ON THE CONJUGACY GROWTH FUNCTIONS OF GROUPS

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Dedicated to Paul Schupp

ABSTRACT. To every finitely generated group, one can assign the conjugacy growth function that counts the number of conjugacy classes intersecting a ball of radius n. Results of Ivanov and Osin show that the conjugacy growth function may be constant even if the (ordinary) growth function is exponential. The aim of this paper is to provide conjectures, examples and statements that show that in "normal" cases, groups with exponential growth functions also have exponential conjugacy growth functions.

1. Introduction

The investigation of the growth of conjugacy classes was motivated by counting closed geodesics (up to free homotopy) on complete Riemannian manifolds. In the case of a negative upper bound for the sectional curvature of a complete Riemannian manifold M, there is only one closed geodesic in each free homotopy class. Strengthening a result of Sinai [Sinai], Margulis [Mar, MarS] proved that for compact manifolds of pinched negative curvature and exponential volume growth $\exp(hn)$, the number of primitive closed geodesics of period $\leq t$ is approximately

$$(1) \qquad \frac{\exp(ht)}{ht}.$$

In group theoretic terms, this implies that the number of primitive conjugacy classes intersecting the ball of radius n in the Cayley graph of the fundamental group of M (with respect to some finite generating set) is between $\frac{1}{Cn}\exp(\frac{1}{C}hn)$ and $\frac{C}{n}\exp(Chn)$ for some constant $C > 1$. (Recall that a conjugacy class is *primitive* if it does not contain elements that are proper

powers.) Basically it means that two cyclically reduced products of generators are conjugate "almost" only if they are cyclic shifts of each other. So this result should not be very surprising to those who know the theory of Gromov-hyperbolic and relatively hyperbolic groups. And indeed, this result has been generalized several times to larger classes of manifolds and groups (see, for example, [Link, CK] and references therein).

DEFINITION 1.1. Let $G = \langle X \rangle$ be a group generated by a finite set X. For every n, let $\gamma^c(n)$ be the number of conjugacy classes of G intersecting the ball of radius n in G. The function $\gamma^c(n)$ will be called the conjugacy growth function of G with respect to X.

Clearly, conjugacy growth functions corresponding to different finite generating sets of the same group are equivalent because a ball of radius n with respect to one generating set contains a ball of radius Cn with respect to the other generating set for some constant C depending only on the two generating sets but not on n. Here, as usual, we say that two functions $f, g : \mathbb{N} \to \mathbb{N}$ are *equivalent* if $f(n) \leq Cg(Cn) + C$ and $g(n) \leq Cf(Cn) + C$ for some constant C and all n. Thus, we can talk about the conjugacy growth function of a finitely generating group.

The definition differs from that in [Mar, MarS] and other papers because we do not consider only primitive conjugacy classes. In the case of hyperbolic, relatively hyperbolic groups or CAT(0)-groups considered in [Mar, MarS, CK, Link] and other papers these definitions give equivalent functions (in the usual sense of asymptotic group theory) because the number of nonprimitive conjugacy classes is small comparing to the number of all conjugacy classes. But, for example, in a torsion group without involutions every element is a square, so there are no primitive conjugacy classes at all.

It is known that the conjugacy growth function for arbitrary groups can differ dramatically from the (ordinary) growth function. Ivanov [Olbook] constructed the first example of a finitely generated infinite group with finite number of conjugacy classes, and Osin [Osin1], answering an old group theory question (one of the origins of this question is also from dynamics), constructed a finitely generated infinite group with just two conjugacy classes. The conjugacy growth functions for these groups are eventually constants while the ordinary growth functions are exponential. Still it is not known how widespread this phenomenon is. For example, there are no examples of finitely presented groups with exponential growth function and subexponential conjugacy growth function. Conjectures, examples and theorems of this paper show that for "ordinary" groups, exponential growth should imply exponential conjugacy growth.

Our results are intentionally not of the most general form. We just demonstrate the ideas which clearly can be used in more general situations.

Updates. During the time between the first appearance of the paper in the arXiv and the day the paper was accepted in the *Illinois Journal of Mathematics*, several problems mentioned in the paper have been solved. For each of these problems, we include references to (and in one case short description of) the solution as an update.

2. Amenable groups

CONJECTURE 2.1. For every amenable group of exponential growth, the conjugacy growth function is exponential. An amenable group with polynomial conjugacy growth function is virtually nilpotent.

For solvable groups, Conjecture 2.1 should follow from the proof of Milnor [Milnor] and Wolf [Wolf] of the fact that solvable groups of subexponential growth are virtually nilpotent. In the case of polycyclic groups, one can use the method of Wolf [Wolf]. In the case of solvable nonpolycyclic groups, Milnor proved that if such a group has exponential growth, it contains a free subsemigroup. One "only" needs to show that some of the free subsemigroups are Frattini-embedded (i.e., two elements of the subsemigroup are conjugate in the ambient groups if and only if they are cyclic shifts of each other). One can also use a result of Kropholler [Kro] characterizing finitely generated solvable groups which do not have sections which are wreath products of a cyclic group with $\mathbb{Z}$ and results of Osin [Osin2] about the uniform growth of solvable groups.

The second part of Conjecture 2.1 is of course an analog of the celebrated theorem of Gromov [Gromov] about groups with polynomial growth. In this regard, it would be interesting to compute the conjugacy growth functions of Grigorchuk (and similar) groups of subexponential growth [Grig1]. It is hard to believe that any of these groups have polynomial conjugacy growth, but these groups served as counterexamples to many other conjectures before.

Update 1. Laurent Bartholdi and independently but a few months later Ievgen Bondarenko proved that the conjugacy growth of every Grigorchuk's group is super-polynomial. Their proofs are essentially the same. A slightly modified proof is presented below.

We denote the conjugacy growth function of G by $\gamma_G^c(n)$.

THEOREM 2.2 (Bartholdi, Bondarenko). *If G is the Grigorchuk 2-group of intermediate growth, then for some $\alpha > 0$ we have $\gamma_G^c(n) \geq e^{n^\alpha}$.*

Proof. The Grigorchuk 2-group G contains a subgroup K of finite index, such that K contains a copy of $K^d = K \times K \times \cdots \times K$ (d times, $d > 1$) [Grig2]. Let ψ be the injection map K^d to K. It is shown in [Grig3] that for any elements $u_1, \ldots, u_d, v_1, \ldots, v_d$ in K, the element $\psi(u_1, \ldots, u_d)$ is conjugate to the element $\psi(v_1, \ldots, v_d)$ if and only if for some permutation σ, the element u_i is conjugate to $v_{\sigma(i)}$ in K for all $i = 1, \ldots, d$. Hence $\gamma_G^c(n) \geq L\gamma_G^c(Ln)^d$

for some constant L. Note also that G has infinitely many conjugacy classes since it has elements of arbitrary large finite orders, hence γ_G^c is unbounded. Now a standard argument (see, for example, [GP, Lemma 2.1]) implies that $\gamma_G^c(n) \geq e^{n^\alpha}$ for some $\alpha > 0$. $\qquad\square$

Note that a very similar proof works for all so called *regularly branch* groups (see [Grig2] for the definition).

2.1. Two examples. Below, we are using the standard Computer Science notation: we write $g(x) = \Theta(f(x))$ (resp. $g(x) = O(f(x))$) if $\frac{1}{C} f(x) \leq g(x) \leq Cf(x)$ (resp. $g(x) \leq Cf(x)$) for some constant $C > 1$ and all sufficiently large x.

EXAMPLE 2.3. The Baumslag–Solitar group $BS(1,n) = \langle a, b \mid b^{-1}ab = a^n \rangle$ ($n \geq 2$) has exponential conjugacy growth function.

Proof. It is easy to see that for numbers $k \neq l$ not divisible by n, the elements a^k, a^l are not conjugate in $BS(1,n)$. The length of a^k in $BS(1,n)$ is $\Theta(\log k)$. Indeed, if $k = m_1 n^{t_1} + \cdots + m_s$ is the n-ary representation of k, then $a^k = b^{-t_1} a^{m_1} b^{t_1} b^{-t_2} a^{m_2} b^{t_2} \cdots$, so the length of a^k is at most $2(t_1 + t_2 + \cdots) + m_1 + m_2 + \cdots$. Hence, the conjugacy growth function is exponential. $\qquad\square$

The next example shows that free subsemigroups are indeed helpful in proving that the conjugacy growth is exponential.

EXAMPLE 2.4. Let S_∞ be the group of all permutations of $\mathbb{Z}$ with finite supports. Then the cyclic group $\mathbb{Z} = \langle b \rangle$ acts on S_∞ by shift. Let $G = S_\infty \rtimes \mathbb{Z}$. This group is clearly generated by b and the transposition $a = (1,2)$. The conjugacy growth of G is exponential.

Proof. Indeed, consider the subsemigroup generated by b^2 and b^2a. Every element in this semigroup has the form $b^n ab^{n_1} a \cdots$ where all n, n_i are positive even integers. By taking a cyclic shift, we can assume that the word ends with a and $n \geq n_1, n_2, \ldots$. Let us prove that no two of such elements are conjugate (this will obviously imply that the conjugacy growth is exponential). Indeed, the element

$$(2) \qquad\qquad b^n ab^{n_1} a \cdots b^{n_k} a$$

is equal to

$$(3) \qquad b^{n+n_1+\cdots+n_k} a(n_1 + \cdots + n_k) a(n_2 + \cdots + n_k) \cdots a,$$

where $a(m) = (1+m, 2+m)$, so $a = a(0)$. Note that since all n_i are even and positive, the transpositions $a(n_i + \cdots + n_k)$ are all independent, so they pairwise commute. Therefore, the form (3) completely determines the word (2). If we conjugate an element $b^s p$ where $p \in S_\infty$ by b^m, the result is $b^s p(m)$ where $p(m)$ is obtained from p by shifting all numbers in the permutation by m. If we conjugate $b^s p$ by $t \in S_\infty$, the result equals $b^s t(s)^{-1} pt$. Note that in

(3), the absolute value of the exponent of b is always bigger than all numbers involved in the permutation $a(n_1 + \cdots + n_k)a(n_2 + \cdots + n_k)\cdots a$. Therefore, if $b^m t$ $(t \in S_\infty, m > 0)$ is conjugate to (3) and not equal to (3), then either some numbers in the support of the permutation t are negative or some of them are bigger than m. In both cases $b^m t$ is not of the form (3). So two different elements of the form (3) are not conjugate in G. It remains to note that the word length of (2) in the alphabet $\{a, b\}$ is at most $n + n_1 + \cdots + n_k + k$. □

Example 2.4 is also interesting because the number of conjugacy classes of elements of S_∞ intersecting the ball of radius n grows subexponentially. In contrast, as we showed above, for $BS(1,n)$, the growth of conjugacy classes of the kernel of the natural homomorphism $BS(1,n) \to \mathbb{Z}$ is exponential.

2.2. Nilpotent groups. Recall that by a theorem of Bass [Bass] if G is a finitely generated nilpotent group, and $\gamma_S(n)$ its growth function for some generating set S, then there are constants $A, B > 0$ such that $An^d \le \gamma_S(n) \le Bn^d$ for all $n \ge 1$, where $d = d(G) = \sum_{h \ge 1} h r_h$, r_h is the torsion-free rank of G_h/G_{h+1} and $G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \supseteq G_h \supseteq \cdots$ is the lower central series of G. In particular, for the Heisenberg group $H = \langle x, y, z \mid [x,y] = z, zx = xz, zy = yz \rangle$ with two generators (x and y), its growth function is $\Theta(n^4)$.

EXAMPLE 2.5. The conjugacy growth function of H is $\Theta(n^2 \log n)$.

Proof. Indeed, every element of H is uniquely expressed in the form $x^k y^l z^m$. The ball B of radius $4n$ consists of such elements with $k \le n, l \le n, m \le n^2$ (up to a big O). Conjugating this element by x amounts to adding k to m (and preserving k, l), conjugating by y amounts to adding l to m. Therefore, if $k \ne 0$ or $l \ne 0$, every element of the form $x^k y^l z^m$ can be conjugated into an element of the form $x^k y^l z^s$ where $0 \le s < d = \gcd(k,l)$. The number of elements of the form $x^k y^l z^m$ in B where either $k = l = 0$, $m \le n^2$ or $kl = 0, |m| \le \max\{|k|, |l|\}$ does not exceed $O(n^2)$, so these elements can be ignored. So we can assume that $k \ne 0$ and $l \ne 0$. Then $k = dk_0$, $l = dl_0$ where k_0, l_0 are co-prime numbers with absolute values bounded by $O(n/d)$. The number of such pairs (k_0, l_0) is at most Cn^2/d^2 for some constant C. Therefore, for every d between 1 and n, the number of triples (k, l, s) with $\gcd(k, l) = d$, $0 \le s < d$ is at most Cn^2/d for some constant C. Summing up for all d from 1 to n, we get $Cn^2(1 + 1/2 + \cdots) \le C'n^2 \log n$. On the other hand, for every $n > 1$ and $d < n$ consider all elements $x^{k_0 d} y^{l_0 d} z^s$ where $0 < k_0 \le n/d, 0 < l_0 \le n/d, 0 \le s < d$, $\gcd(k_0, l_0) = 1$. Again it is easy to see that conjugating such an element by any nontrivial element of H produces an element not of this form. Hence, these elements are pairwise nonconjugate. The length of each of these elements is at most $C_1 n$ for some constant C_1. By the well-known number theory result (see, for example, [HW, page 354]), the number of such triples is $\Theta(n^2/d)$. Hence, the total number of conjugacy classes intersecting a ball of radius n in

H is at least $C_2 n^2(1 + 1/2 + \cdots + 1/n) \geq C_3 n^2 \log n$ for some constant $C_3 > 0$ and large enough n. $\qquad\square$

Similarly, one can prove that under the assumptions and notation of Bass' theorem above the conjugacy growth function $\gamma^c(m)$ does not exceed Cm^s where $s = \sum_{h \geq 1} r_h$.

PROBLEM 2.6. Find more precise estimates for the conjugacy growth functions of finitely generated nilpotent groups.

Update 2. In the case of solvable groups, Conjecture 2.1 has been proved. In the case of polycyclic groups, it was proved by Hull [Hull]. In the general solvable case, it was proved by Breuillard and Cornulier [BC].

3. Linear groups

One of the major steps towards describing groups of polynomial growth was the Tits alternative: a finitely generated linear group (over a field) of subexponential growth is virtually solvable [Tits].

CONJECTURE 3.1. Every finitely generated linear group (over a field) of subexponential conjugacy growth is virtually solvable.

EXAMPLE 3.2. The conjugacy growth of $SL_n(\mathbb{Z})$ for $n \geq 2$ is exponential.

Proof. proof Indeed, for every positive integer m consider the matrix A_m with $A_m[1,1] = m, A_m[1,2] = m - 1, A_m[i,i] = 1, i \geq 2$, and all other $A[i,j]$ equal 0. The word length of A_m is approximately $\log m$ by [LMR], and all these matrices are pairwise nonconjugate because they have different eigenvalues. This immediately implies that the conjugacy growth is exponential. $\qquad\square$
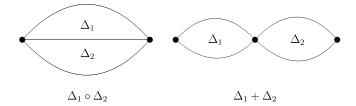
Update 3. Conjecture 3.1 has been proved by Breuillard, Cornulier, Lubotzky and Meiri [BCLM].

4. Diagram groups

Let us recall the definition of a *diagram group* (see [GS1, GS3, GS4] for more formal definitions). A (semigroup) *diagram* is a planar directed labeled graph tesselated into cells, defined up to an isotopy of the plane. Each diagram Δ has the top path top(Δ), the bottom path bot(Δ), the initial and terminal vertices $\iota(\Delta)$ and $\tau(\Delta)$. These are common vertices of top(Δ) and bot(Δ). The whole diagram is situated between the top and the bottom paths, and every edge of Δ belongs to a (directed) path in Δ between $\iota(\Delta)$ and $\tau(\Delta)$. More formally, let X be an alphabet. For every $x \in X$, we define the *trivial diagram* $\varepsilon(x)$ which is just an edge labeled by x. The top and bottom paths of $\varepsilon(x)$ are equal to $\varepsilon(x)$, $\iota(\varepsilon(x))$ and $\tau(\varepsilon(x))$ are the initial and terminal vertices of the edge. If u and v are words in X, a *cell* $(u \to v)$ is a planar graph consisting of two directed labeled paths, the top path labeled by u

and the bottom path labeled by v, connecting the same points $\iota(u \to v)$ and $\tau(u \to v)$. There are three operations that can be applied to diagrams in order to obtain new diagrams.

1. *Addition.* Given two diagrams Δ_1 and Δ_2, one can identify $\tau(\Delta_1)$ with $\iota(\Delta_2)$. The resulting planar graph is again a diagram denoted by $\Delta_1 + \Delta_2$, whose top (bottom) path is the concatenation of the top (bottom) paths of Δ_1 and Δ_2. If $u = x_1 x_2 \cdots x_n$ is a word in X, then we denote $\varepsilon(x_1) + \varepsilon(x_2) + \cdots + \varepsilon(x_n)$ (i.e., a simple path labeled by u) by $\varepsilon(u)$ and call this diagram also *trivial*.

2. *Multiplication.* If the label of the bottom path of Δ_2 coincides with the label of the top path of Δ_1, then we can *multiply* Δ_1 and Δ_2, identifying $\mathbf{bot}(\Delta_1)$ with $\mathbf{top}(\Delta_2)$. The new diagram is denoted by $\Delta_1 \circ \Delta_2$. The vertices $\iota(\Delta_1 \circ \Delta_2)$ and $\tau(\Delta_1 \circ \Delta_2)$ coincide with the corresponding vertices of Δ_1, Δ_2, $\mathbf{top}(\Delta_1 \circ \Delta_2) = \mathbf{top}(\Delta_1), \mathbf{bot}(\Delta_1 \circ \Delta_2) = \mathbf{bot}(\Delta_2)$.

$$\Delta_1 \circ \Delta_2 \qquad\qquad\qquad \Delta_1 + \Delta_2$$

3. *Inversion.* Given a diagram Δ, we can flip it about a horizontal line obtaining a new diagram Δ^{-1} whose top (bottom) path coincides with the bottom (top) path of Δ. Note that a cell can have the form $u \to u$ (i.e., v and u can coincide). In this case, in order to distinguish the top from the bottom, we draw a vertical arrow inside the cell from the top to the bottom. If the inverse cell $(u \to u)^{-1}$, then the arrow points in the opposite direction. So the inverse cell is not the same as the cell itself.
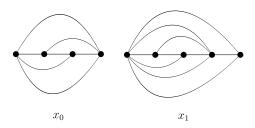
DEFINITION 4.1. A diagram over a collection of cells P is any planar graph obtained from the trivial diagrams and cells of P by the operations of addition, multiplication and inversion. If the top path of a diagram Δ is labeled by a word u and the bottom path is labeled by a word v, then we call Δ a (u, v)-diagram over P.

Two cells in a diagram form a *dipole* if the bottom part of the first cell coincides with the top part of the second cell, and the cells are inverses of each other. In this case, we can obtain a new diagram removing the two cells and replacing them by the top path of the first cell. This operation is called *elimination of dipoles*. The new diagram is called *equivalent* to the initial one. A diagram is called *reduced* if it does not contain dipoles. It is proved in [GS1, Theorem 3.17] that every diagram is equivalent to a unique reduced diagram.

Now let $P = \{c_1, c_2, \ldots\}$ be a collection of cells. We can view it either as a semigroup presentation as in [GS1] or as the collection of cells of a directed 2-complex obtained by identifying all edges of the cells with the same label as in [GS3]. We shall use the first point of view here. If the top and the bottom paths of a diagram are labeled by the same word u, we call it a *spherical* (u, u)-*diagram*. The diagram group $\mathcal{D}(P, u)$ corresponding to the collection of cells P and a word u consists of all reduced spherical (u, u)-diagrams obtained from these cells and trivial diagrams by using the three operations mentioned above. The product $\Delta_1 \Delta_2$ of two diagrams Δ_1 and Δ_2 is the reduced diagram obtained by removing all dipoles from $\Delta_1 \circ \Delta_2$. The fact that $\mathcal{D}(P, u)$ is a group is proved in [GS1].

Examples: (Both examples can be found in [GS1].) If X consists of one letter x and P consists of one cell $x \to x^2$, then the group $\mathcal{D}(P, x)$ is the R. Thompson group F. If X consists of three letters a, b, c and P consists of three cells $ab \to a, b \to b, bc \to c$, then the diagram group $\mathcal{D}(P, ac)$ is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$ [GS4].

Here are the diagrams representing the two standard generators x_0, x_1 of the R. Thompson group F. All edges are labeled by x and oriented from left to right, so we omit the labels and orientation of edges.

$$x_0 \qquad\qquad\qquad x_1$$

It is easy to represent, say, x_0 as a product of sums of cells and trivial diagrams:

$$x_0 = (x \to x^2) \circ \big(\varepsilon(x) + (x \to x^2)\big) \circ \big((x \to x^2)^{-1} + \varepsilon(x)\big) \circ \big((x \to x^2)^{-1}\big).$$

There is a natural *diagram metric* on every diagram group $\mathcal{D}(P, u)$: $\mathrm{dist}(\Delta, \Delta')$ is the number of cells in the diagram $\Delta^{-1}\Delta'$. For some finitely generated diagram groups (such as F or $\mathbb{Z} \wr \mathbb{Z}$) this metric is quasi-isometric to the word metric [AGS]. In this case, we say that the group satisfies property B (after J. Burillo). We do not know whether every finitely generated diagram group satisfies B (see [AGS, Question 1.5]).

We shall need the following description of the conjugacy relation in diagram groups.

Let Q be a collection of cells. A spherical (u, u)-diagram Δ over Q is called *absolutely reduced* if all its $\circ$-powers $\Delta \circ \Delta \circ \cdots \circ \Delta$ are reduced. Every absolutely reduced diagram Δ is canonically decomposed as a sum $\Delta_1 + \Delta_2 +$

$\cdots + \Delta_n$ where each Δ_i is either a trivial diagram $\varepsilon(u_i)$ or a spherical (u_i, u_i)-diagram which is further indecomposable as a sum of spherical diagrams. Indecomposable absolutely reduced spherical diagrams are called *simple*.

THEOREM 4.2 (see [GS1, Lemmas 15.14, 15.15, 15.20]). (i) *Every spherical* (u, u)-*diagram is conjugate to an absolutely reduced spherical* (v, v)-*diagram.*

(ii) *Suppose that two absolutely reduced diagrams* A *and* B *have canonical decompositions* $A_1 + \cdots + A_m$ *and* $B_1 + \cdots + B_n$ *(where* A_i *is a* (u_i, u_i)-*diagram,* B_j *is a* (v_j, v_j)-*diagram). Suppose further that* A *and* B *are conjugate. Then* $m = n$, *and* A_i *is conjugate to* B_i, *that is* $A_i = \Gamma_i^{-1} B_i \Gamma_i$ *for some* (v_i, u_i)-*diagram* Γ_i, $i = 1, \ldots, m$.

(iii) *If two simple diagrams* A, B *are conjugate, then they have the same number of cells. Two trivial diagrams* $\varepsilon(u)$ *and* $\varepsilon(v)$ *are conjugate, if and only if* $u = v$ *modulo the semigroup presentation* P.

CONJECTURE 4.3. The following conditions for a finitely generated diagram group G are equivalent:

(1) G contains a non-Abelian free subsemigroup.
(2) The growth function of G is exponential.
(3) The conjugacy growth function of G is exponential.

It is clear that $(3) \to (2)$, $(1) \to (2)$. The implication $(2) \to (1)$ (and hence equivalence of these two conditions) is proved as follows. Let G be a finitely generated diagram group of exponential growth. By [GS5, Theorem 5.7], G is embedded into a (split) extension of a right angled Artin group A by the Thompson group F. A subgroup of a right angled Artin group is either Abelian, or contains a non-Abelian free subgroup [GS5, Lemma 7.6]. Hence, we can assume that $G \cap A$ is Abelian. A subgroup of F either contains a copy of $\mathbb{Z} \wr \mathbb{Z}$ or is Abelian [GS2, Theorem 21]. Since $\mathbb{Z} \wr \mathbb{Z}$ contains a non-Abelian free semigroup, we can assume that the the subgroup $G/(G \cap A)$ of F is Abelian. Hence, G is solvable of class at most 2. Since G has exponential growth, it is not virtually nilpotent. By the result of Milnor and and Wolf cited above, G contains a non-Abelian free subsemigroup.

The next theorem proves a weaker statement than the implication $(1) \to (3)$.

THEOREM 4.4. *Every finitely generated diagram group containing the wreath product* $\mathbb{Z} \wr \mathbb{Z}$ *(in particular, the Thompson group* F *[GS3]) has exponential conjugacy growth function.*

Proof. Consider first the group $\mathbb{Z} \wr \mathbb{Z}$ itself, that is the diagram group $\mathcal{D}(P, ac)$ for $P = \{ab \to a,\ b \to b,\ bc \to c\}$. Let π be the cell $b \to b$, and $n_0, \ldots, n_k$ be positive integers. Let $\Delta(n_0, \ldots, n_k)$ be the following diagram:

$$(4) \qquad \varepsilon(a) + \pi^{n_0} + \cdots + \pi^{n_k} + \varepsilon(c).$$

Let $n = n_0 + \cdots + n_k$, $\Gamma(n)$ be the diagram obtained as the following product $((ab,a)^{-1} + \varepsilon(c)) \circ ((ab,a)^{-1} + \varepsilon(bc)) \circ \cdots \circ ((ab,a)^{-1} + \varepsilon(b^k c))$. It is clearly an $(ac, ab^k c)$-diagram. Finally, let

$$A(n_0, \ldots, n_k) = \Gamma(n)\Delta(n_0, \ldots, n_k)\Gamma(n)^{-1}.$$

This is a spherical (ac, ac)-diagram from $\mathcal{D}(P, ac)$.

Note that each component π^{n_i} is absolutely reduced, hence simple. Thus, the decomposition (4) is the canonical decomposition of the absolutely reduced diagram $\Delta(n_0, \ldots, n_k)$. Since $A(n_0, \ldots, n_k)$ is a conjugate of $\Delta(n_0, \ldots, n_k)$, two diagrams $A(n_0, \ldots, n_k)$ and $A(n_0', \ldots, n_l')$ are conjugate in $\mathcal{D}(Q, u)$ if and only if the diagrams $\Delta(n_0, \ldots, n_k)$ and $\Delta'(n_0', \ldots, n_l')$ are conjugate. By Theorem 4.2, then $k = l$ and $n_0 = n_0', \ldots, n_k = n_k'$. Note also that each diagram $A(n_0, \ldots, n_k)$ has $n + 2(k+1) \leq 3n$ cells, whence by property B, it belongs to the ball of radius $O(n)$ in $\mathbb{Z} \wr \mathbb{Z}$. The number of solutions $n = n_0 + \cdots + n_k$ in positive integers is equal to 2^{n-1} $(k \geq 0)$. Hence, the conjugacy growth of $\mathbb{Z} \wr \mathbb{Z}$ is exponential.

Now suppose that for some collection of cells (semigroup presentation) Q and some word u we have $\mathcal{D}(Q, u) \geq \mathbb{Z} \wr \mathbb{Z}$. We shall use the fact that $\mathbb{Z} \wr \mathbb{Z}$ is a *rigid* diagram group in terminology of [GS2, GS4]: by [GS2, Theorem 24] (for another formulation see [GS4, Section 10]), there exists an embedding Ψ of $\mathbb{Z} \wr \mathbb{Z}$ into $\mathcal{D}(Q, u)$ *induced* by a $(\psi(ac), u)$-diagram Γ, and a map ψ that takes letters a, b, c to words $\psi(a), \psi(b), \psi(c)$ over the alphabet of Q, and each of the three cells $x \to y$ of P to a nontrivial $(\psi(x), \psi(y))$-diagram $\psi(x \to y)$ over Q. The map Ψ takes each (ac, ac)-diagram Δ of $\mathcal{D}(P, ac)$ to the diagram $\Gamma^{-1}\psi(\Delta)\Gamma$ where $\psi(\Delta)$ is obtained from Δ by replacing every edge $\varepsilon(e)$ by the path $\varepsilon(\psi(e))$ and every cell π by the diagram $\psi(\pi)$.

Note that the number of cells in $\Psi(\Delta)$ is big-O of the number of cells in Δ. Hence if, as above, we let $n = n_0 + \cdots + n_k$, then by property B, the diagram $\Psi(A(n_0, \ldots, n_k))$ is in the ball of radius $O(n)$. Also note that $\Psi(A(n_0, \ldots, n_k))$ is conjugate to the diagram $\varepsilon(\psi(a)) + \psi(\pi)^{n_0} + \cdots + \psi(\pi)^{n_0} + \varepsilon(\psi(c))$. By Theorem 4.2, part (i), $\psi(\pi)$ is conjugate to an absolutely reduced diagram that is a sum of simple or trivial diagrams $\Pi_1 + \cdots + \Pi_s$, and the number of simple diagrams among Π_i is not zero. Therefore, $\Psi(A(n_0, \ldots, n_k))$ is conjugate to the diagram

$$\varepsilon(\psi(a)) + (\Pi_1^{n_0} + \cdots + \Pi_s^{n_0}) + \cdots + (\Pi_1^{n_k} + \cdots + \Pi_s^{n_k}) + \varepsilon(\psi(c)).$$

By Theorem 4.2, parts (ii) and (iii), different diagrams of the form $\Psi(A(n_0, \ldots, n_k))$ are not conjugate in $\mathcal{D}(Q, u)$. This proves that the conjugacy growth of $\mathcal{D}(Q, u)$ is exponential. $\qquad\square$

5. Groups acting on simplicial trees

Groups acting nontrivially on simplicial trees form a large and well studied class of groups. "Most" of these groups contain free noncyclic subgroups. It is

quite possible that some of these free subgroups embed as Frattini subgroups. So we expect their conjugacy growth to be exponential.

CONJECTURE 5.1. Suppose that G acts on a simplicial tree nontrivially and faithfully. Then the conjugacy growth function of G is exponential provided the growth function of G is exponential.

THEOREM 5.2. *Let G be the HNN-extension of a group H with associated subgroups A, B such that $AB \cup BA \neq H$. Then the conjugacy growth function of G is exponential.*

REMARK 5.3. Note that the Baumslag–Solitar group $BS(m, n) = \langle a, b \mid b^{-1}a^m b = a^n \rangle$, satisfies the conditions of Theorem 5.2 if and only if m, n are not co-prime. There is no doubt that the conclusion of the theorem remains true for every m, n except $m = n = 1$ (in which case $BS(m, n) = \mathbb{Z}^2$ and the conjugacy growth function is quadratic).
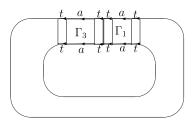
Proof. Let $a \in G \setminus (AB \cup BA)$, t be the stable letter of the HNN-extension. Consider the subsemigroup S generated by t, ta. Every word in S has the form

$$(5) \qquad\qquad t^{n_1} a t^{n_2} \cdots a t^{n_{k+1}}$$

were all $n_i \geq 0$, and $n_1, \ldots, n_k > 0$.

Taking a cyclic shift if necessary, we will assume that $n_{k+1} = 0$.

We fix the presentation of G which consists of all relations of H plus the conjugacy relations $ut = tv$ of the HNN-extension (here $u \in A, v \in B$). Suppose that two words of the form (5) are conjugate in G. Then there exists an annular (Schupp) diagram Δ for this conjugacy with the inner (outer) label of the form $t^{n_1} a t^{n_2} \cdots t^{n_k} a$ (resp. $t^{m_1} a t^{m_2} \cdots t^{m_l} a$) (see [MS]). We can assume that Δ has minimal possible number of cells. Since G is an HNN-extension, following Miller and Schupp [MS], we can consider the t-bands, that is sequences of cells corresponding to the HNN-relations such that every two consecutive cells have a t-edge in common. Since our HNN-relations include all relations of the form $ut = tv$, $u \in A, v \in B$, and since Δ has minimal possible number of cells, every t-band must have length 1 or 0. It is proved in [MS], that Δ cannot contain t-annuli (i.e., t-bands with same first and last t-edges). Also a t-band cannot connect two t-edges on the same boundary component of Δ because all $n_i \geq 0$. Hence, every t-band connects a t-edge of the inner boundary of Δ with a t-edge on the outer boundary of Δ. Therefore, the t-bands subdivide Δ into a number of disc van Kampen diagrams $\Gamma_1, \ldots, \Gamma_s$ without HNN-cells, hence without t-edges. Thus, each Γ_i is a diagram over H, and so it consists of one cell corresponding to a relation of H. Note that the boundary of Γ_i is of the form $p_1(\Gamma_i)^{-1} q_1(\Gamma_i) p_2(\Gamma_i) q_2(\Gamma_i)$ (read counterclockwise) where $p_1(\Gamma_i)$ is a subpath of the inner boundary of Δ, $p_2(\Gamma_i)$ is a subpath of the outer boundary of Δ, $q_1(\Gamma_i)$ and $q_2(\Gamma_i)$ are sides of two consecutive t-bands.

Since $p_j(\Gamma_i)$ does not contain t-edges, we obtain that either $p_j(\Gamma_i)$ is empty or it is an a-edge, $j = 1, 2$. Since all $n_1, \ldots, n_k$ are nonnegative, we have that the label u_i of the path $q_1(\Gamma_i)$ is from A, the label v_i of the path $q_2(\Gamma_i)$ is from B. Hence, each diagram Γ_i corresponds to a relation of H of the form $a^{-\varepsilon_i} u_i a^{\delta_i} v_i$ where $\varepsilon_i, \delta_i \in \{0, 1\}$. Since $a \notin AB \cup BA$, we conclude that for every i, $\varepsilon_i = \delta_i$. Therefore, the t-bands of Δ provide a correspondence between letters of the words written on the inner and outer boundaries of Δ. This correspondence shows that these two words are cyclic shifts of each other, so $l = k$ and for some j we have $n_1 = m_j, n_2 = m_{j+1}, \ldots$ (addition in the indices is modulo k). Thus, we prove that two words of the form (5) are conjugate in G if and only if they are cyclic shifts of each other. Hence, the conjugacy growth function of G is exponential. $\qquad\square$

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