathfrak{M}_2 = \{4, 9, 10, 12, 14\}$, $\mathfrak{M}_3 = \{6, 11, 15, 16, 17, 19, 21\}$, $\mathfrak{M}_4 = \{8, 13, 18, 20, 22, 23, 24, 26, 28\}$, $\mathfrak{M}_5 = \{25, 27, 29, 30, 31, 33, 35\}$. We observe that $\mathfrak{M}_5 \subseteq 7 + \mathfrak{M}_4$, so $i_0 = 5$ and, by Theorem 4.1, $\operatorname{reg}(R_{\mathscr{S}}) = 4$. In this case $\mathfrak{M}_4$ contains $8, 13 \in \operatorname{UStb}(\mathscr{S})$, $20 \in \operatorname{StbB}(\mathscr{S})$, and $8 \in \operatorname{StdB}(\mathscr{S})$.

Sometimes one can explicitly describe all the sets $\mathfrak{M}_i$ for an infinite class of examples. In [11] this was done for the curves $\mathscr{S} = \{\delta, m_0, \ldots, m_{p+1}\}$ defined in the next proposition. From the $\mathfrak{M}_i$ it is straightforward to obtain explicitly $\operatorname{UStb}(\mathscr{S})$ and $\operatorname{StbB}(\mathscr{S})$ for all $\mathscr{S}$ of this class. This yields the following proposition.

PROPOSITION 4.3. *Let* $\mathscr{S} = \{\delta, m_0, \dots, m_{p+1}\}$ *where* $m_i = m + i\delta$ *for* $0 \le i \le p+1, p \ge 0, \delta \ge 1, m \ge 2$, *and* $\gcd(\delta, m) = 1$. *If* $\delta < m-1$, *then* $\mathrm{UStb}(\mathscr{S}) = \{cm_{p+1} + i\delta \mid \delta < i < m, 0 \le c < \lceil \frac{m-i}{p+1} \rceil\}$, *and* $\max\{\mathrm{ord}_{\mathscr{S}}(n) \mid n \in \mathrm{UStb}(\mathscr{S})\} = m-1$. *If* $\delta > m+p+1$, *then* $\mathrm{UStb}(\mathscr{S}) = \{cm_{p+1} + i\delta \mid m+p+1 \le i \le \delta - 1, \lceil \frac{m-i}{p+1} \rceil \le c < 0\}$ *and* $\max\{\mathrm{ord}_{\mathscr{S}}(n) \mid n \in \mathrm{UStb}(\mathscr{S})\} = \delta - 1$. *For other* δ, $\mathrm{UStb}(\mathscr{S}) = \emptyset$. *Furthermore,* $\mathrm{StbB}(\mathscr{S}) = \{i\delta \mid 0 \le i \le \delta - 1\} \cup \{\min(\delta^2, \delta^2 + \lceil (m-\delta)/(p+1) \rceil m_{p+1})\} \cup \{i\delta + \lceil (m-i)/(p+1) \rceil m_{p+1} \mid \delta < i < m_{p+1}\}$ *and*

$$\max\{\mathrm{ord}_{\mathscr{S}}(n) \mid n \in \mathrm{StbB}(\mathscr{S})\} = \begin{cases} \delta + \lceil \big(m - (\delta+1)\big)/(p+1) \rceil; & \text{if } \delta < m, \\ \delta; & \text{if } m < \delta \le m+p, \\ \delta - 1; & m+p < \delta < m_{p+1}. \end{cases}$$

There is a similar explicit expression for the $\mathrm{StdB}(\mathscr{S})$. From Proposition 4.3 and Theorem 3.4, we deduce the following theorem.

THEOREM 4.4. *Let* $\mathscr{S} = \{\delta, m_0, \dots, m_{p+1}\}$ *where* $m_i = m + i\delta$ *for* $0 \le i \le p+1, p \ge 0, \delta \ge 1, m \ge 2$, *and* $\gcd(\delta, m) = 1$. *Then*

$$\mathrm{reg}(R_{\mathscr{S}}) = \begin{cases} m-1; & \text{if } \delta < m, \\ \delta; & \text{if } m < \delta \le m+p, \\ \delta - 1; & \text{otherwise.} \end{cases}$$

The curves $\mathscr{S}$ in Theorem 4.4 are an example of almost arithmetic progression curves (those in which all but one of the a_i are consecutive terms of an arithmetic progression). It would be interesting to find a similar explicit expression for the regularity of all almost arithmetic progression curves. However, we have not been able to do this. The curves in Theorem 4.4 seem to be very special.

We have implemented in Mathematica the algorithm of Theorem 4.1 and used it to investigate the regularity of all projective monomial curves up to degree 19. (This took several hours of computer time. With more patience one could go a bit higher. However, the number of projective monomial curves of degree d is approximately 2^{d-1} so this seemed to be a reasonable place to stop.) We have observed that frequently (as with Example 4.2) one has $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\} = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StdB}(\mathscr{S})\} = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StbB}(\mathscr{S})\}$. There are also many $\mathscr{S}$ for which $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\} = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StbB}(\mathscr{S})\} > \max\{\mathrm{ord}(a) \mid a \in \mathrm{StdB}(\mathscr{S})\}$ (and similarly with StbB and StdB interchanged), for example $\mathscr{S} = \{1, 2, 5, 6\}$. Much less frequently, we found cases where $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\} > \max\{\mathrm{ord}(a) \mid a \in \mathrm{StdB}(\mathscr{S})\} = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StbB}(\mathscr{S})\}$ (this first happens for $d = 13$, e.g. $\mathscr{S} = \{2, 10, 12, 13\}$, and also for $\mathscr{S} = \{2, 12, 15\}$) and where $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\} < \max\{\mathrm{ord}(a) \mid a \in \mathrm{StdB}(\mathscr{S})\} = \max\{\mathrm{ord}(a) \mid a \in \mathrm{StbB}(\mathscr{S})\}$ so the unstable elements and either the standard or stable basis are needed in the statement of Theorem 3.4. We were most interested in the last case, so would like to say a bit more about it.

Let $\mathscr{R}$ be the set of non-Cohen–Macaulay projective monomial curves $\mathscr{S}$ such that $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\} < \max\{\mathrm{ord}(a) \mid a \in \mathrm{StdB}(\mathscr{S})\}$. By Remark 3.3(2) and Theorem 3.4, $\mathscr{R}$ is exactly the set of non-Cohen–Macaulay projective monomial curves for which regularity is not attained at the last step of the resolution. We have observed a curious relation between $\mathscr{R}$ and the set $\mathscr{N}$ of non-Cohen–Macaulay projective monomial curves with no negative entry in their h-vector (studied in [13]). We have been able to show that $\mathscr{N} \subseteq \mathscr{R}$. We found that both $\mathscr{R}$ and $\mathscr{N}$ are empty in degrees less than or equal to 11 and that $\mathscr{N} = \mathscr{R}$ in degrees 12 through 17 (for example, $\{1,2,5,8,12\} \in \mathscr{N}$), but in degrees 18 and 19 there are a few cases of curves in $\mathscr{R}\backslash\mathscr{N}$ (for example, $\{6,10,11,14,15,17,18\} \in \mathscr{R}\backslash\mathscr{N}$).

Let S' be the set of all elements $\boldsymbol{\alpha} \in \mathbb{N}^2$ such that there exist $m_1, m_2 \in \mathbb{N}$ with $\boldsymbol{\alpha} + m_1(d,0) \in S, \boldsymbol{\alpha} + m_2(0,d) \in S$. We can show that (in the non-Cohen–Macaulay case) $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\} = 1 + \max\{\deg(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in S'\backslash S\}$. Bruns, Gubeladze, and Trung in [5], p. 207, give an expression for $\mathrm{reg}(\mathbb{K}[S])$ in terms of S', which we interpret in our notation as $\mathrm{reg}(\mathbb{K}[S]) = 1 + \max\{\deg(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in S'\backslash S\}$. But the latter is equal to $\max\{\mathrm{ord}(a) \mid a \in \mathrm{UStb}(\mathscr{S})\}$ so their claim (as we have interpreted it) will fail for curves in $\mathscr{R}$. In any case their expression for $\mathrm{reg}(\mathbb{K}[S])$ gives value 1 if $S' = S$ (the Cohen–Macaulay case), which is false. In [9] it is proved that $\mathrm{reg}(\mathbb{K}[S]) \le d+1-n$. It is remarked in [5] that "It would be nice if we could find a combinatorial proof for this bound." (The same question is also posed in [10], p. 721.) We tried quite hard to do this using the methods of the present paper and [14], but could only obtain a bound of about $2d$ instead of $d+1-n$. (Note that the regularity bound is stated in [9] as $d+2-n$, but they define the regularity of a curve to be that of its ideal $I_{\mathscr{S}}$, which is one more than that of $R/I_{\mathscr{S}}$, which we use.)

The results of [2] have been programmed in Singular ([4], [8]). Our algorithm is often faster. As one example, if $\mathscr{S} = \{7,13,16,127,321,433,761\}$ it gave regularity (13 for $R_{\mathscr{S}}$) in 0.11 seconds compared with about 10 seconds with regMonCurve from [4]. However, we have found regMonCurve to sometimes be faster if $n = 3$ or 4. (If $n = 2$, $\mathrm{reg}(R_{\mathscr{S}}) = d-1$, so no algorithm is needed.)

Acknowledgment. The authors would like to thank the referee for a number of helpful suggestions, which made a substantial improvement to the paper.

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