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THE REGULATED PRIMITIVE INTEGRAL

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ABSTRACT. A function on the real line is called regulated if it has a left limit and a right limit at each point. If f is a Schwartz distribution on the real line such that $f = F'$ (distributional or weak derivative) for a regulated function F then the regulated primitive integral of f is $\int_{(a,b)} f = F(b-) - F(a+)$, with similar definitions for other types of intervals. The space of integrable distributions is a Banach space and Banach lattice under the Alexiewicz norm. It contains the spaces of Lebesgue and Henstock–Kurzweil integrable functions as continuous embeddings. It is the completion of the space of signed Radon measures in the Alexiewicz norm. Functions of bounded variation form the dual space and the space of multipliers. The integrable distributions are a module over the functions of bounded variation. Properties such as integration by parts, change of variables, Hölder inequality, Taylor's theorem and convergence theorems are proved.

1. Introduction

One way of defining an integral is via its primitive. The primitive is a function whose derivative is in some sense equal to the integrand. For example, if f and F are functions on the real line and F is absolutely continuous (AC) such that $F'(x) = f(x)$ for almost all $x \in (a, b)$, then the Lebesgue integral of f is $\int_a^b f = F(b) - F(a)$. The same method can be used to define the Henstock–Kurzweil integral, for which $F \in ACG^*$. We get the wide Denjoy integral if $F \in ACG$ and we use the approximate derivative. See, for example, [25] for the definitions of these function spaces and the wide Denjoy integral. The Henstock–Kurzweil integral is equivalent to the Denjoy integral and is defined

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further in this introduction. Note that $C^1 \subsetneq AC \subsetneq ACG^* \subsetneq ACG \subsetneq C^0$. The symbols $\subset$ and $\supset$ allow set equality. If we use the distributional derivative, then the primitives need not have any pointwise differentiation properties. See [30] for an integral with continuous functions as primitives. In this paper, we will describe an integral whose primitives are regulated functions. This integral will contain all of the integrals above. As well, it can integrate signed Radon measures.

We will now describe the space of primitives for the regulated primitive integral. Function $F:\mathbb{R}\to\mathbb{R}$ is *regulated* if it has left and right limits at each point in $\mathbb{R}$. For $a\in\mathbb{R}$ write $F(a-)=\lim_{x\to a^-}F(x)$, $F(a+)=\lim_{x\to a^+}F(x)$, $F(-\infty)=\lim_{x\to-\infty}F(x)$, $F(\infty)=\lim_{x\to\infty}F(x)$. Then F is left continuous at $a\in\mathbb{R}$ if $F(a)=F(a-)$ and right continuous if $F(a)=F(a+)$. We define $\mathcal{B}_R=\{F:\mathbb{R}\to\mathbb{R}\mid F \text{ is regulated and left continuous on } \mathbb{R}, F(-\infty)=0, F(\infty)\in\mathbb{R}\}$. Hence, elements of $\mathcal{B}_R$ are real-valued functions defined on the extended real line $\overline{\mathbb{R}}=[-\infty,\infty]$. It will be shown below that $\mathcal{B}_R$ is a Banach space under the uniform norm. The space of integrable distributions will be those distributions that are the distributional derivative of a function in $\mathcal{B}_R$. We will see that elementary properties of distributions can be used to prove that the set of integrable distributions is a Banach space isometrically isomorphic to $\mathcal{B}_R$. Most of the usual properties of integrals hold: fundamental theorem of calculus, addivity over intervals, integration by parts, change of variables, Hölder inequality, Taylor's theorem, convergence theorems. The multipliers and dual space are the functions of bounded variation. This defines a product that makes the integrable distributions into a module over the functions of bounded variation. There is a Banach lattice structure. The subspace of primitives of bounded variation corresponds to absolutely convergent integrals. Each integrable distribution is a finitely additive measure defined on the algebra of sets of bounded variation. We get a finite measure if and only if the convergence is absolute. These distributions are signed Radon measures. The regulated functions are continuous in the topology of half-open intervals. It is shown that the space of integrable distributions is the completion of the space of signed Radon measures in the Alexiewicz norm. See the paragraph preceding Theorem 4 for the definition. This embedding is continuous. Note, however, that the spaces of Lebesgue and Henstock–Kurzweil integrable functions are separable in the Alexiewicz norm topology while our space of integrable distributions is not separable. Hence, it is not the completion of these function spaces.

The integral in the present paper has several possible extensions to Euclidean spaces. In $\mathbb{R}^n$, geometrical considerations change the character of the integral. And there is the problem of which differential operator to invert. There are integrals associated with inverting the nth order distributional differential operator $\partial^n/\partial x_1\cdots\partial x_n$. For continuous primitives, this type of integral was introduced in [22] and developed systematically in [1]. At the other extreme, there are integrals that invert the first order distributional divergence operator. See [23]. If a set $S \subset \mathbb{R}^n$ has a normal vector at almost all points of its boundary then we can use this direction to define limits along the normal from within and without S. This then defines functions that are regulated on the boundary of this particular set S. The divergence theorem for sets of finite perimeter ([8], [35]) can then be used to define an integral over S for distributions that are the distributional derivative of a regulated function. If S is a hypercube in $\mathbb{R}^n$, then its boundary is a hypercube in $\mathbb{R}^{n-1}$ so that the divergence theorem in $\mathbb{R}^n$ yields the integral of [1]. Details will be published elsewhere.

If μ is a Borel measure on the real line, we use the notation $L^p(\mu)$ ($1 \le p < \infty$) for the functions $f : \mathbb{R} \to \mathbb{R}$ such that the Lebesgue integral $\int_{\mathbb{R}} |f|^p \, d\mu$ exists. For Lebesgue measure λ we write L^p.

To proceed further, we will fix some notation for distributions. The *test functions* are $\mathcal{D} = C_c^\infty(\mathbb{R})$, that is, the smooth functions with compact support. The *support* of a function ϕ is the closure of the set on which ϕ does not vanish. Denote this as $\operatorname{supp}(\phi)$. There is a notion of continuity in $\mathcal{D}$. If $\{\phi_n\} \subset \mathcal{D}$, then $\phi_n \to \phi \in \mathcal{D}$ if there is a compact set $K \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\operatorname{supp}(\phi_n) \subset K$, and for each integer $m \ge 0$, $\phi_n^{(m)} \to \phi^{(m)}$ uniformly on K as $n \to \infty$. The distributions are the continuous linear functionals on $\mathcal{D}$, denoted $\mathcal{D}'$. If $T \in \mathcal{D}'$, then $T : \mathcal{D} \to \mathbb{R}$ and we write $\langle T, \phi \rangle \in \mathbb{R}$ for $\phi \in \mathcal{D}$. If $\phi_n \to \phi$ in $\mathcal{D}$, then $\langle T, \phi_n \rangle \to \langle T, \phi \rangle$ in $\mathbb{R}$. And, for all $a_1, a_2 \in \mathbb{R}$ and all $\phi, \psi \in \mathcal{D}$, $\langle T, a_1\phi + a_2\psi \rangle = a_1\langle T, \phi \rangle + a_2\langle T, \psi \rangle$. If $f \in L^p_{\rm loc}$ for some $1 \le p \le \infty$, then $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f\phi$ defines a distribution. The differentiation formula $\langle T', \phi \rangle = -\langle T, \phi' \rangle$ ensures that all distributions have derivatives of all orders which are themselves distributions. This is known as the *distributional derivative* or *weak derivative*. The formula follows by mimicking integration by parts in the case of T_f where $f \in C^1$. We will usually denote distributional derivatives by F' and pointwise derivatives by $F'(t)$. For $T \in \mathcal{D}'$ and $t \in \mathbb{R}$ the *translation* τ_t is defined by $\langle \tau_t T, \phi \rangle = \langle T, \tau_{-t}\phi \rangle$ where $\tau_t\phi(x) = \phi(x - t)$ for $\phi \in \mathcal{D}$. Most of the results on distributions we use can be found in [10].

Schwartz [28] defined the integral of distribution T as $\langle T, 1 \rangle$, provided this exists. This agrees with $\int_{-\infty}^{\infty} f$ if $T = T_f$ for $f \in L^1$ (i.e., f is integrable with respect to Lebesgue measure). In Schwartz's definition, the integral is then a linear functional on the constant functions. We will see that, as a result of the Hölder inequality (Theorem 13), our integral can be viewed as a continuous linear functional on functions of bounded variation (Theorem 17). It then extends Schwartz's definition. Other methods of integrating distributions have been considered by Mikusiński, Musielak and Sikorski. See the last paragraph in Section 11 of [30] for references. As will be seen in Section 5 below, the integration by parts formula connects the regulated primitive integral with a type of Stieltjes integral that has been studied by Kurzweil, Schwabik and Tvrdý [31].

Denjoy's original constructive approach to a nonabsolute integral that integrated all pointwise and approximate derivatives was a type of transfinite induction applied to sequences of Lebesgue integrals. The method of ACG^* functions and ACG functions is due to Lusin. See [15] and [25]. In what follows, we define an integral that integrates weak derivatives of regulated functions. It also has a definition based on a previously defined integral; in this case from Riemann integrals on compact intervals.

Another approach to nonabsolute integration is through Riemann sums. The Henstock–Stieltjes integral is defined as follows ([21, Section 7.1] where it is called the gauge integral). A *gauge* on $\overline{\mathbb{R}}$ is a mapping γ from $\overline{\mathbb{R}}$ to the open intervals in $\overline{\mathbb{R}}$ (cf. Remark 3) with the property that for each $x\in\overline{\mathbb{R}}$, $\gamma(x)$ is an open interval containing x. Note that this requires $\gamma(\pm\infty)=\overline{\mathbb{R}}$ or $\gamma(-\infty)=[-\infty,a)$ or $\gamma(\infty)=(b,\infty]$ for some $a,b\in\mathbb{R}$. A *tagged partition* is a finite set of pairs of closed intervals and tag points in the extended real line, $\mathcal{P}=\{([x_{n-1},x_n],z_n)\}_{n=1}^N$ for some $N\in\mathbb{N}$ such that $z_n\in[x_{n-1},x_n]$ for each $1\leq n\leq N$ and $-\infty=x_0<x_1<x_2<\cdots<x_N=\infty$. In addition, $z_0=-\infty$ and $z_N=\infty$. Given a gauge, γ, the partition $\mathcal{P}$ is said to be γ-*fine* if $[x_{n-1},x_n]\subset\gamma(z_n)$ for each $1\leq n\leq N$. If $F,g:\mathbb{R}\to\mathbb{R}$ then F is integrable with respect to g if there is $A\in\mathbb{R}$ such that for all $\epsilon>0$ there is gauge γ such that for each γ-fine tagged partition we have $|\sum_{n=1}^N F(z_n)[g(x_n)-g(x_{n-1})]-A|<\epsilon$. We will use the Henstock–Stieltjes integral only for regulated functions F and g so we can use limits to define the values of these functions at $\pm\infty$. If $g(x)=x$, we have the Henstock–Kurzweil integral, that is, integration with respect to Lebesgue measure. In this case, we take $F(\pm\infty)=0$. For integration over a compact interval, a function is Riemann integrable if and only if the gauge γ can be taken to be constant, that is, $\gamma(x)=(x-\delta,x+\delta)$ for some constant $\delta>0$.

If function F has a pointwise derivative at each point in $[a,b]$, then the derivative is integrable in the Henstock–Kurzweil sense and $\int_a^b F'(x)\,dx=F(b)-F(a)$. In this sense, the Henstock–Kurzweil integral inverts the pointwise derivative operator. It is well known that the Riemann and Lebesgue integrals do not have this property. For details, see [21]. There are functions for which this fundamental theorem of calculus formula holds and yet these functions do not have a pointwise derivative at each point. In this sense, the Henstock–Kurzweil integral is not the inverse of the pointwise derivative. The C-integral of Bongiorno, Di Piazza and Priess is defined using Riemann sums and a modification of the gauge process above. A function has a C-integral if and only if it is everywhere the pointwise derivative of its primitive. See [4]. In this sense, the C-integral is the inverse of the pointwise derivative. The integral defined in the present paper inverts the distributional derivative but only for primitives that are regulated functions. The restriction to regulated primitives is useful as it leads to a Banach space of integrable distributions.

2. The regulated primitive integral

Define $\mathcal{A}_R = \{f \in \mathcal{D}' \mid f = F' \text{ for some } F \in \mathcal{B}_R\}$. A distribution f is integrable if it is the distributional derivative of some primitive $F \in \mathcal{B}_R$, that is, for all $\phi \in \mathcal{D}$ we have $\langle f, \phi\rangle = \langle F', \phi\rangle = -\langle F, \phi'\rangle = -\int_{-\infty}^{\infty} F(t)\phi'(t)\,dt$. Since F is regulated and ϕ is smooth with compact support, the last integral exists as a Riemann integral. We will use the following convention in labeling primitives of elements in $\mathcal{A}_R$.

Convention 1. When $f, g, f_1, \tilde{h}$, etc. are in $\mathcal{A}_R$ we will denote their respective primitives in $\mathcal{B}_R$ by $F, G, F_1, \tilde{H}$, etc.

It will be shown in Theorem 4 below that primitives are unique and that the spaces $\mathcal{A}_R$ and $\mathcal{B}_R$ are isometrically isomorphic, the integral constituting a linear isometry. If $f \in \mathcal{A}_R$ and $-\infty < a < b < \infty$, then

$$\int_{(a,b)} f = \int_{a+}^{b-} f = F(b-) - F(a+) = F(b) - F(a+), \tag{1}$$

$$\int_{(a,b]} f = \int_{a+}^{b+} f = F(b+) - F(a+), \tag{2}$$

$$\int_{[a,b)} f = \int_{a-}^{b-} f = F(b-) - F(a-) = F(b) - F(a), \tag{3}$$

$$\int_{[a,b]} f = \int_{a-}^{b+} f = F(b+) - F(a-) = F(b+) - F(a). \tag{4}$$

If F is continuous, then these four integrals agree. For $a = -\infty$ and $b = \infty$, we write these four integrals as $\int_{-\infty}^{\infty} f = F(\infty)$. We can also define $\int_{\{a\}} f = \int_{[a,a]} f = \int_{a-}^{a+} f = F(a+) - F(a-)$.

Elements of $\mathcal{B}_R$ are tempered distributions of order one, while elements of $\mathcal{A}_R$ are tempered distributions of order two. See [10] for the definitions.

3. Examples

(a) If $F \in AC$ and $F'(t) = f(t)$ for almost all $t \in \mathbb{R}$, then for $\phi \in \mathcal{D}$ we can integrate by parts to get

$$\begin{aligned}\langle F', \phi\rangle = -\langle F, \phi'\rangle &= -\int_{-\infty}^{\infty} F(t)\phi'(t)\,dt \\ &= \int_{-\infty}^{\infty} F'(t)\phi(t)\,dt - [F(t)\phi(t)]_{t=-\infty}^{\infty} \\ &= \int_{-\infty}^{\infty} f(t)\phi(t)\,dt = \langle f, \phi\rangle.\end{aligned}$$

Each of the integrals above is a Lebesgue integral. It then follows that if $F(-\infty)=0$ and $F(\infty)$ exists then f is Lebesgue integrable and $L^1 \subsetneq \mathcal{A}_R$. Similarly, the regulated primitive integral contains the Henstock–Kurzweil integral and wide Denjoy integral. See [5, pp. 33–34] for the integration by parts formula for these integrals.

(b) Suppose $F\in\mathcal{B}_R$ is continuous but differentiable nowhere. Then f defined by $f=F'\in\mathcal{A}_R$ and $\int_I f=F(b)-F(a)$ for $I=(a,b),(a,b],[a,b),[a,b]$ whenever $-\infty\leq a<b\leq\infty$. Note that for $\phi\in\mathcal{D}$ we have $\langle f,\phi\rangle=\langle F',\phi\rangle=-\langle F,\phi'\rangle=-\int_{-\infty}^{\infty}F(t)\phi'(t)\,dt$. This last integral exists in the Riemann sense.

(c) If $F:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $F'(x)=0$ for almost all $x\in\mathbb{R}$ then for all $[a,b]\subset\mathbb{R}$ the Lebesgue integral $\int_a^b F'(t)\,dt$ exists and is zero, while $\int_a^b F'=F(b)-F(a)$. An example of such a function F is the Cantor–Lebesgue function (devil's staircase).

(d) Let $\mathcal{BV}$ denote the functions of *bounded variation*, that is, functions F for which $VF:=\sup\sum|F(x_i)-F(y_i)|$ is bounded, where the supremum is taken over all disjoint intervals $\{(x_i,y_i)\}$. Note that if $F\in\mathcal{BV}$ then F is regulated and $F(\pm\infty)$ exist. Hence, $F'\in\mathcal{A}_R$. Although $F'(t)$ exists for almost all $t\in\mathbb{R}$, and the Lebesgue integral $\int_a^b F'(t)\,dt$ exists, it need not equal $F(b)-F(a)$. If $F\in L^1_{\mathrm{loc}}$, then the *essential variation* of F is $\operatorname{ess\,var}F:=\sup\int_{-\infty}^{\infty}F(t)\phi'(t)\,dt$ where the supremum is taken over all $\phi\in\mathcal{D}$ with $\|\phi\|_\infty\leq 1$. And, $\operatorname{ess\,var}F=\inf VG$ such that $F=G$ almost everywhere. The essential variation can also be computed by restricting the points x_i,y_i above to be points of approximate continuity of F. Denote the functions with bounded essential variation as $\mathcal{EBV}$. If $F\in\mathcal{EBV}$, then the distributional derivative of F is a signed Radon measure, that is, there is a signed Radon measure μ such that for all $\phi\in\mathcal{D}$ we have $\langle F',\phi\rangle=-\langle F,\phi'\rangle=\int_{-\infty}^{\infty}\phi(t)\,d\mu(t)$. Radon measures are Borel measures that are finite on compact sets, inner regular with respect to compact sets ($\mu(E)=\sup\mu(K)$ where E is a Borel set and the supremum is taken over all compact sets $K\subset E$) and outer regular with respect to open sets ($\mu(E)=\inf\mu(G)$ where E is a Borel set and the infimum is taken over all open sets $G\supset E$). In $\mathbb{R}$, the Radon measures are the Borel measures that are finite on compact sets. See, for example, [3, Section 26]. A signed Radon measure is then the difference of two finite Radon measures. If μ is a signed Radon measure, then F defined by $F(x)=\int_{(-\infty,x)}d\mu$ is a function of bounded variation. For, if $\{(x_i,y_i)\}$ are disjoint intervals then $\sum|F(x_i)-F(y_i)|=\sum|\int_{[x_i,y_i)}d\mu|\leq\sum\int_{[x_i,y_i)}d|\mu|\leq|\mu|(\mathbb{R})<\infty$. The regularity of μ shows $F\in\mathcal{B}_R$. Hence, each signed Radon measure is in $\mathcal{A}_R$. Since functions of bounded variation can have a pointwise derivative that vanishes almost everywhere, we cannot use a descriptive definition of the integral of a measure using the pointwise derivative of its primitive.

If ν is a Radon measure and $f\in L^1(\nu)$, then the set function μ defined by $\mu(E)=\int_E f\,d\nu$ is a signed Radon measure. Hence, $\mu\in\mathcal{A}_R$. If ν is absolutely continuous with respect to Lebesgue measure ($\nu \ll \lambda$) and $f \in L^1(\nu)$, then it follows from the Radon–Nikodým theorem that $f\, d\nu/d\lambda \in L^1 \subset \mathcal{A}_R$.

(e) A distribution T is said to be *positive* if $\langle T, \phi\rangle \geq 0$ whenever $\phi \in \mathcal{D}$ with $\phi \geq 0$. It is known that positive distributions correspond to Radon measures, that is, $T \in \mathcal{D}'$ is positive if and only if there is a Radon measure μ such that for all $\phi \in \mathcal{D}$ we have $\langle T, \phi\rangle = \int_{-\infty}^{\infty} \phi(t)\, d\mu(t)$. For example, [34, p. 17]. An example of a positive distribution in $\mathcal{B}_R$ is the Dirac distribution. Define the Heaviside step function by $H_1(x) = 0$ for $x \leq 0$ and $H_1(x) = 1$ for $x > 0$. The Dirac distribution is then given by $\langle \delta, \phi\rangle = \phi(0)$ ($\phi \in \mathcal{D}$). And, $\langle H_1', \phi\rangle = -\int_0^{\infty} \phi'(t)\, dt = \phi(0)$ so $H_1' = \delta$, $H_1 \in \mathcal{B}_R$, $\delta \in \mathcal{A}_R$. We have $\int_{(0,1)} \delta = \int_{(0,1)} H_1' = H_1(1-) - H_1(0+) = 1 - 1 = 0$ while $\int_{[0,1)} \delta = H_1(1-) - H_1(0-) = 1 - 0 = 1$. Define $H_2(x) = 0$ for $x < 0$ and $H_2(x) = 1$ for $x \geq 0$. In $\mathcal{D}$, $H_1 = H_2$ and $H_1' = H_2' = \delta$. Note that $H_2 \notin \mathcal{B}_R$ but $\int_I H_2' = \int_I H_1'$ for every interval $I \subset \mathbb{R}$. This discrepancy in $\mathcal{B}_R$ is discussed in Remark 5 below. Note also that δ is a Radon measure defined by $\delta(E) = \chi_E(0)$ for all $E \subset \mathbb{R}$. And, $\int_{\{0\}} \delta = 1$.

(f) If $\{a_k\}$ is a sequence in $\mathbb{R}$ such that $\sum_1^{\infty} a_k$ converges (absolutely or conditionally), then we can define a function $F : [0, \infty) \to \mathbb{R}$ by $F(x) = \sum_1^n a_k$ if $x \in (n, n+1]$ for some $n \in \mathbb{N}$ and $F(x) = 0$ if $x \leq 1$. Then F is regulated, left continuous, $F(0) = 0$ and $F(\infty) = \sum_1^{\infty} a_k$. We have $F' = f$ where $f \in \mathcal{A}_R$ is the distribution $f = \sum_1^{\infty} a_k(\tau_k \delta)$. (See the Introduction for the definition of translation.) This gives $\int_{[1,N]} f = \sum_1^N a_k$ for each $N \in \mathbb{N}$ and for $N = \infty$. Hence, integration in $\mathcal{A}_R$ includes series.

(g) Some finitely additive measures are also in $\mathcal{A}_R$. For example, if $f(t) = \sin(t^2)$ define $F(x) := \int_{-\infty}^x f(t)\, dt$. Then $F(I) = F(b) - F(a)$ for interval I with endpoints $-\infty \leq a < b \leq \infty$ defines a finitely additive measure on the algebra generated by intervals on the real line. And, $F \in \mathcal{B}_R$ with $F' = T_f$. Since the integral converges conditionally, F is not countably additive. Thus, $\sum_0^{\infty} F([\sqrt{2n\pi}, \sqrt{(2n+1)\pi})) = \infty$ while $\sum_1^{\infty} F([\sqrt{(2n-1)\pi}, \sqrt{2n\pi})) = -\infty$ but $F([0,\infty)) = \int_0^{\infty} \sin(t^2)\, dt = \sqrt{\pi}/2^{3/2}$. A similar example is obtained with $f(t) = (d/dt)[t^2 \sin(t^{-4})]$.

(h) If $F : \overline{\mathbb{R}} \to \mathbb{R}$ is any function then the Riemann–Stieltjes integral $\int_a^b dF = F(b) - F(a)$ exists for all $(a, b) \subset \mathbb{R}$. The Riemann–Stieltjes integral then contains the regulated primitive integral. We will see that $\mathcal{A}_R$ is a useful restriction since it is a Banach space. Below it will be shown that we can define $\int_{-\infty}^{\infty} f\, dg$ under more general conditions than can be done for the Riemann–Stieltjes or Lebesgue–Stieltjes integrals.

4. Properties of the integral

First, we have some properties of our space of primitives.

THEOREM 2 (Properties of $\mathcal{B}_R$). (a) *If $F\in\mathcal{B}_R$ then it is uniformly regulated, i.e., for each $\epsilon>0$ there exists $\delta>0$ such that for each $x\in\mathbb{R}$, if $y\in(x-\delta,x)$ then $|F(x-)-F(y)|<\epsilon$ and if $y\in(x,x+\delta)$ then $|F(x+)-F(y)|<\epsilon$. If $y<1/\delta$ then $|F(y)|<\epsilon$. And, if $y>1/\delta$ then $|F(\infty)-F(y)|<\epsilon$. Similarly, F is uniformly left continuous.* (b) *If $F\in\mathcal{B}_R$, then F is bounded and has at most a countable number of discontinuities.* (c) *Using pointwise operations, $\mathcal{B}_R$ is a Banach space under the uniform norm: $\|F\|_\infty=\sup_{x\in\mathbb{R}}|F(x)|$, for $F\in\mathcal{B}_R$.* (d) *$\mathcal{B}_R$ is not separable.*

Proof. (a) Let $\epsilon>0$. There is $\alpha<0$ such that if $y\leq\alpha$ then $|F(y)|<\epsilon$. For each $x\in\mathbb{R}$, there is $\eta_x>0$ such that if $y\in(x-\eta_x,x]$ then $|F(y)-F(x)|<\epsilon$. There is $\gamma_x>0$ such if $y\in(x,x+\gamma_x)$ then $|F(y)-F(x+)|<\epsilon$. There is $\beta>0$ such that if $y\geq\beta$ then $|F(y)-F(\infty)|<\epsilon$. Let $\zeta_x=\min(\eta_x,\gamma_x)$. The family of open intervals $\{(x-\zeta_x,x+\zeta_x)\}_{x\in\mathbb{R}}$ forms an open cover of the compact interval $[\alpha,\beta]$. There is then a finite index set $J\subset\mathbb{R}$ such that $\{(x-\zeta_x,x+\zeta_x)\}_{x\in J}$ is again an open cover of $[\alpha,\beta]$. Now let $\delta=\min(-1/\alpha,1/\beta,\min_{x\in J}\zeta_x)$. Since $\delta>0$, this shows F is uniformly regulated and uniformly left continuous.

(b) In (a) let $\epsilon=1$. Then

$$|F(x)|\leq 1+\max\Big(|F(\alpha)|,\max_{x\in J}\big(|F(x)|,|F(x+)|\big),|F(\beta)|\Big)<\infty.$$

See [21, p. 225] for a proof that there are at most countably many points of discontinuity.

(c) By (b), if $F\in\mathcal{B}_R$ then F is bounded and measurable. To prove $\mathcal{B}_R$ is a Banach space, first note it is a linear subspace of $L^\infty(\mathbb{R})$ since $\mathcal{B}_R$ is clearly closed under linear combinations. And, if $F\in\mathcal{B}_R$ such that $\|F\|_\infty=0$ then $F(x)=0$ for almost all $x\in\mathbb{R}$. But F is left continuous so if there were $b\in\mathbb{R}$ such that $F(b)>0$ then there is an interval $(a,b]$ on which F is positive, which is a contradiction, so $F(x)=0$ for all $x\in\mathbb{R}$. Positivity, homogeneity and the triangle inequality are inherited from $L^\infty(\mathbb{R})$. To show $\mathcal{B}_R$ is complete, suppose $\{F_n\}$ is a Cauchy sequence in $\mathcal{B}_R$. Then $\{F_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$ so there is $F\in L^\infty(\mathbb{R})$ such that $\|F-F_n\|_\infty\to 0$. To show F is left continuous, suppose $a\in\mathbb{R}$. For $x<a$ and $n\in\mathbb{N}$,

$$\begin{aligned}(5)\quad |F(a)-F(x)|&\leq|F(a)-F_n(a)|+|F_n(a)-F_n(x)|+|F_n(x)-F(x)|\\&\leq 2\|F-F_n\|_\infty+|F_n(a)-F_n(x)|.\end{aligned}$$

Given $\epsilon>0$, fix n large enough so that $\|F-F_n\|_\infty<\epsilon/3$. Then let $x\to a-$. Hence, F is left continuous on $\mathbb{R}$. Using $|F(a)|\leq\|F-F_n\|_\infty+|F_n(a)|$, we see that $F(-\infty)=0$. We can see that F has a right limit at $a\in\mathbb{R}$ by taking $x,y>a$ and letting $x,y\to a+$ in $|F(x)-F(y)|\leq 2\|F-F_n\|_\infty+\|F_n(x)-F_n(y)|$. And, $F(\infty)$ is seen to exist by letting $x,y\to\infty$ in this inequality. Therefore, $F\in\mathcal{B}_R$ and the space is complete.

(d) To see that $\mathcal{B}_R$ is not separable, consider the family of translations $\{\tau_tH_1\mid t\in\mathbb{R}\}$. The function H_1 is defined in Example 3(e). Given $0<\epsilon<1/2$, for each $t \in \mathbb{R}$ a dense subset of $\mathcal{B}_R$ would have to contain a function F_t with $|F_t| < \epsilon$ on $(-\infty, t]$ and $|F_t - 1| < \epsilon$ on (t, ∞). Hence, no such dense set can be countable. □

Further properties of regulated functions can be found in [9] and [14].

REMARK 3. Note that the construction in (a) gives a compactification of $\mathbb{R}$. A topological base for $\overline{\mathbb{R}}$ consists of the usual open intervals (a, b) with $-\infty \leq a < b \leq \infty$, as well as $[-\infty, a)$ with $-\infty < a \leq \infty$, and $(a, \infty]$ with $-\infty \leq a < \infty$. This makes $\overline{\mathbb{R}}$ into a compact Hausdorff space. A different topology is introduced in Section 10, under which all functions in $\mathcal{B}_R$ are continuous.

We now present some of the basic properties of the integral. One of the main results is that $\mathcal{A}_R$ is a Banach space under the Alexiewicz norm. For $f \in \mathcal{A}_R$, this is defined as $\|f\| = \|F\|_\infty$ where, as usual, F is the unique primitive in $\mathcal{B}_R$ (Convention 1). Linear combinations are defined by $\langle a_1 f_1 + a_2 f_2, \phi\rangle = \langle a_1 F_1' + a_2 F_2', \phi\rangle$ for $\phi \in \mathcal{D}$; $a_1, a_2 \in \mathbb{R}$; $f_1, f_2 \in \mathcal{A}_R$ with primitives $F_1, F_2 \in \mathcal{B}_R$.

THEOREM 4 (Basic properties of the integral). (a) *The integral is unique.* (b) *Addivity over intervals. If $f \in \mathcal{A}_R$, then for all $-\infty \leq a < b < c < \infty$ we have $\int_{(a,b]} f + \int_{(b,c]} f = \int_{(a,c]} f$. There are similar formulas for other intervals.* (c) *With the Alexiewicz norm, $\mathcal{A}_R$ is a Banach space. The integral provides a linear isometry and isomorphism between $\mathcal{A}_R$ and $\mathcal{B}_R$.* (d) *$\mathcal{A}_R$ is not separable.* (e) *Linearity. If $f_1, f_2 \in \mathcal{A}_R$ and $a_1, a_2 \in \mathbb{R}$, then $a_1 f_1 + a_2 f_2 \in \mathcal{A}_R$ and $\int_{-\infty}^{\infty}(a_1 f_1 + a_2 f_2) = a_1 \int_{-\infty}^{\infty} f_1 + a_2 \int_{-\infty}^{\infty} f_2$.* (f) *Reverse limits of integration. Let $-\infty \leq a_1 < a_2 \leq \infty$ and $\epsilon_1, \epsilon_2 \in \{+, -\}$. Then $\int_{a_1\epsilon_1}^{a_2\epsilon_2} f = -\int_{a_2\epsilon_2}^{a_1\epsilon_1} f$. If $a_1 = -\infty$, then we don't need ϵ_1 and if $a_2 = \infty$ then we don't need ϵ_2.*

Proof. (a) To prove the integral is unique we need to prove primitives in $\mathcal{B}_R$ are unique. Suppose $F, G \in \mathcal{B}_R$ and $F' = G'$. Then $(F - G)' = 0$ and the only solutions of this distributional differential equation are the constant distributions [10, Section 2.4]. The only constant distribution in $\mathcal{B}_R$ is the zero function.

(b) Note that $[F(b+) - F(a+)] + [F(c+) - F(b+)] = F(c+) - F(a+)$.

(c) Linearity of the distributional derivative shows $\mathcal{A}_R$ is a linear subspace of $\mathcal{D}'$. To prove $\|\cdot\|$ is a norm, let $f, g \in \mathcal{A}_R$.

(i) By uniqueness of the primitive, $\|0\| = \|0\|_\infty = 0$. If $\|f\| = 0$, then $\|F\|_\infty = \sup_{x\in\mathbb{R}} |F(x)| = 0$ so $F(x) = 0$ for all $x \in \mathbb{R}$ and therefore $f = F' = 0$.

(ii) Let $k \in \mathbb{R}$. Then $(kF)' = kF'$ so $\|kf\| = \|kF\|_\infty = |k|\|F\|_\infty = |k|\|f\|$.

(iii) Since $f + g = F' + G' = (F + G)'$ we have $\|f + g\| = \|F + G\|_\infty \leq \|F\|_\infty + \|G\|_\infty = \|f\| + \|g\|$.

This shows $\mathcal{A}_R$ is a normed linear space. To prove it is complete, suppose $\{f_n\}$ is a Cauchy sequence in $\mathcal{A}_R$. Then $\|F_n - F_m\|_\infty = \|f_n - f_m\|$ so $\{F_n\}$ is a Cauchy sequence in $\mathcal{B}_R$. There is $F \in \mathcal{B}_R$ such that $\|F_n - F\|_\infty \to 0$. And then $\|f_n - F'\| = \|F_n - F\|_\infty \to 0$. Since $F \in \mathcal{B}_R$, we have $F' \in \mathcal{A}_R$ and $\mathcal{A}_R$ is complete.

A linear bijection $\psi : \mathcal{A}_R \to \mathcal{B}_R$ is given by $\psi(f) = F$ where $f \in \mathcal{A}_R$ and F is its unique primitive in $\mathcal{B}_R$. Since the integral is linear, so is ψ. It is an isometry because $\|f\| = \|F\|_\infty = \|\psi(f)\|_\infty$.

(d) To show that $\mathcal{A}_R$ is not separable, consider the set $\{\tau_t\delta \mid t \in \mathbb{R}\}$. Now proceed as in the proof of Theorem 2(d).

(e) Since $a_1 f_1 + a_2 f_2 = (a_1 F_1 + a_2 F_2)'$, we have $\int_{-\infty}^{\infty}(a_1 f_1 + a_2 f_2) = (a_1 F_1 + a_2 F_2)(\infty) = a_1 F_1(\infty) + a_2 F_2(\infty) = a_1 \int_{-\infty}^{\infty} f_1 + a_2 \int_{-\infty}^{\infty} f_2$.

(f) $\int_{a_1\epsilon_1}^{a_2\epsilon_2} f = F(a_2\epsilon_2) - F(a_1\epsilon_1) = -[F(a_1\epsilon_1) - F(a_2\epsilon_2)] = -\int_{a_2\epsilon_2}^{a_1\epsilon_1} f$. □

No space of integrable functions or distributions for which primitives are continuous can be dense in $\mathcal{A}_R$. If G is a continuous primitive, then $\|G' - H_1'\| = \|G - H_1\|_\infty \geq 1/2$. Thus, L^1 is not dense in $\mathcal{A}_R$, nor are the spaces of Henstock–Kurzweil or wide Denjoy integrable functions. The completion of these spaces in the Alexiewicz norm is the Banach space $\mathcal{A}_C = \{f \in \mathcal{D}' \mid f = F'$ for some $F \in \mathcal{B}_C\}$, where $\mathcal{B}_C = \{F \in C^0(\overline{\mathbb{R}}) \mid F(-\infty) = 0$ and $F(\infty) \in \mathbb{R}\}$. If $f \in \mathcal{A}_C$ and $F' = f$ where F is its unique primitive in $\mathcal{B}_C$, then the *continuous primitive integral* of f is $\int_a^b f = F(b) - F(a)$. This integral is discussed in [30], where further references can also be found. Note that the spaces of Henstock–Kurzweil and wide Denjoy integrable functions are barrelled but not complete under the Alexiewicz norm.

REMARK 5. In defining $\mathcal{B}_R$, we have chosen the primitives to be left continuous. This is convenient but somewhat arbitrary. If two regulated functions have the same left and right limit at each point, then the functions can be different on a countable set but will still define the same distribution and thus have the same distributional derivative. This does not affect the integral since it only depends on limits at endpoints of an interval and not on the value of the primitive at the endpoints. It is clear that an equivalence relation between such primitives could be established, namely, $F \equiv G$ if and only if $F(x-) = G(x-)$ for all $-\infty < x \leq \infty$ and $F(x+) = G(x+)$ for all $-\infty \leq x < \infty$. An advantage of using left continuous functions rather than just regulated functions is that the norm on $\mathcal{B}_R$ can be taken as $\|F\|_\infty = \sup_{x\in\mathbb{R}} |F(x)|$ rather than essential supremum. This choice also affects the lattice operations in Section 9. Other obvious conventions are to take primitives that are right continuous or for which $F(x) = [F(x-) + F(x+)]/2$. As pointed out in [16], any normalising condition $F(x) = (1-\lambda)F(x-) + \lambda F(x+)$ suffices for fixed $0 \leq \lambda \leq 1$. In Lebesgue and Henstock–Kurzweil integration, we have equivalence classes of functions that agree almost everywhere. In $\mathcal{A}_R$ there are no such equivalence classes, for two distributions are equal if they agree on all test functions. For example, if $f,g \in L^1_{\rm loc}$ and $f = g$ almost everywhere then $T_f = T_g$.

Our definition of the integral builds in half of the fundamental theorem of calculus. The other half follows easily from uniqueness.

THEOREM 6 (Fundamental theorem of calculus). (a) *Let $f \in \mathcal{A}_R$. Define $G_1(x) = \int_{(-\infty,x)} f$. Then $G_1 = F$ on $\mathbb{R}$ and $G_1' = f$ in $\mathcal{D}$. Define $G_2(x) = \int_{(-\infty,x]} f$. Then G_2 is right continuous, $G_2(-\infty) = 0$, $G_2(\infty)$ exists and $G_2' = f$.* (b) *Let G be a regulated function with limits at $\pm\infty$. Then $G' \in \mathcal{A}_R$ and, for all $x \in \mathbb{R}$, $\int_{(-\infty,x)} G' = G(x-) - G(-\infty)$ and $\int_{(-\infty,x]} G' = G(x+) - G(-\infty)$.*

Proof. (a) Since $f \in \mathcal{A}_R$ there is a unique function $F \in \mathcal{B}_R$ such that $F' = f$ and $G_1(x) = \int_{(-\infty,x)} f = F(x-) = F(x)$ for all $x \in \mathbb{R}$. For $x \in \mathbb{R}$, we have $G_2(x) = \int_{(-\infty,x]} f = F(x+)$. It follows that G_2 is right continuous. And, $\lim_{x\to-\infty} G_2(x) = \lim_{x\to-\infty} F(x+) = F(-\infty) = 0$. As well, $\lim_{x\to\infty} G_2(x) = \lim_{x\to\infty} F(x+) = F(\infty)$. Therefore, $G_2 = F$ except perhaps on a countable set. They then define the same distribution and $G_2' = F' = f$.

(b) Define $F_1(x) = G(x-) - G(-\infty)$. Then $F_1 \in \mathcal{B}_R$ and $F_1' = G'$ so $G' \in \mathcal{A}_R$. Since $G((x-)-) = G(x-)$, we have $\int_{(-\infty,x)} G' = F_1(x-) = G(x-) - G(-\infty)$. As well, $G((x-)+) = G(x+)$ so $\int_{(-\infty,x]} G' = F_1(x+) = G(x+) - G(-\infty)$. □

As with the Henstock–Kurzweil integral, there are no improper integrals.

THEOREM 7 (Hake theorem). *Suppose $f \in \mathcal{D}'$ and $f = F'$ for some regulated function F. If $F(-\infty)$ and $F(\infty)$ exist in $\mathbb{R}$, then $f \in \mathcal{A}_R$ and $\int_{-\infty}^{\infty} f = \lim_{x\to\infty} \int_{(0,x)} f + \lim_{x\to-\infty} \int_{(x,0]} f = F(\infty) - F(-\infty)$.*

There are similar versions of this theorem on compact intervals.

If T is a distribution and $G : \mathbb{R} \to \mathbb{R}$ is an increasing C^∞ bijection, then for test function ϕ define $\psi \in \mathcal{D}$ by $\psi = (\phi \circ G^{-1})/(G' \circ G^{-1})$. The composition $T \circ G \in \mathcal{D}'$ is then defined by $\langle T \circ G, \phi\rangle = \langle T, \psi\rangle$. This follows from the change of variables formula for integration of smooth functions. See [10, Section 7.1].

For the case at hand, we can reduce the requirements on G. If $F \in \mathcal{B}_R$ and the real line can be partitioned into a finite number of intervals, on each of which G is monotonic (i.e., G is piecewise monotonic), then $F \circ G$ is regulated so we can integrate its derivative. This gives a change of variables formula.

THEOREM 8 (Change of variables). (a) *Let $F \in \mathcal{B}_R$. For each point in $\mathbb{R}$ let $G : \mathbb{R} \to \mathbb{R}$ have left and right limits with values in $\overline{\mathbb{R}}$. Let G be piecewise monotonic with $\lim_{\pm\infty} G$ existing in $\overline{\mathbb{R}}$. Then $F \circ G$ is regulated on $\mathbb{R}$ with real limits at $\pm\infty$. Define $(F' \circ G)G' := (F \circ G)'$.* (b) *Let $f \in \mathcal{A}_R$. In addition to* (a), *assume G is increasing, left continuous, $\lim_{-\infty} G = -\infty$ and $\lim_{\infty} G \in (-\infty, \infty]$. Then $F \circ G \in \mathcal{B}_R$ and $\int_{-\infty}^{\infty} (f \circ G)G' = \int_{-\infty}^{\infty} (F \circ G)' =$*

$\int_{(-\infty,G(\infty))} f = F(G(\infty)-)$. (c) *Let* $f \in \mathcal{A}_R$. *Assume* G *as in* (a). *Let* $-\infty < a_1 < a_2 < \infty$. *For each* $i \in \{1,2\}$, *let* $\sigma_i, \epsilon_i \in \{+,-\}$. *Then*

$$\begin{aligned}\int_{a_1\epsilon_1}^{a_2\epsilon_2} (f\circ G)G' &= \int_{a_1\epsilon_1}^{a_2\epsilon_2} (F\circ G)' = \int_{G(a_1\epsilon_1)\sigma_1}^{G(a_2\epsilon_2)\sigma_2} f \\ &= (F\circ G)(a_2\epsilon_2) - (F\circ G)(a_1\epsilon_1) \\ &= F(G(a_2\epsilon_2)\sigma_2) - F(G(a_1\epsilon_1)\sigma_1).\end{aligned} \tag{6}$$

For each $i \in \{1,2\}$, $\sigma_i = \epsilon_i$ *if* G *is increasing on an interval with endpoints* a_i *and* $a_i\epsilon_i\delta$ *for some* $\delta > 0$, *and* $\sigma_i \neq \epsilon_i$ *if* G *is decreasing on an interval with endpoints* a_i *and* $a_i\epsilon_i\delta$ *for some* $\delta > 0$. *If* $G(a_i\epsilon_i) = \pm\infty$, *then we don't need* σ_i.

If $a_1 = -\infty$, *then replace* $a_1\epsilon_1$ *with* $-\infty$ *in* (6). *If* $a_2 = \infty$ *then replace* $a_2\epsilon_2$ *with* ∞ *in* (6). *If* G *is increasing in a neighbourhood of* $-\infty$, *then* $\sigma_1 = +$. *If* G *is decreasing in a neighbourhood of* $-\infty$, *then* $\sigma_1 = -$. *If* G *is increasing in a neighbourhood of* ∞, *then* $\sigma_2 = -$. *If* G *is decreasing in a neighbourhood of* ∞, *then* $\sigma_2 = +$. *If* $G(-\infty) \in \{-\infty,\infty\}$, *then we don't need* σ_1 *and if* $G(\infty) \in \{-\infty,\infty\}$ *then we don't need* σ_2. (d) *Let* $f \in \mathcal{A}_C$. (*See the paragraph preceding Remark* 5 *for the definition.*) *Let* G *be regulated with* $\lim_{\pm\infty} G$ *existing in* $\overline{\mathbb{R}}$. *Then* $F\circ G$ *is regulated and*

$$\begin{aligned}\int_{a_1\epsilon_1}^{a_2\epsilon_2} (f\circ G)G' &= \int_{a_1\epsilon_1}^{a_2\epsilon_2} (F\circ G)' = \int_{G(a_1\epsilon_1)}^{G(a_2\epsilon_2)} f \\ &= (F\circ G)(a_2\epsilon_2) - (F\circ G)(a_1\epsilon_1).\end{aligned}$$

The last integral exists as a continuous primitive integral [30].

Proof. (a) Let $x \in \mathbb{R}$. For small enough $\delta > 0$, G is monotonic on intervals with endpoints x and $x \pm \delta$. Suppose G is decreasing on $(x, x+\delta)$. If $\lim_{y\to x^+} G(y) \in \mathbb{R}$, then for each $\nu > 0$ there exists $\eta(\nu) > 0$ such that if $y \in (x, x+\eta(\nu))$ then $G(y) \in (G(x+) - \eta(\nu), G(x+))$. Since $\lim_{z\to G(x+)-} F(z)$ exists we have that for every $\epsilon > 0$ there is $\nu(\epsilon) > 0$ such that if $z \in (G(x+) - \nu(\epsilon), G(x+))$ then $|F(z) - F(G(x+)-)| < \epsilon$. To show $\lim_{z\to x+}(F\circ G)(z)$ exists, let $\epsilon > 0$ and let $y \in (x, x+\eta(\nu(\epsilon)))$. Then $|(F\circ G)(y) - F(G(x+)-)| < \epsilon$. Other cases are similar with only minor modifications, including showing left or right continuity of $F\circ G$. Similarly in part (c).

(c) Suppose $-\infty < a < b < \infty$ and for some $\delta > 0$ we have G increasing on $(a-\delta, a)$ and decreasing on $(b, b+\delta)$. Then $a_1 = a$, $a_2 = b$, $\epsilon_1 = \sigma_1 = -$, $\epsilon_2 = +$, $\sigma_2 = -$ and we have

$$\begin{aligned}\int_{a-}^{b+} (f\circ G)G' &= \int_{[a,b]} (F\circ G)' = (F\circ G)(b+) - (F\circ G)(a-) \\ &= F\big(G(b+)-\big) - F\big(G(a-)-\big) = \int_{G(a-)-}^{G(b+)-} f.\end{aligned}$$

Other cases are similar and (b) is included in (c).

(d) There need be no interval on which G is monotonic. However, since F is continuous we have

$$\begin{aligned}\int_{a_1\epsilon_1}^{a_2\epsilon_2}(f\circ G)G' &= (F\circ G)(a_2\epsilon_2)-(F\circ G)(a_1\epsilon_1)\\ &= \lim_{x\to G(a_2\epsilon_2)}F(x)-\lim_{x\to G(a_1\epsilon_1)}F(x)\\ &= F(G(a_2\epsilon_2))-F(G(a_1\epsilon_1))\\ &= \int_{G(a_1\epsilon_1)}^{G(a_2\epsilon_2)}f.\end{aligned}$$

The cases when $a_1=-\infty$ or $a_2=\infty$ are similar. □

Note that in (c) there are 16 cases, depending on whether G is increasing or decreasing at each of the endpoints for the four types of integrals in (1). There are four cases for endpoints at $\pm\infty$.

Note that G need not be strictly monotonic but then we have to use the left continuity of F to interpret the integral. For example, if $f\in\mathcal{A}_R$ and $G=H_1$ then $\int_{-\infty}^{\infty}(F\circ H_1)'=\int_{-\infty}^{\infty}(f\circ H_1)\delta=(F\circ H_1)(\infty)-(F\circ H_1)(-\infty)=F(1)-F(0)=F(1-)-F(0-)=\int_{[0,1)}f$. And, G need not be bounded. For example, let $G(x)=1+x^{-2}$ for $x\neq 0$. The value of G at 0 is immaterial. Let $f\in\mathcal{A}_R$. Then $\int_{(-\infty,0)}(f\circ G)G'=(F\circ G)(0-)-(F\circ G)(-\infty)=F(\infty)-F(1+)$. We have $a_1=-\infty$, $a_2=0$, $\epsilon_2=-$, $\sigma_1=+$, which gives $\int_{G(-\infty)+}^{G(0-)}f=\int_{1+}^{\infty}f=F(\infty)-F(1+)=\int_{(-\infty,0)}(f\circ G)G'$.

THEOREM 9 (Translations). (a) $\mathcal{A}_R$ *is invariant under translation, that is,* $f\in\mathcal{A}_R$ *if and only if* $\tau_t f\in\mathcal{A}_R$ *for all* $t\in\mathbb{R}$. (b) $\|\tau_t f\|=\|f\|$ *for all* $f\in\mathcal{A}_R$ *and all* $t\in\mathbb{R}$.

Proof. (a) Let $f\in\mathcal{A}_R$. Then $f=F'$ for $F\in\mathcal{B}_R$. For $\phi\in\mathcal{D}$ we have

$$\begin{aligned}\langle(\tau_t F)',\phi\rangle &= -\langle\tau_t F,\phi'\rangle=-\langle F,\tau_{-t}\phi'\rangle=-\langle F,(\tau_{-t}\phi)'\rangle\\ &= \langle F',\tau_{-t}\phi\rangle=\langle\tau_t F',\phi\rangle=\langle\tau_t f,\phi\rangle.\end{aligned}$$

If $f\in\mathcal{D}'$ such that $\tau_t f\in\mathcal{A}_R$, reverse the above steps.

(b) Note that $\|\tau_t f\|=\sup_{x\in\mathbb{R}}|\tau_t F(x)|=\sup_{x\in\mathbb{R}}|F(x-t)|=\|F\|_\infty=\|f\|$. □

We have *continuity in norm* if for $f\in\mathcal{A}_R$ we have $\|f-\tau_x f\|\to 0$ as $x\to 0$. But this is not true in $\mathcal{A}_R$. For example, $\|H_1'-\tau_x H_1'\|=1$ if $x\neq 0$.

5. Integration by parts

An integration by parts formula is obtained using the Henstock–Stieltjes integral. (See the Introduction.) This allows us to prove versions of the Hölder inequality and Taylor's theorem.

The integration by parts formula in $\mathcal{A}_R$ follows from integration by parts for the Henstock–Stieltjes integral [21, p. 199]. The integrals $\int_{-\infty}^{\infty} F\,dg$ and $\int_{-\infty}^{\infty} g\,dF$ exist when one of F and g is regulated and one is of bounded variation. See also [31] and [33] where various properties of these integrals are established.

We first need to define the product of $f\in\mathcal{A}_R$ and $g\in\mathcal{BV}$.

Proposition 10. *For $f\in\mathcal{A}_R$ and $g\in\mathcal{BV}$, let $\{c_n\}$ contain all $t\in\mathbb{R}$ such that both F and g are not right continuous at t. Define $\Psi(x)=F(x)g(x)-\int_{-\infty}^{x} F\,dg-\sum_{c_n<x}[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]$. Then $\Psi\in\mathcal{B}_R$. The sum is over all $n\in\mathbb{N}$ such that $c_n<x$. The integral and series defining Ψ converge absolutely.*

Proof. There is $M\in\mathbb{R}$ such that $|g|\leq M$ and $Vg\leq M$. Let $x\in\mathbb{R}$. Then

$$\begin{aligned}|\Psi(x)| &\leq |F(x)|M+\left\|F\chi_{(-\infty,x]}\right\|_\infty Vg+2\left\|F\chi_{(-\infty,x+1]}\right\|_\infty Vg\\ &\to 0 \quad \text{as } x\to-\infty.\end{aligned}$$

This also shows that the integral and series defining Ψ converge absolutely. Let $y<x$. Then

$$\begin{aligned}\Psi(y)-\Psi(x)=[F(y)-F(x)]g(y)+\int_y^x [F(t)-F(x)]\,dg(t)\\ -\sum_{c_n\in[y,x)}[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]\end{aligned}$$

so that, using the uniform left continuity of F (Theorem 2),

$$\begin{aligned}\lim_{y\to x^-}|\Psi(y)-\Psi(x)| &\leq \lim_{y\to x^-}|F(y)-F(x)|M+2\lim_{y\to x^-}\sup_{s,t\in[y,x]}|F(s)-F(t)|Vg\\ &=0.\end{aligned}$$

Therefore, Ψ is left continuous. Similarly, using the uniform right regularity of F we see that Ψ has a right limit at each point. Letting $x,y\to\infty$ in the above inequality shows $\Psi(\infty)$ exists. □

If F is taken as regulated but not left continuous, then there is an additional term in Ψ involving $F(c_n)-F(c_n-)$. See [21, p. 199].

For an arbitrary distribution $T\in\mathcal{D}'$, we have the product $T\psi$ defined for all $\psi\in C^\infty(\mathbb{R})$ by $\langle T\psi,\phi\rangle=\langle T,\psi\phi\rangle$ for $\phi\in\mathcal{D}$. Distributions in $\mathcal{A}_R$ can be multiplied by functions of bounded variation.

Definition 11 (Product). With Ψ as in Proposition 10 and $\phi\in\mathcal{D}$, the product of $f\in\mathcal{A}_R$ and $g\in\mathcal{BV}$ is defined by $\langle fg,\phi\rangle=\langle\Psi',\phi\rangle=-\langle\Psi,\phi'\rangle$.

This defines $fg\in\mathcal{A}_R$ since $\Psi\in\mathcal{B}_R$. Each of the three terms in Ψ is regulated so the product $\Psi(t)\phi(t)$ is Riemann integrable.

Definition 12 (Integration by parts). Let $f \in \mathcal{A}_R$ and $g \in \mathcal{BV}$ and use the notation of Proposition 10. Define the integral of fg by

$$\int_{-\infty}^{\infty} fg = \int_{-\infty}^{\infty} g\,dF$$
$$= F(\infty)g(\infty) - \int_{-\infty}^{\infty} F\,dg - \sum_{n\in\mathbb{N}}[F(c_n) - F(c_n+)][g(c_n) - g(c_n+)].$$

Notice that if F is continuous or if g is right continuous then the sum in the integration by parts formula vanishes and we recover the more familiar formula $\int_{-\infty}^{\infty} fg = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F\,dg$. Note also that we have defined the integration by parts formula to agree with the Stieltjes integral but we have no way of proving the formula. However, when F is appropriately smooth it reduces to the usual formula for Lebesgue ($F \in AC$), Henstock–Kurzweil ($F \in ACG^*$) and wide Denjoy integrals ($F \in ACG$). Density arguments show we have the correct formula in $\mathcal{A}_R$. Since step functions are dense in the regulated functions [21, Section 7.13], given $f \in \mathcal{A}_R$ there is a sequence of step functions $\{F_n\} \in \mathcal{B}_R$ such that $\|F_n - F\|_\infty \to 0$. Definition 12 certainly holds for $f_n = F_n'$ and $g \in \mathcal{BV}$. To see this it suffices, to prove the formula for $f = \delta$, $F = H_1$ and $g \in \mathcal{BV}$. We have $F(\infty)g(\infty) = g(\infty)$. To evaluate $\int_{-\infty}^{\infty} F\,dg$, take a gauge γ that forces 0 to be a tag. If $\mathcal{P} = \{([x_{n-1}, x_n], z_n)\}_{n=1}^N$ is γ-fine then

$$\sum_{n=1}^{N} H_1(z_n)[g(x_n) - g(x_{n-1})] = \sum_{z_n>0} [g(x_n) - g(x_{n-1})]$$
$$= g(\infty) - g(z),$$

where z is the smallest positive tag in $\mathcal{P}$. We can take γ so that z is as close to 0 as we like. Therefore, $\int_{-\infty}^{\infty} F\,dg = g(\infty) - g(0+)$. And, $-\sum[H_1(c_n) - H_1(c_n+)][g(c_n) - g(c_n+)] = g(0) - g(0+)$. Hence, $F(\infty)g(\infty) - \int_{-\infty}^{\infty} F\,dg - \sum_{n\in\mathbb{N}}[F(c_n) - F(c_n+)][g(c_n) - g(c_n+)] = g(0)$. And, $\sum_{n=1}^N g(z_n)[H_1(x_n) - H_1(x_{n-1})] = g(0)$ so that $\int_{-\infty}^{\infty} g\,dF = g(0)$. The Hölder inequality (Theorem 13 below) then gives $|\int_{-\infty}^{\infty}(f_n - f)g| \le \|F_n - F\|_\infty \|g\|_{\mathcal{BV}} \to 0$ as $n \to \infty$. This justifies the integration by parts formula.

The calculation above shows that $\int_{-\infty}^{\infty} \delta g = g(0)$ for each function g that has a right limit at 0 and a limit at ∞. Thus, the integration by parts formula is in accordance with the action of δ as a measure. For example, let $a \in \mathbb{R}$ and define

$$g_a(x) = \begin{cases} 0, & x < 0, \\ a, & x = 0, \\ 1, & x > 0. \end{cases} \tag{7}$$

Then for $\phi \in \mathcal{D}$, we have $\langle \delta g_a, \phi\rangle = \langle \delta, g_a\phi\rangle = a\phi(0)$. Putting $a=0$ gives $\langle \delta H_1, \phi\rangle = 0$ so $\delta H_1 \in \mathcal{A}_R$ and $\delta H_1 = 0$; $a = 1$ gives $\langle \delta H_2, \phi\rangle = \phi(0)$ so $\delta H_2 \in \mathcal{A}_R$ and $\delta H_2 = \delta$. See [24] for references to other methods of multiplying the Dirac and Heaviside distributions. We also see that changing g at even one point can affect the value of the integral of fg, that is, for $f = \delta$ the integral depends on the value of $g(0)$. And, defining a function F to be the right side of (7) we see that the integration by parts formula does not depend on our convention of using left continuous primitives. For such F and any $g \in \mathcal{BV}$, both right sides of Definition 12 give zero, provided we use the more general formula [21, p. 199] that allows discontinuities from the left and right.

The integration by parts formula leads to a version of the Hölder inequality. Note that $\mathcal{BV}$ is a Banach space under the norm $\|g\|_{\mathcal{BV}} = \|g\|_\infty + Vg$.

THEOREM 13 (Hölder). *Let $f \in \mathcal{A}_R$ and $g \in \mathcal{BV}$. Then $|\int_{-\infty}^{\infty} fg| \leq |\int_{-\infty}^{\infty} f||g(\infty)| + \|f\|Vg \leq \|f\|\|g\|_{\mathcal{BV}}$. The inequality is sharp in the sense that if $|\int_{-\infty}^{\infty} fg| \leq \|f\|(\alpha|g(\infty)| + \beta Vg)$ for all $f \in \mathcal{A}_R$ and all $g \in \mathcal{B}_R$ then $\alpha, \beta \geq 1$. For each $-\infty \leq a \leq \infty$ there is the inequality $|\int_{-\infty}^{\infty} fg| \leq \|f\|(|g(a)| + 2Vg)$.*

Proof. Use the fact that $\int_{-\infty}^{\infty} fg = \int_{-\infty}^{\infty} g\,dF$. Given $\epsilon > 0$ there is a partition $\{(z_n, [x_{n-1}, x_n])\}_{n=1}^N$ so that $|\int_{-\infty}^{\infty} g\,dF - \sum_{n=1}^N g(z_n)[F(x_n) - F(x_{n-1})]| < \epsilon$. Since $F(x_0) = F(-\infty) = 0$,

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} fg\right| &\leq \epsilon + \left|\sum_{n=1}^N g(z_n)[F(x_n) - F(x_{n-1})]\right| \\
&= \epsilon + \left|\sum_{n=1}^N g(z_n)F(x_n) - \sum_{n=1}^{N-1} g(z_{n+1})F(x_n)\right| \\
&= \epsilon + \left|F(\infty)g(\infty) - \sum_{n=1}^{N-1} F(x_n)[g(z_{n+1}) - g(z_n)]\right| \\
&\leq \epsilon + \left|\int_{-\infty}^{\infty} f\right||g(\infty)| + \|F\|_\infty Vg \\
&\leq \epsilon + \|f\|\|g\|_{\mathcal{BV}}.
\end{aligned}
$$

The final estimate follows upon noting that $|g(\infty)| \leq |g(a)| + |g(\infty) - g(a)| \leq |g(a)| + Vg$.

We can see the estimate is sharp by letting $F(x) = (P/\pi)(\pi/2 + \arctan(x))$ and $g(x) = Q$ for $x \leq a$ and $g(x) = R$ for $x > a$, where $P > 0$ and $Q > R > 0$. Then $\int_{-\infty}^{\infty} fg = PR + (Q-R)F(a)$. As $a \to \infty$ we see this approaches $F(\infty) \times g(\infty) + \|f\|Vg$. ☐

For a proof using the Henstock–Stieltjes integral, see [31, Theorem 2.8] and [33].

Integration by parts and the fundamental theorem can be used to prove a version of Taylor's theorem.

THEOREM 14 (Taylor). *Let $f:[a,\infty)\to\mathbb{R}$. Let $n\in\mathbb{N}$. If $f\in C^{n-1}([a,\infty))$ so that $f^{(n)}$ is regulated and right continuous on $[a,\infty)$, then for all $x\in(a,\infty)$ we have $f(x)=P_n(x)+R_n(x)$ where*

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)(x-a)^k}{k!} \quad \textit{and} \quad R_n(x)=\frac{1}{n!}\int_{(a,x]}f^{(n+1)}(t)(x-t)^n\,dt.$$

We have the estimates $|R_n(x)|\le\sup_{a\le t<x}|f^{(n)}(t)-f^{(n)}(a)|(x-a)^n/n!$ for $x\in(a,\infty)$ and $\|R_n\chi_{(a,b)}\|\le\|R_n\chi_{(a,b)}\|_1\le(b-a)^{n+1}\sup_{a\le t<b}|f^{(n)}(t)-f^{(n)}(a)|/(n+1)!$.

Integration by parts gives an induction proof. The remainder exists because the function $t\mapsto(x-t)^n$ is in $\mathcal{BV}$ for each x. The estimates on the remainder follow from the Hölder inequality (Theorem 13). Note that $R_n(x)=o((x-a)^n)$ as $x\to a+$. If $f^{(n)}$ is left continuous on $(-\infty,a]$, then we expand f in powers of $a-x$ and $R_n(x)=o((a-x)^n)$ as $x\to a-$. Usual versions of Taylor's theorem require $f^{(n+1)}$ to be integrable. For the Lebesgue integral, this means taking $f^{(n)}$ to be absolutely continuous. Here we only require $f^{(n)}$ to be regulated. The case $n=0$ corresponds to Theorem 6.

6. Norms and dual space

Multipliers are those functions g for which fg is integrable for all integrable f. In this section, we consider some equivalent norms on $\mathcal{A}_R$ and then show that the space of multipliers of $\mathcal{A}_R$ and the dual space of $\mathcal{A}_R$ are both given by $\mathcal{BV}$.

THEOREM 15 (Equivalent norms). (a) *The following norms are equivalent to $\|\cdot\|$ in $\mathcal{A}_R$. For $f\in\mathcal{A}_R$, define $\|f\|'=\sup_I|\int_I f|$, where the supremum is taken over all finite intervals $I\subset\mathbb{R}$; $\|f\|''=\sup_g\int fg$, where the supremum is taken over all $g\in\mathcal{BV}$ such that $\|g\|_\infty\le 1$ and $Vg\le 1$.* (b) *Let $g\in\mathcal{BV}$ and be normalised so that $g(x)=(1-\lambda)g(x-)+\lambda g(x+)$ for fixed $0\le\lambda\le 1$ and all $x\in\mathbb{R}$. Then for $f\in\mathcal{A}_R$ we have $|\int_{-\infty}^{\infty}fg|\le|\int_{-\infty}^{\infty}f|\inf|g|+\|f\|'Vg$.* (c) *Let $a\in\overline{\mathbb{R}}$. The norms $\|g\|'_{\mathcal{BV}}=|g(a)|+Vg$ and $\|g\|_{\mathcal{BV}}=\|g\|_\infty+Vg$ are equivalent on $\mathcal{BV}$.*

Proof. (a) Note that $\|f\|\le\|f\|'$. And, we have $\|f\|'=\sup_{a<b}|\int_{(a,b)}f|=\sup_{a<b}|F(b-)-F(a+)|\le 2\|f\|$. Similarly, for other types of intervals. Hence, $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. Let $g\in\mathcal{BV}$ such that $\|g\|_\infty\le 1$ and $Vg\le 1$. By the Hölder inequality (Theorem 13),

$$\left|\int_{-\infty}^{\infty}fg\right|\le\|f\|[|g(\infty)|+Vg]\le 2\|f\|.$$

And,
$$\|f\|'' \geq \max\left(\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty} f\chi_{(-\infty,x]}, -\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty} f\chi_{(-\infty,x]}\right).$$
It follows that $\frac{1}{2}\|f\|'' \leq \|f\| \leq \|f\|''$.

(b) The alternative Hölder inequality is proved as Lemma 24 in [29].

(c) Clearly, $\|g\|'_{\mathcal{BV}} \leq \|g\|_{\mathcal{BV}}$. Let $x \in \mathbb{R}$. The inequality $|g(x)| \leq |g(a)| + |g(a) - g(x)| \leq |g(a)| + Vg$ shows $\|g\|_{\mathcal{BV}} \leq 2\|g\|'_{\mathcal{BV}}$. □

The Hölder inequality can be reformulated in any of these equivalent norms.

For the Henstock–Kurzweil and continuous primitive integral [30] the multipliers and dual space are the functions of essential bounded variation. (See Example 3(d) for the definition.) For $\mathcal{A}_R$, the multipliers and dual space are the functions of bounded variation.

THEOREM 16. *The set of multipliers for $\mathcal{A}_R$ is $\mathcal{BV}$.*

Proof. The multipliers are defined in Definition 12. Hence, every function of bounded variation is a multiplier. In order for a function g to be a multiplier the integral $\int_{-\infty}^{\infty} g\,dF$ must exist for every $F \in \mathcal{B}_R$. Taking F to be a step function $F = \sum \sigma_n \chi_{(a_n,b_n]}$ for disjoint intervals $\{(a_n, b_n)\}$ and $\sigma_n \in \mathbb{R}$, we see that $\int_{-\infty}^{\infty} g\,dF = \sum \sigma_n[g(a_n) - g(b_n)]$. Taking $\sigma_n = \mathrm{sgn}(g(a_n) - g(b_n))$ shows $g \in \mathcal{BV}$. □

If $\{f_n\}$ is a sequence in $\mathcal{A}_R$ such that $\|f_n\| \to 0$ then the Hölder inequality shows that $\int_{-\infty}^{\infty} f_n g \to 0$ for each $g \in \mathcal{BV}$. Hence, for each fixed $g \in \mathcal{BV}$, $T_g(f) := \int_{-\infty}^{\infty} fg$ defines a continuous linear functional on $\mathcal{A}_R$. Hence, $\mathcal{A}_R^* \supset \mathcal{BV}$. In fact, all continuous linear functionals on $\mathcal{A}_R$ are of this form, i.e., $\mathcal{A}_R^* = \mathcal{BV}$. We can prove this by using the representation of the dual of the space of regulated functions.

Various authors have used different specialised integrals to represent the continuous linear functionals on regulated functions. See Kaltenborn [16] (Dushnik interior integral) for compact intervals (also [14]), Hildebrandt [11] (refinement or Young integral) for $\mathbb{R}$, Tvrdý [31], [32], [33] (Henstock–Stieltjes integral, where it is called the Perron–Stieltjes integral). Tvrdý gives a representation for such a functional acting on regulated function F on compact interval $[a,b]$ as $T(F) = qF(a) + \int_a^b g\,dF$ for some function $g \in \mathcal{BV}$ and $q \in \mathbb{R}$. See also [27]. Extension to regulated functions on $\mathbb{R}$ follows by replacing a with $-\infty$ and b with ∞, using our definition of the Henstock–Stieltjes integral (Section 5) and compactification of $\mathbb{R}$ (Remark 3). For $F \in \mathcal{B}_R$, the functional then becomes $T_g(F) = \int_{-\infty}^{\infty} g\,dF = \int_{-\infty}^{\infty} F'g$. The connection between the Dushnik interior and Young integrals is given in [13]. Equality of Young and Henstock–Stieltjes integrals for one function regulated and one of bounded variation is established in [26].

THEOREM 17. *The dual space of $\mathcal{A}_R$ is $\mathcal{BV}$ ($\mathcal{A}_R^* = \mathcal{BV}$).*

Proof. Let $\psi : \mathcal{A}_R \to \mathcal{B}_R$ be given by $\psi(f) = F$. Then $\psi^{-1} : \mathcal{B}_R \to \mathcal{A}_R$ is given by $\psi^{-1}(F) = F'$. Let $\{f_n\} \subset \mathcal{A}_R$ such that $\|f_n\| \to 0$. Then $\|F_n\|_\infty \to 0$. If $T \in \mathcal{A}_R^*$ then $T(f_n) = T(\psi^{-1}(F_n)) \to 0$. Hence, $T \circ \psi^{-1} \in \mathcal{B}_R^*$. Using the result of the previous paragraph, we have $\mathcal{B}_R^* = \mathcal{BV}$. There exists $g \in \mathcal{BV}$ such that $T \circ \psi^{-1}(F_n) = \int_{-\infty}^{\infty} F_n' \, dg = \int_{-\infty}^{\infty} f_n g$. Hence, $T(f_n) = \int_{-\infty}^{\infty} f_n g$. □

The integration by parts formula also shows that $\langle f, g \rangle = \int_{-\infty}^{\infty} fg$ for all $f \in \mathcal{A}_R$ and all $g \in \mathcal{BV}$ so that we could use integration by parts as a starting point to define the integral as a continuous linear functional on $\mathcal{BV}$.

In the space of Henstock–Kurzweil integrable functions, we identify functions almost everywhere so the dual of this space is $\mathcal{EBV}$ rather than $\mathcal{BV}$, that is, if T is a continuous linear functional on the space of Henstock–Kurzweil integrable functions then there exists a function $g \in \mathcal{BV}$ such that $T(f) = \int_{-\infty}^{\infty} fg$ for each Henstock–Kurzweil integrable function f. The integral is that of Henstock–Kurzweil. Changing g on a set of measure zero does not affect the value of this integral so the dual space is $\mathcal{EBV}$.

In $\mathcal{A}_R$, we do not have this equivalence relation so the dual of $\mathcal{A}_R$ is $\mathcal{BV}$ and not $\mathcal{EBV}$. Similarly, for no normalisation in $\mathcal{BV}$ (see Remark 5) is the dual of $\mathcal{A}_R$ equal to functions of normalised bounded variation. To see this, note that the function $g = \chi_{\{0\}}$ is not equivalent to 0 since $\int_{-\infty}^{\infty} fg = F(0+) - F(0-)$. But every normalisation makes $g = 0$.

No concrete description of $\mathcal{BV}^*$ seems to be known. But note that if $\{g_n\} \subset \mathcal{BV}$ such that $\|g_n\|_{\mathcal{BV}} \to 0$ then $\int_{-\infty}^{\infty} fg_n \to 0$ for each $f \in \mathcal{BV}$. Hence, $T_f(g) = \int_{-\infty}^{\infty} fg$ defines a continuous linear functional on $\mathcal{BV}$. The Hölder inequality shows that for each regulated function F the linear functional

$$
\begin{aligned}
(8) \qquad T_F(g) = \int_{-\infty}^{\infty} F \, dg = -\int_{-\infty}^{\infty} F'g + F(\infty)g(\infty) - F(-\infty)g(-\infty) \\
- \sum_{n \in \mathbb{N}} [F(c_n) - F(c_n+)][g(c_n) - g(c_n+)] \\
+ \sum_{n \in \mathbb{N}} [F(c_n) - F(c_n-)][g(c_n) - g(c_n-)]
\end{aligned}
$$

is in $\mathcal{BV}^*$, that is, if $\|g_n\|_{\mathcal{BV}} \to 0$ then $T_F(g_n) \to 0$ in $\mathbb{R}$. Hence, $\mathcal{BV}^*$ contains the space of regulated functions. If we let $F = \chi_{\{0\}}$, then $T_F(g) = \int_{-\infty}^{\infty} F \, dg = g(0+) - g(0-)$ and $|T_F(g)| \le Vg$ so $T_F \in \mathcal{BV}^*$ but as an element of $\mathcal{A}_R$, $F' = 0$. Hence, $\mathcal{BV}^* \supsetneq \mathcal{A}_R$. And, consider the following example. Let $S = \{1/n \mid n \in \mathbb{N}\}$, $F = \chi_S$ and define $U_F : \mathcal{BV} \to \mathbb{R}$ by $U_F(g) = \int_{-\infty}^{\infty} F \, dg$. Then F is not of bounded variation and since $\lim_{x \to 0+} F(x)$ does not exist, F is not regulated. But, for $g \in \mathcal{BV}$, $U_F(g) = \sum_{n=1}^{\infty} [g(n^{-1}+) - g(n^{-1}-)]$. This can be seen by taking a gauge that forces 0 to be a tag and forces n^{-1} to be a tag for some N_0 and all $1 \le n \le N_0$. We then have $|U_F(g)| \le Vg$. This shows that $U_F \in \mathcal{BV}^*$. Hence, $\mathcal{BV}^*$ properly contains the space of regulated functions. More precisely, the space of regulated functions is identified with finitely additive measures defined by $\mu([a,b]) = F(b+) - F(a-)$ for regulated function F. Similarly for other intervals. These measures are defined on the algebra generated by intervals.

Hildebrandt [13] and Aye and Lee [2] have given explicit representation of the dual of $\mathcal{BV}^*$ in the topology of uniform bounded variation with uniform convergence. A sequence $\{g_n\} \subset \mathcal{BV}$ converges to 0 in this sense if $\|g_n\|_\infty \to 0$ and there is $M \in \mathbb{R}$ so that for all $n \in \mathbb{N}$, $Vg_n \leq M$. These authors show that the dual of $\mathcal{BV}$ in this topology contains only (pairs of) regulated functions. This dual must then be a proper subset of $\mathcal{BV}^*$ since U_F from the preceding paragraph is not continuous in the topology of uniform bounded variation with uniform convergence. For example, define the piecewise linear functions $g_n(x) = n^{-1}\sum_{m=1}^{n}(1-m+m^2x)\chi_{[m^{-1},m^{-1}+m^{-2}]}(x)$. Then $\|g_n\|_\infty = 1/n$ and $Vg_n = 1$. Hence, $g_n \to 0$ in the topology of [13] and [2]. But,

$$U_F(g_n) = n^{-1}\sum_{m=1}^{n} F(m^{-1})[g(m^{-1}+) - g(m^{-1}-)] = 1 \not\to 0.$$

Mauldin ([20] and references therein) and Hildebrandt [12] have given representations of $\mathcal{BV}^*$ in terms of abstract integrals.

7. $\mathcal{BV}$-module

In Definition 11, we have a product defined from $\mathcal{A}_R \times \mathcal{BV}$ onto $\mathcal{A}_R$. It has distributive, commutative and associative properties that make $\mathcal{A}_R$ into a Banach $\mathcal{BV}$-module. See [6] for the definition. Properties of the integral of fg then follow from properties of the product.

Theorem 18 (Products). *Let $f, f_1, f_2 \in \mathcal{A}_R$; $g, g_1, g_2 \in \mathcal{BV}$; $k \in \mathbb{R}$. The product has the following properties.* (a) *Distributive.* $(f_1 + f_2)g = f_1g + f_2g$, $f(g_1 + g_2) = fg_1 + fg_2$. (b) *Homogeneous.* $(kf)g = f(kg) = k(fg)$. (c) *Commutative.* $f(g_1g_2) = f(g_2g_1)$. (d) *Compatible with distribution product.* $\langle fg, \phi\rangle = \langle f, g\phi\rangle$ *for all* $\phi \in \mathcal{D}$. (e) *Associative.* $(fg_1)g_2 = f(g_1g_2)$. (f) *Zero divisors. There are* $f \neq 0$ *and* $g \neq 0$ *such that* $fg = 0$. (g) *Compatible with pointwise product. If f and g are functions that are continuous at $a \in \mathbb{R}$, then $\langle fg, \phi_n\rangle \to f(a)g(a)$ for any δ-sequence supported at $\{a\}$.*

Proof. Properties (a), (b) and (c) follow immediately from the definition.

Notice we can include terms in the sum (also labeled c_n but not necessarily points were F and g are simultaneously discontinuous from the right) so that $\sup c_n = \infty$. To prove (d), let $f \in \mathcal{A}_R$, $g \in \mathcal{BV}$ and $\phi \in \mathcal{D}$. Then

$$\langle fg, \phi\rangle = -\int_{-\infty}^{\infty} F(x)g(x)\phi'(x)\,dx + \int_{-\infty}^{\infty}\int_{-\infty}^{x} F(t)\,dg(t)\,\phi'(x)\,dx$$
$$+ \int_{-\infty}^{\infty}\sum_{c_n<x}[F(c_n) - F(c_n+)][g(c_n) - g(c_n+)]\phi'(x)\,dx.$$

Note that $|\int_{-\infty}^{\infty}\int_{-\infty}^{x} F(t)\,dg(t)\,\phi'(x)\,dx| \leq \|F\|_\infty Vg\|\phi'\|_1$ so by the Fubini–Tonelli theorem, $\int_{-\infty}^{\infty}\int_{-\infty}^{x} F(t)\,dg(t)\,\phi'(x)\,dx = -\int_{-\infty}^{\infty} F(t)\phi(t)\,dg(t)$. And, $|\int_{-\infty}^{\infty}\sum_{c_n<x}[F(c_n) - F(c_n+)][g(c_n) - g(c_n+)]\phi'(x)\,dx| \leq 2\|F\|_\infty Vg\|\phi'\|_1$. Again,

$$\begin{aligned}\int_{-\infty}^{\infty}\sum_{c_n<x}&[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]\phi'(x)\,dx\\ &=\sum_{n=1}^{\infty}[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]\int_{x>c_n}\phi'(x)\,dx\\ &=-\sum_{n=1}^{\infty}[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]\phi(c_n).\end{aligned}$$

Therefore,

$$\begin{aligned}(9)\qquad \langle fg,\phi\rangle = -\int_{-\infty}^{\infty}& F(t)g(t)\phi'(t)\,dt - \int_{-\infty}^{\infty} F(t)\phi(t)\,dg(t)\\ &-\sum_{n=1}^{\infty}[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]\phi(c_n).\end{aligned}$$

And, $g\phi \in \mathcal{BV}$ with compact support, so using the continuity of ϕ,

$$\begin{aligned}\langle f, g\phi\rangle &= \int_{-\infty}^{\infty} f(g\phi)\\ &= -\int_{-\infty}^{\infty} F\,d(g\phi) - \sum[F(c_n)-F(c_n+)][g(c_n)\phi(c_n) - g(c_n+)\phi(c_n+)]\\ &= -\int_{-\infty}^{\infty} F\,d(g\phi) - \sum[F(c_n)-F(c_n+)][g(c_n)-g(c_n+)]\phi(c_n).\end{aligned}$$

Now show that $\int_{-\infty}^{\infty} F\,d(g\phi) = \int_{-\infty}^{\infty} Fg\,d\phi + \int_{-\infty}^{\infty} F\phi\,dg$. Each of these integrals exists because, in each case, one of the integrands and integrators is of bounded variation and one is regulated. For $\epsilon > 0$ there is then a tagged partition $\{(z_n, [x_{n-1}, x_n])\}_{n=1}^{N}$ such that $|S_N - \int_{-\infty}^{\infty} F\,d(g\phi) + \int_{-\infty}^{\infty} Fg\,d\phi + \int_{-\infty}^{\infty} F\phi\,dg| < \epsilon$, where

$$\begin{aligned}S_N = \sum_{n=1}^{N} F(z_n)\{&[g(x_n)\phi(x_n) - g(x_{n-1})\phi(x_{n-1})]\\ &- g(z_n)[\phi(x_n)-\phi(x_{n-1})] - \phi(z_n)[g(x_n)-g(x_{n-1})]\}.\end{aligned}$$

But,

$$\begin{aligned}|S_N| \leq \sum_{n=1}^{N}|F(z_n)|\{&|[g(x_n)-g(x_{n-1})||\phi(x_n)-\phi(z_n)|\\ &+|g(x_{n-1})-g(z_n)||\phi(x_n)-\phi(x_{n-1})|\}.\end{aligned}$$

Since ϕ is uniformly continuous we can arrange the partition so that the maximum of $|\phi(x_n)-\phi(t_n)|$ for $t_n\in[x_{n-1},x_n]$ is less than ϵ for each $1\leq n\leq N$. Then $|S_N|\leq 2\epsilon\|F\|_\infty Vg$. Hence, $\int_{-\infty}^{\infty}F\,d(g\phi)=\int_{-\infty}^{\infty}Fg\,d\phi+\int_{-\infty}^{\infty}F\phi\,dg$. Now using (9) we see that $\langle fg,\phi\rangle=\langle f,g\phi\rangle$.

Associativity (e) then follows by writing $\langle f(g_1g_2),\phi\rangle=\langle f,(g_1g_2)\phi\rangle=\langle f, g_1(g_2\phi)\rangle=\langle fg_1,g_2\phi\rangle=\langle (fg_1)g_2,\phi\rangle$.

To prove (f), let $F\in\mathcal{B}_R$ and $g\in\mathcal{BV}$ be continuous with disjoint support. Then $F'g=0$.

A δ-sequence supported at a is a sequence $\{\phi_n\}\subset\mathcal{D}$ such that $\phi_n\geq 0$, $\int_{-\infty}^{\infty}\phi_n=1$, $\mathrm{supp}(\phi_n)$ is an interval containing a in its interior such that $\mathrm{supp}(\phi_n)\to\{a\}$. For such a sequence, suppose $\mathrm{supp}(\phi_n)\subset[a-\delta,a+\delta]$. Then for (g),

$$\begin{aligned}\left|f(a)g(a)-\int_{-\infty}^{\infty}fg\phi_n\right| &= \left|\int_{-\infty}^{\infty}[f(a)g(a)\phi_n-fg\phi_n]\right|\\ &\leq \int_{a-\delta}^{a+\delta}|f(a)g(a)-f(x)g(x)|\phi_n(x)\,dx\\ &\to 0 \quad \text{using the continuity of } f \text{ and } g.\end{aligned}$$ □

If $g\in C^\infty$, then we see the product reduces to the usual product of a distribution and a smooth function (cf. the paragraph preceding Definition 11).

Each result in Theorem 18 concerning a product can be integrated. For example, $\int_{-\infty}^{\infty}f(gh)=\int_{-\infty}^{\infty}(fg)h$. Taking g to be the characteristic function of an interval and integrating by parts recovers each of the four integrals defined in (1)–(4): $\int_{-\infty}^{\infty}f\chi_I=\int_I f$ for any interval I. Each of the integrals $\int_{-\infty}^{\infty}F\,d\chi_I$ and $\int_{-\infty}^{\infty}\chi_I\,dF$ exists as a Henstock–Stieltjes integral because we can take a gauge that forces endpoints of I to be tags. If F is not continuous at the endpoints of I, then these integrals will not exist as Riemann–Stieltjes integrals.

The usual pointwise product makes $\mathcal{BV}$ into an algebra with unit $g=1$. Our product on $\mathcal{A}_R\times\mathcal{BV}$ makes $\mathcal{A}_R$ into a (left) Banach $\mathcal{BV}$-module.

THEOREM 19 (Banach $\mathcal{BV}$-module). *$\mathcal{BV}$ is a Banach algebra. $\mathcal{A}_R$ is a Banach $\mathcal{BV}$-module.*

Proof. For $g_1,g_2\in\mathcal{BV}$, the inequalities

$$\begin{aligned}\|g_1g_2\|_{\mathcal{BV}} &= \|g_1g_2\|_\infty+V(g_1g_2)\\ &\leq \|g_1\|_\infty\|g_2\|_\infty+\|g_1\|_\infty Vg_2+Vg_1\|g_2\|_\infty\\ &\leq \|g_1\|_{\mathcal{BV}}\|g_2\|_{\mathcal{BV}}\end{aligned}$$

show that $\mathcal{BV}$ is closed under multiplication. It then follows easily that $\mathcal{BV}$ is a Banach algebra.

The second statement follows from (a), (b), (c) and (e) of Theorem 18 and the inequality $\|fg\|\leq\|f\|\|g\|_{\mathcal{BV}}$, valid for all $f\in\mathcal{A}_R$ and $g\in\mathcal{BV}$. □

Notice that $\mathcal{BV}$ is not a division ring since $\chi_{[0,1]} \neq 0$ has no multiplicative inverse. There are zero divisors. For example, $\chi_{[0,1]}\chi_{[2,3]} = 0$.

Notice that if $g_1, g_2 \in \mathcal{BV}$ then $(g_1 g_2)' = g_1' g_2 + g_1 g_2' \in \mathcal{A}_R$. The product on the left is pointwise in $\mathcal{BV}$ while the products on the right are as per Definition 11. Hence, the distributional derivative is a *derivation* on the algebra $\mathcal{BV}$ into the Banach $\mathcal{BV}$-module $\mathcal{A}_R$. See [6].

8. Absolute integrability

The primitives of an L^1 function are absolutely continuous and hence are functions of bounded variation. Whereas, if function f is Henstock–Kurzweil or wide Denjoy integrable but $|f|$ is not integrable in this sense then the primitives of f are not of bounded variation. We use this observation to define absolute integrability in $\mathcal{A}_R$. We also show that L^1 and the space of signed Radon measures are embedded continuously in $\mathcal{A}_C$ and $\mathcal{A}_R$, respectively.

Definition 20 (Absolute integrability, $\mathcal{NBV}$). Define the functions of normalised bounded variation as $\mathcal{NBV} = \mathcal{B}_R \cap \mathcal{BV}$. A distribution $f \in \mathcal{A}_R$ is *absolutely integrable* if it has a primitive $F \in \mathcal{NBV}$. Denote the space of absolutely integrable distributions by $\mathcal{A}_{\mathcal{NBV}}$.

Hence, $\mathcal{A}_{\mathcal{NBV}}$ is isometrically isomorphic to the space of signed Radon measures under the Alexiewicz norm. For $f \in \mathcal{A}_{\mathcal{NBV}}$, let its primitive in $\mathcal{NBV}$ be F. As in Example 3(d), there is a unique signed Radon measure μ such that $F' = \mu$, i.e., $\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} \phi\, d\mu$ for all $\phi \in \mathcal{D}$. And, if μ is a signed Radon measure then a function defined by $F(x) = \mu((-\infty, x))$ is in $\mathcal{NBV}$. The Alexiewicz norm of μ identified with $f \in \mathcal{A}_{\mathcal{NBV}}$ is $\|\mu\| = \sup_{x \in \mathbb{R}} |\mu((-\infty, x))|$.

This then gives an alternative definition of the regulated primitive integral. It is the completion of the space of signed Radon measures in the Alexiewicz norm. Integration in $\mathcal{A}_{\mathcal{NBV}}$ is thus Lebesgue integration.

Denote the space of signed Radon measures by $\mathcal{M}$. A norm is given by $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}) = \mu^+(\mathbb{R}) + \mu^-(\mathbb{R})$, which is the total variation of μ.

Theorem 21. (a) *$\mathcal{NBV}$ is a Banach subspace of $\mathcal{B}_R$ under the norm $\|g\|_{\mathcal{BV}} = \|g\|_\infty + Vg$.* (b) *For each $a \in \overline{\mathbb{R}}$, the norms $\|g\|_{\mathcal{BV}}$ and $\|g\|_{\mathcal{BV}a} := |g(a)| + Vg$ are equivalent.* (c) *$\mathcal{NBV}$ is not a Banach space under $\|\cdot\|_\infty$.* (d) *$\mathcal{NBV}$ is dense in $\mathcal{B}_R$. The completion of $\mathcal{NBV}$ in $\|\cdot\|_\infty$ is $\mathcal{B}_R$.* (e) *$\mathcal{A}_{\mathcal{NBV}}$ is a Banach subspace of $\mathcal{A}_R$.* (f) *$\mathcal{A}_R$ is the completion of the space of signed Radon measures in the Alexiewicz norm.* (g) *The embeddings $L^1 \hookrightarrow \mathcal{A}_C$ and $\mathcal{M} \hookrightarrow \mathcal{A}_R$ are continuous.*

Proof. (a) It is a classical result that functions of bounded variation form a Banach space. For example, see [17]. The case of $\mathcal{NBV}$ is similar, as in the proof of Theorem 2. (b) See Theorem 15(c). (c) Let $g(x) = x \sin(x^{-2})$ for $x > 0$ and $g(x) = 0$ for $x \leq 0$. Then $g \in C^0 \setminus \mathcal{BV}$. Let $g_n = [1 - \chi_{[0,(n\pi)^{-1/2}]}]g$.

Each $g_n \in \mathcal{NBV}$. And, $\|g_n - g\|_\infty \le (n\pi)^{-1/2} \to 0$. In $\|\cdot\|_\infty$, the sequence $\{g_n\}$ converges to $g \notin \mathcal{BV}$. (d) Let $F \in \mathcal{B}_R$ and let $\epsilon > 0$ be given. There exists $M > 0$ such that $|F(x)| < \epsilon$ for all $x \le -M$ and $|F(x) - F(\infty)| < \epsilon$ for all $x \ge M$. For each $x \in [-M, M]$ there is $\delta_x > 0$ such that if $y \in (x - \delta_x, x]$, then $|F(y) - F(x)| < \epsilon$ and if $y \in (x, x + \delta_x)$ then $|F(y) - F(x+)| < \epsilon$. Let $I_x = (x - \delta_x, x + \delta_x)$. The collection $\{I_x\}_{x \in [-M,M]}$ is an open cover of the compact interval $[-M, M]$. There is then a finite subcover, $\{I_x\}_{x \in J}$ for some finite set $J \in [-M, M]$. We can then take open subintervals $I'_x \subset I_x$ such that each point in $[-M, M]$ is in either one or two of these intervals. Then we can define $g(x) = 0$ for $x \le -M$, $g(x) = F(\infty)$ for $x > M$ and g is piecewise constant on each interval I'_x such that $g \in \mathcal{NBV}$ and $\|g - F\|_\infty < \epsilon$. Hence, $\mathcal{NBV}$ is dense in $\mathcal{B}_R$ and its completion is $\mathcal{B}_R$. (e), (f) These follow from the isomorphism between $\mathcal{A}_R$ and $\mathcal{B}_R$ given by the integral. (g) For $\mu \in \mathcal{M}$ we have

$$\|\mu\| = \sup_{x \in \mathbb{R}} |\mu((-\infty, x))| \le \sup_{x \in \mathbb{R}} [\mu^+((-\infty, x)) + \mu^-((-\infty, x))] = \|\mu\|_{\mathcal{M}}.$$

If $f \in L^1$ then $\|f\| = \sup_{x \in \mathbb{R}} |\int_{-\infty}^x f| \le \|f\|_1$. □

In Proposition 24 below, it is shown that distributions in $\mathcal{A}_R$ are finitely additive measures that are finite when their primitives are of bounded variation.

Here is an alternative way of defining functions of normalised bounded variation. Fix $0 \le \lambda \le 1$. For $g \in \mathcal{BV}$ define $g_\lambda(x) = (1 - \lambda) g(x-) + \lambda g(x+)$, $g_\lambda(-\infty) = g(-\infty)$ and $g_\lambda(\infty) = g(\infty)$. Define $\mathcal{NBV}_\lambda = \{g_\lambda \mid g \in \mathcal{BV}\}$. Then $\mathcal{NBV}_\lambda$ is a Banach space under $\|g\|_{\mathcal{BV}} = \|g\|_\infty + Vg$. The connection with the functions of essential bounded variation is the following. As in Example 3(d), we have $\mathcal{EBV} = \{g \in L^1_{\text{loc}} \mid \operatorname{ess\,var} g < \infty\}$. This is a Banach space under the norm $\|g\|_{\mathcal{EBV}} = \operatorname{ess\,sup} |g| + \operatorname{ess\,var} g$. The space $\mathcal{EBV}$ consists of equivalence classes of functions identified almost everywhere. For each $g \in \mathcal{EBV}$, there is a unique $g_\lambda \in \mathcal{NBV}_\lambda$ such that $\operatorname{ess\,sup} g = \|g_\lambda\|_\infty$ and $\operatorname{ess\,var} g = V g_\lambda$. For each $0 \le \lambda \le 1$, the Banach spaces $\mathcal{NBV}_\lambda$ and $\mathcal{EBV}$ are isometrically isomorphic. These spaces are distinct from $\mathcal{BV}$. For example, $\chi_{\{0\}}$ is equivalent to 0 in $\mathcal{EBV}$, its normalisation is 0 in $\mathcal{NBV}_\lambda$ but $V\chi_{\{0\}} = 2$ in $\mathcal{BV}$. Note that $\mathcal{NBV}$ is isometrically isomorphic to the signed Radon measures and to $\mathcal{A}_{\mathcal{NBV}}$, whereas $\mathcal{NBV}_\lambda$ is isometrically isomorphic to $\mathcal{A}_{\mathcal{NBV}} \times \mathbb{R}$. If $g \in \mathcal{NBV}_\lambda$ then its distributional derivative is a Radon measure μ and $g(-\infty) \in \mathbb{R}$. For more on essential variation see [35].

If $F \in \mathcal{NBV}$, then there are increasing functions of normalised bounded variation G and H such that $F = G - H$. A distribution T is *positive* if $\langle T, \phi \rangle \ge 0$ for each $\phi \in \mathcal{D}$ with $\phi \ge 0$. Suppose $\phi \ge 0$. Let $[a, b]$ contain the support of ϕ. By the second mean value theorem for integrals [21, p. 211], there is $\xi \in [a,b]$ such that

$$\langle G', \phi \rangle = -\int_a^b G\phi' = -\left[G(a)\int_a^\xi \phi' + G(b)\int_\xi^b \phi'\right]$$
$$= [G(b) - G(a)]\phi(\xi) \geq 0.$$

Hence, $f \in \mathcal{A}_{\mathcal{NBV}}$ if and only if it can be written as $f = G' - H'$ for $G, H \in \mathcal{NBV}$ with $G', H' \geq 0$.

In the next section, we will introduce an ordering suitable for all distributions in $\mathcal{A}_R$.

9. Banach lattice

In $\mathcal{B}_R$, there is the partial order: $F \leq G$ if and only if $F(x) \leq G(x)$ for all $x \in \mathbb{R}$. Note that this order depends on our choice that functions in $\mathcal{B}_R$ be left continuous. Since $\mathcal{A}_R$ is isomorphic to $\mathcal{B}_R$ it inherits this partial order. For $f, g \in \mathcal{A}_R$, define $f \preceq g$ if and only if $F \leq G$, where F and G are the respective primitives in $\mathcal{B}_R$. This order is not compatible with the usual order on distributions: if $T, U \in \mathcal{D}'$ then $T \geq U$ if and only if $\langle T - U, \phi \rangle \geq 0$ for all $\phi \in \mathcal{D}$ such that $\phi \geq 0$. Nor is it compatible with pointwise ordering in the case of functions in $\mathcal{A}_R$. For example, if $f(t) = H_1(t)\sin(t^2)$ then $F \geq 0$ so $f \succeq 0$ in $\mathcal{A}_R$ but not pointwise. And, f is not positive in the distributional sense. Note, however, that if $f \in \mathcal{A}_R$ is a measure or a nonnegative function or distribution then $f \succeq 0$ in $\mathcal{A}_R$.

The importance of this ordering is that it interacts with the Alexiewicz norm so that $\mathcal{A}_R$ is a Banach lattice. If $\preceq$ is a binary operation on set S, then it is a *partial order* if for all $x, y, z \in S$ it is *reflexive* ($x \preceq x$), *antisymmetric* ($x \preceq y$ and $y \preceq x$ imply $x = y$) and *transitive* ($x \preceq y$ and $y \preceq z$ imply $x \preceq z$). If S is a Banach space with norm $\|\cdot\|_S$ and $\preceq$ is a partial order on S, then S is a *Banach lattice* if for all $x, y, z \in S$

(1) $x \vee y$ and $x \wedge y$ are in S. The *join* is $x \vee y = \sup\{x, y\} = w$ such that $x \preceq w$, $y \preceq w$ and if $x \preceq \tilde{w}$ and $y \preceq \tilde{w}$ then $w \preceq \tilde{w}$. The *meet* is $x \wedge y = \inf\{x, y\} = w$ such that $w \preceq x$, $w \preceq y$ and if $\tilde{w} \preceq x$ and $\tilde{w} \preceq y$ then $\tilde{w} \preceq w$.
(2) $x \preceq y$ implies $x + z \preceq y + z$.
(3) $x \preceq y$ implies $kx \preceq ky$ for all $k \in \mathbb{R}$ with $k \geq 0$.
(4) $|x| \preceq |y|$ implies $\|x\|_S \leq \|y\|_S$.

If $x \preceq y$, we write $y \succeq x$. We also define $|x| = x \vee (-x)$, $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Then $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

We have absolute integrability: if $f \in \mathcal{A}_R$ so is $|f|$. The lattice operations are defined for $F, G \in \mathcal{B}_R$ by $(F \vee G)(x) = \sup(F, G)(x) = \max(F(x), G(x))$. And, $(F \wedge G)(x) = \inf(F, G)(x) = \min(F(x), G(x))$.

THEOREM 22 (Banach lattice). (a) *$\mathcal{B}_R$ is a Banach lattice.* (b) *For $f, g \in \mathcal{A}_R$, define $f \preceq g$ if $F \leq G$ in $\mathcal{B}_R$. Then $\mathcal{A}_R$ is a Banach lattice isomorphic to $\mathcal{B}_R$.* (c) *Let $F, G \in \mathcal{B}_R$. Then $(F \vee G)' = F' \vee G'$, $(F \wedge G)' = F' \wedge G'$,*

$|F'| = |F|'$, $(F^+)' = (F')^+$, *and* $(F^-)' = (F')^-$. (d) *If* $f \in \mathcal{A}_R$ *then* $|f| \in \mathcal{A}_R$ *with primitive* $|F| \in \mathcal{B}_R$. *For each interval* $I \subset \mathbb{R}$, *we have* $|\int_I f| \geq |\int_I |f||$. *For each* $-\infty < x \leq \infty$, *we have* $|\int_{(-\infty,x)} f| = \int_{(-\infty,x)} |f|$. *And,* $\||f|\| = \|f\|$, $\|f^\pm\| \leq \|f\|$. (e) *If* $f \in \mathcal{A}_R$ *then* $f^\pm \in \mathcal{A}_R$ *with respective primitives* $F^\pm \in \mathcal{B}_R$. *Jordan decomposition*: $f = f^+ - f^-$. *And,* $\int_I f = \int_I f^+ - \int_I f^-$ *for every interval* $I \subset \overline{\mathbb{R}}$. (f) $\mathcal{A}_R$ *is distributive*: $f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$ *and* $f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h)$ *for all* $f, g, h \in \mathcal{A}_R$. (g) $\mathcal{A}_R$ *is modular*: *For all* $f, g \in \mathcal{A}_R$, *if* $f \preceq g$ *then* $f \vee (g \wedge h) = g \wedge (f \vee h)$ *for all* $h \in \mathcal{A}_R$. (h) *Let* F *and* G *be regulated functions on* $\mathbb{R}$ *with real limits at* $\pm\infty$. *Then*

$$F' \preceq G' \iff F(x-) - F(-\infty) \leq G(x-) - G(-\infty) \quad \forall x \in \mathbb{R} \tag{10}$$

$$\iff F(x+) - F(-\infty) \leq G(x+) - G(-\infty) \quad \forall x \in \mathbb{R}. \tag{11}$$

Proof. (a) Let $F, G \in \mathcal{B}_R$. Define $\Phi = (F \vee G)$ and $\Psi = (F \wedge G)$. We need to prove $\Phi, \Psi \in \mathcal{B}_R$. Let $a \in \mathbb{R}$ and prove Φ is left continuous at a. Suppose $F(a) > G(a)$. Given $\epsilon > 0$ there is $\delta > 0$ such that $|F(x) - F(a)| < \epsilon$, $|G(x) - G(a)| < \epsilon$ and $F(x) > G(x)$ whenever $x \in (a - \delta, a)$. For such x, $|\Phi(x) - \Phi(a)| = |F(x) - F(a)| < \epsilon$. If $F(a) = G(a)$, then $|\Phi(x) - \Phi(a)| \leq \max(|F(x) - F(a)|, |G(x) - G(a)|) < \epsilon$. Therefore, Φ is left continuous on $(-\infty, \infty]$. For $x \in (-\infty, 1/\delta)$ we can assume $\max(|F(x)|, |G(x)|) < \epsilon$. Therefore, $|\Phi(x)| < \epsilon$. Similarly, Φ has a right limit at each point so that $\Phi \in \mathcal{B}_R$. Similarly with the infimum. Hence, $\Phi, \Psi \in \mathcal{B}_R$.

The following properties follow immediately from the definition. If $F \leq G$, then for all $H \in \mathcal{B}_R$ we have $F + H \leq G + H$. If $F \leq G$ and $a \geq 0$ then $aF \leq aG$. If $|F| \leq |G|$, then $\|F\|_\infty \leq \|G\|_\infty$. Hence, $\mathcal{B}_R$ is a Banach lattice.

(b), (c) First, we show that $\mathcal{A}_R$ is closed under the operations $f \vee g$ and $f \wedge g$. For $f, g \in \mathcal{A}_R$, we have $f \vee g = \sup(f, g)$. This is h such that $f \preceq h$, $g \preceq h$, and if $f \preceq \tilde{h}$, $g \preceq \tilde{h}$, then $h \preceq \tilde{h}$. This last statement is equivalent to $F \leq H$, $G \leq H$, and if $F \leq \tilde{H}$, $G \leq \tilde{H}$, then $H \leq \tilde{H}$. But then $H = \max(F, G)$ and $h = H'$ so $f \vee g = (F \vee G)' \in \mathcal{A}_R$. Similarly, $f \wedge g = (F \wedge G)' \in \mathcal{A}_R$. And, $|F'| = F' \vee (-F') = F' \vee (-F)' = (F \vee (-F))' = |F|'$. The proofs that $(F^+)' = (F')^+$ and $(F^-)' = (F')^-$ are similar.

If $f, g \in \mathcal{A}_R$ and $f \preceq g$, then $F \leq G$. Let $h \in \mathcal{A}_R$. Then, $F + H \leq G + H$. But then $(F + H)' = F' + H' = f + h \preceq g + h$. If $k \in \mathbb{R}$ and $k \geq 0$, then $(kF)' = kF' = kf$ so $kf \preceq kg$. And, if $|f| \preceq |g|$ then $|F|' \preceq |G|'$ so $|F| \leq |G|$, that is, $|F(x)| \leq |G(x)|$ for all $x \in \mathbb{R}$. Then $\|f\| = \|F\|_\infty \leq \|G\|_\infty = \|g\|$. And, $\mathcal{A}_R$ is a Banach lattice that is isomorphic to $\mathcal{B}_R$.

(d) Note that $|\int_{(a,b)} f| = |F(b-) - F(a+)| \geq ||F(b-)| - |F(a+)|| = |\int_{(a,b)} |F|'| = |\int_{(a,b)} |F'||$. Similarly for other intervals. The other parts of (d) and (e) follow from (c) and the definitions. (f) The real-valued functions on any set form a distributed lattice due to inheritance from $\leq$ in $\mathbb{R}$. Therefore, $\mathcal{B}_R$ is a distributed lattice and hence so is $\mathcal{A}_R$. See [19, p. 484] for an elementary proof and for another property of distributed lattices. (g) Modularity is also inherited from $\leq$ in $\mathbb{R}$ via $\mathcal{B}_R$. (h) We have $F', G' \in \mathcal{A}_R$ with respective primitives $\Phi_F, \Phi_G \in \mathcal{B}_R$ given by $\Phi_F(x) = F(x-) - F(-\infty)$ and $\Phi_G(x) = G(x-) - G(-\infty)$. The definition of order then gives (10). The relations $F(x\pm) = \lim_{y\to x^\pm} F(y) = \lim_{y\to x^\pm} F(y-)$ then give (11). $\square$

For the function $f(t) = H_1(t)\sin(t^2)$, we have $f^+ = |f| = f$ and $f^- = 0$.

Notice that the definition of order allows us to integrate both sides of $f \preceq g$ in $\mathcal{A}_R$ to get $F \leq G$ in $\mathcal{B}_R$. The isomorphism allows us to differentiate both sides of $F \leq G$ in $\mathcal{B}_R$ to get $F' \preceq G'$ in $\mathcal{A}_R$. If F and G are regulated functions on $\mathbb{R}$ with real limits at $\pm\infty$, then the inequality $|F'| \preceq G'$ lets us prove that $|F(x\pm)| - |F(-\infty)| \leq G(x\pm) - G(-\infty)$ for all $x \in \mathbb{R}$. This is then a type of mean value theorem. See [18] where the inequality $|F'(x)| \leq G'(x)$ yields $|F(b) - F(a)| \leq G(b) - G(a)$ under the assumption that F is continuous or absolutely continuous and the first inequality holds except on a countable set or set of measure zero. In [18], G is required to be increasing.

A lattice is *complete* if every subset that is bounded above has a supremum. But $\mathcal{B}_R$ is not complete. Let $F_n(x) = H_1(x - 1/n)\sin(\pi/x)$ and let $S = \{F_n \mid n \in \mathbb{N}\}$. Then an upper bound for S is H_1 but $\sup(S)(x) = H_1(x)\sin(\pi/x)$, which is not regulated. Hence, $\mathcal{A}_R$ is also not complete.

In this section, we have considered only the most elementary lattice properties. Other questions, such as the relation of $\mathcal{A}_R$ and $\mathcal{B}_R$ to abstract L spaces and abstract M spaces, will be dealt with elsewhere.

10. Topology and measure

In this section, we define a topology on $\mathbb{R}$ so that regulated functions are continuous. We then describe $\mathcal{A}_R$ in terms of finitely additive measures.

The topology of half-open intervals or Sorgenfrey topology on the real line is defined by taking a base to be the collection of all intervals $(a, b]$ for all $-\infty < a < b < \infty$. See, for example, [3, p. 156]. Call the resulting topology τ_L. Then $(\mathbb{R}, \tau_L)$ is separable and first countable but not second countable. This topology is finer than the usual topology on $\mathbb{R}$, hence it is a Hausdorff space. However, it is not locally compact. Each interval $(a, b]$ is also closed. Observe that $[0, 1] \subset (-1, 0] \cup \bigcup_{n=1}^\infty (1/(n+1), 1/n]$, so $[0, 1]$ is not compact in τ_L. In fact, each compact set is countable.

All functions in $\mathcal{B}_R$ are continuous in $(\mathbb{R}, \tau_L)$. This follows from the fact that every regulated function is the uniform limit of a sequence of step functions [21, Section 7.13]. Hence, it is only necessary to consider H_1. But we have $H_1^{-1}((0.5, 1.5)) = (0, \infty) \in \tau_L$ and $H_1^{-1}((-0.5, 0.5)) = (-\infty, 0] \in \tau_L$. Notice that right continuous functions need not be continuous in $(\mathbb{R}, \tau_L)$. For example, $H_2^{-1}((0.5, 1.5)) = [0, \infty) \notin \tau_L$. See Example 3(e) for the definition of H_2. Functions such as $f(x) = \sin(1/x)$ for $x > 0$ and $f(x) = 0$, otherwise, and $g(x) = 1/x$ for $x > 0$ with $g(x) = 0$, otherwise, are continuous in $(\mathbb{R}, \tau_L)$, that is, left continuous functions are continuous as functions from $(\mathbb{R}, \tau_L)$ to $\mathbb{R}$ with the usual topology.

If X is a nonempty set, then an *algebra* on X is a collection of sets $\mathcal{A} \subset \mathcal{P}(X)$ such that (i) $\varnothing, X \in \mathcal{A}$ and if $E, F \in \mathcal{A}$ then (ii) $E \cup F \in \mathcal{A}$ and (iii) $E \setminus F \in \mathcal{A}$. Since $E \setminus F = (X \setminus F) \setminus (X \setminus E)$, (iii) can be replaced with $X \setminus E \in \mathcal{A}$. By de Morgan's laws, $\mathcal{A}$ is also closed under intersections. Hence, $\mathcal{A}$ is closed under finite unions and intersections. A set $E \subset \mathbb{R}$ is a $\mathcal{BV}$ set if $\chi_E \in \mathcal{BV}$. If $\mathcal{A}$ is an algebra, then $\nu : \mathcal{A} \to \mathbb{R}$ is a *finitely additive measure* if whenever $E, F \in \mathcal{A}$ such that $E \cap F = \varnothing$ then $\nu(E \cup F) = \nu(E) + \nu(F)$. Notice that $\nu(\varnothing) = 0$. We have the following results.

PROPOSITION 23. (a) *The $\mathcal{BV}$ sets form an algebra over* $\mathbb{R}$. (b) *If $f \in \mathcal{A}_R$ define $\nu_f(\varnothing) = 0$ and $\nu_f(E) = \int_{-\infty}^{\infty} f\chi_E$ for a $\mathcal{BV}$ set E. Then ν_f is a finitely additive measure on $\mathcal{BV}$ sets. If S is a $\mathcal{BV}$ set, then $|f(S)| \leq \|f\|(1 + V\chi_S)$.*

Proof. (a) Note that $\chi_\varnothing = 0 \in \mathcal{BV}$ and $\chi_\mathbb{R} = 1 \in \mathcal{BV}$. If E and F are $\mathcal{BV}$ sets, then $\chi_{\mathbb{R}\setminus E} = 1 - \chi_E \in \mathcal{BV}$. And, $\chi_{E\cup F} = \chi_{\mathbb{R}\setminus[(\mathbb{R}\setminus E)\cap(\mathbb{R}\setminus F)]} = 1 - \chi_{(\mathbb{R}\setminus E)\cap(\mathbb{R}\setminus F)} = 1 - \chi_{\mathbb{R}\setminus E}\chi_{\mathbb{R}\setminus F} = 1 - [1 - \chi_E][1 - \chi_F] = \chi_E + \chi_F - \chi_E\chi_F \in \mathcal{BV}$. (b) Since the functions of bounded variation are multipliers for $\mathcal{A}_R$, we have that ν_f is a real-valued function on $\mathcal{BV}$ sets. If E and F are disjoint $\mathcal{BV}$ sets, then $\chi_{E\cup F} = \chi_E + \chi_F$ and $\nu_f(E \cup F) = \int_{-\infty}^{\infty} f\chi_{E\cup F} = \int_{-\infty}^{\infty} f(\chi_E + \chi_F) = \int_{-\infty}^{\infty} f\chi_E + \int_{-\infty}^{\infty} f\chi_F = \nu_f(E) + \nu_f(F)$. We have $|f(S)| = |\int_{-\infty}^{\infty} f\chi_S| \leq \|f\|(1 + V\chi_S)$ using the Hölder inequality Theorem 13. □

A finitely additive measure ν on algebra $\mathcal{A}$ is *finite* if $\sup_{E\in\mathcal{A}} |\nu(E)| < \infty$. As finitely additive measures, elements of $\mathcal{A}_R$ need not be finite. For example, if $g(x) = \sin(x)/x$ for $x \neq 0$ then $f := T_g \in \mathcal{A}_R$. Let $E_n = \bigcup_{k=0}^{n} [2k\pi, (2k+1)\pi]$. Then $\nu_f(E_n) \to \infty$ as $n \to \infty$. We have the following connection between absolute integrability and the finitely additive measures that are finite in $\mathcal{A}_R$.

PROPOSITION 24. *Let $f \in \mathcal{A}_R$. Then ν_f is finite if and only if $F \in \mathcal{NBV}$.*

Proof. Let $F \in \mathcal{NBV}$ and E be a $\mathcal{BV}$ set. Then $|\nu_f(E)| = |\int_{-\infty}^{\infty} \chi_E \, dF| \leq VF$. Hence, $|\nu_f| \leq VF < \infty$.

Suppose $|\nu_f| < \infty$. Let (x_i, y_i) be disjoint intervals. Then $\sum |F(x_i) - F(y_i)| = \nu_f(\bigcup [x_i, y_i))$. Therefore, $F \in \mathcal{NBV}$. □

The $\mathcal{BV}$ sets do not form a σ-algebra. For example, the set $\bigcup_{n=1}^{\infty} [2n, 2n+1]$ is not a $\mathcal{BV}$ set.

Let $F : \overline{\mathbb{R}} \to \mathbb{R}$ be any function. Let I be an interval with endpoints $-\infty \leq a < b \leq \infty$. Define $\nu(\varnothing) = 0$ and $\nu(I) = F(b) - F(a)$. Then ν is a finitely additive measure on $\mathcal{BV}$ sets. But F need not be regulated so $\mathcal{A}_R$ does not contain all finitely additive measures on $\mathcal{BV}$ sets.

11. Convergence theorems

There are different modes of convergence in $\mathcal{A}_R$. If $\{f_n\} \subset \mathcal{A}_R$ then $f_n \to f \in \mathcal{A}_R$ *strongly* if $\|f_n - f\| \to 0$. The convergence is *weak in* $\mathcal{D}$ if $\langle f_n - f, \phi\rangle = \int_{-\infty}^{\infty}(f_n - f)\phi \to 0$ for all $\phi \in \mathcal{D}$ and the convergence is *weak in* $\mathcal{BV}$ if $\int_{-\infty}^{\infty}(f_n - f)g \to 0$ for all $g \in \mathcal{BV}$. Clearly, strong convergence implies weak convergence in $\mathcal{BV}$ (Theorem 15), which implies weak convergence in $\mathcal{D}$. We would like conditions under which $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$. Certainly weak convergence in $\mathcal{BV}$ is sufficient, take $g = 1$. Weak convergence in $\mathcal{D}$ is not sufficient. For example, let $f_n = \tau_n\delta$, for which $F_n(x) = H_1(x - n)$. Then $\{f_n\}$ converges weakly in $\mathcal{D}$ to 0 but $F_n(\infty) = 1$.

Strong convergence is equivalent to uniform convergence of the sequence of primitives.

THEOREM 25. *Let* $\{f_n\} \subset \mathcal{A}_R$ *and let* $\{F_n\} \subset \mathcal{B}_R$ *be the respective primitives. Suppose* $F : \overline{\mathbb{R}} \to \mathbb{R}$ *and* $F_n \to F$ *on* $\overline{\mathbb{R}}$. (a) $F_n \to F$ *uniformly on* $\overline{\mathbb{R}}$ *if and only if* $\|f_n - f\| \to 0$. (b) *If* $F_n \to F$ *uniformly on* $\overline{\mathbb{R}}$ *then* $F' \in \mathcal{A}_R$, $f_n \to F'$ *strongly and* $\int_{-\infty}^{\infty} f_n g \to \int_{-\infty}^{\infty} F'g$ *for each* $g \in \mathcal{BV}$. *In particular,* $\int_I f_n \to \int_I F'$ *for each interval* $I \subset \mathbb{R}$.

Part (b) follows from the Hölder inequality.

If $\{F_n\}$ is a sequence of continuous functions that converges uniformly to function F then F is continuous. A necessary and sufficient condition for F to be continuous is that the convergence be quasi-uniform. Because of our compactification of $\overline{\mathbb{R}}$ (Remark 3), Arzelà's theorem applies. See [7, p. 268]. We have a similar criteria for regulated functions.

THEOREM 26. *Let each function* $F_n : \overline{\mathbb{R}} \to \mathbb{R}$ *be regulated on* $\overline{\mathbb{R}}$. *Suppose* $F_n \to F$ *at each point in* $\overline{\mathbb{R}}$. *We require* $F(\pm\infty) \in \mathbb{R}$ *and* $F_n(\pm\infty) \in \mathbb{R}$ *for each* $n \in \mathbb{N}$ *but do not require* $F_n(\pm\infty) = \lim_{x\to\pm\infty} F_n(x)$. *Let* $\epsilon > 0$. *Suppose that for each* $a \in \overline{\mathbb{R}}$ *and each* $N \in \mathbb{N}$ *there exist* $n \geq N$ *and* $\delta > 0$ *such that if* $x \in (a - \delta, a + \delta)$ *then* $|F_n(x) - F(x)| < \epsilon$, *for* $a \in \mathbb{R}$. *For* $a = -\infty$ *we require* $x \in [-\infty, -1/\delta)$. *For* $a = \infty$ *we require* $x \in (1/\delta, \infty]$. *Then* F *is regulated on* $\overline{\mathbb{R}}$.

Proof. Let $\epsilon > 0$. Write

$$|F(x) - F(y)| \leq |F(x) - F_n(x)| + |F(y) - F_n(y)| + |F_n(x) - F_n(y)|.$$

We have $n \geq 1$ and $\delta_n > 0$ such that if $x \in (1/\delta_n, \infty]$ then $|F(x) - F_n(x)| < \epsilon$. Each F_n has a limit at ∞ so there is $\eta_n > 0$ such that if $x, y \in (1/\eta_n, \infty)$ then $|F_n(x) - F_n(y)| < \epsilon$. Take $\delta = \min(\delta_n, \eta_n)$. If $x, y \in (1/\delta, \infty)$, then $|F(x) - F(y)| < 3\epsilon$. Hence, $\lim_{x\to\infty} F(x)$ exists. Similarly, F has a limit at $-\infty$.

The proof that F has a left limit at $a \in \mathbb{R}$ is similar. Now the intervals become $x \in (a - \delta_n, a)$ and $x, y \in (a - \eta_n, a)$ and finally $x, y \in (a - \delta, a)$. Similarly for the right limit. □

See [9, Proposition 3.6] for a another sufficient condition on $\{F_n\}$, called bounded ε-variation, that ensures F is regulated.

Corollary 27. *If $\{F_n\} \subset \mathcal{B}_R$, then $F \in \mathcal{B}_R$. Now, as usual for functions in $\mathcal{B}_R$, we define $F_n(\pm\infty) = \lim_{x\to\pm\infty} F_n(x)$. Let $f_n = F_n'$ and $f = F'$. Then for each $x \in (-\infty, \infty]$ we have $\int_{(-\infty,x)} f_n \to \int_{(-\infty,x)} f$.*

Proof. Let $\epsilon > 0$. For $a \in \mathbb{R}$, write

$$|F(x) - F(a)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(a)| + |F_n(a) - F(a)|.$$

Since $F_n(a) \to F(a)$ we have $N_a \in \mathbb{N}$ such that if $n \geq N_a$ then $|F_n(a) - F(a)| < \epsilon$. We now have existence of $n \geq N_a$ and $\delta_n > 0$ such that if $x \in (a - \delta_n, a]$ then $|F(x) - F_n(x)| < \epsilon$. And, each F_n is left continuous at a so there is $\eta_n > 0$ such that if $x \in (a - \eta_n, a]$ then $|F_n(x) - F_n(a)| < \epsilon$. Take $\delta = \min(\delta_n, \eta_n)$. If $x \in (a - \delta, a]$, then $|F(x) - F(a)| < 3\epsilon$ so F is left continuous at a. Similarly, $\lim_{x\to-\infty} F(x) = 0 = F(-\infty)$ and $\lim_{x\to\infty} F(x) = F(-\infty) \in \mathbb{R}$. ☐

The Sorgenfrey topology of Section 10 makes each function in $\mathcal{B}_R$ continuous. However, no interval in $\overline{\mathbb{R}}$ is compact in this topology. Hence, Arzelà's theorem ([7, p. 268]), establishing that quasi-uniform convergence is a necessary and sufficient condition for the limit of a sequence of continuous functions to be continuous, is not applicable. We do not know necessary and sufficient conditions under which a sequence in $\mathcal{B}_R$ will converge to a function in $\mathcal{B}_R$. However, Theorem 25 and Theorem 32 give a sufficient condition while Theorem 26 and Theorem 29 with their corollaries give conditions under which left continuity is preserved.

Example 28. The example $f_n = \tau_n\delta$ in the first paragraph of this section shows the condition at ∞ cannot be dropped. For then, we have $F_n(x) = H_1(x - n)$. For each $x \in \mathbb{R}$, we have $F_n(x) \to 0$ but $F_n(\infty) = 1$ so $F(x) = 0$ for $x \in [-\infty, \infty)$ and $F(\infty) = 1$. Hence, $F \notin \mathcal{B}_R$. Although $f_n \to 0$ weakly in $\mathcal{D}$, we have $\int_{-\infty}^{\infty} f_n = F_n(\infty) = 1 \not\to 0$. Note that if $n < x < \infty$ then $|F_n(x) - F(x)| = 1$ so the condition at infinity in Theorem 26 is not satisfied.

Weak convergence in $\mathcal{D}$ of f_n to f is not sufficient for $\{F_n\}$ to converge to a function in $\mathcal{B}_R$. The following theorem gives conditions in addition to weak convergence in $\mathcal{D}$ so that $\int_{(-\infty,x)} f_n \to \int_{(-\infty,x)} f$.

Theorem 29. *Let $\{f_n\} \subset \mathcal{A}_R$ and let $F : \mathbb{R} \to \mathbb{R}$ be regulated and left continuous on $\mathbb{R}$ with real limits at $\pm\infty$. Suppose $\{F_n\}$ is uniformly bounded on each compact interval in $\mathbb{R}$ and $F_n \to F$ on $\overline{\mathbb{R}}$. Then $f_n \to F'$ weakly in $\mathcal{D}$ and $\int_{(-\infty,x)} f_n \to \int_{(-\infty,x)} F'$ for each $x \in (-\infty, \infty]$.*

The ordering introduced in Section 9 restores absolute convergence to the integral. Using this order, we can rephrase part of the above conditions in terms of dominated convergence.

COROLLARY 30. *Let* $\{f_n\} \subset \mathcal{A}_R$ *and let* $F : \mathbb{R} \to \mathbb{R}$ *be regulated and left continuous on* $\mathbb{R}$ *with real limits at* $\pm\infty$. *Suppose there is* $g \in \mathcal{A}_R$ *such that* $|f_n| \preceq g$ *for all* $n \geq 1$. *Suppose* $f_n \to f$ *weakly in* $\mathcal{D}$ *for some* $f \in \mathcal{D}'$. *Suppose* $F_n \to F$ *on* $\overline{\mathbb{R}}$. *Then* $f = F' \in \mathcal{A}_R$ *and* $\int_{(-\infty,x)} f_n \to \int_{(-\infty,x)} f$ *for each* $x \in (-\infty, \infty]$.

The proofs are easy modifications of Theorems 8 and 9 in [1]. See also Theorem 17 in [30].

EXAMPLE 31. Let $f_n = n\chi_{(0,1/n)} - \tau_{1/n}\delta$. Then $F_n(x) = nx$ for $0 \leq x \leq 1/n$ and $F_n(x) = 0$, otherwise. We have $F_n \to 0$ on $\overline{\mathbb{R}}$. The convergence is not uniform, since $F_n(1/n) = 1$. Theorem 25 is not applicable. The convergence $F_n \to 0$ on $\overline{\mathbb{R}}$ satisfies the conditions of Corollary 27. This then gives $\int_{(-\infty,x)} f_n \to 0$ for each $x \in (-\infty, \infty]$. Note that $|F_n(x)| \leq 1$ so $\{F_n\}$ is uniformly bounded. Theorem 29 then gives the same conclusion. Note that $0 \leq F_n \leq H_1$, so $|f_n| \preceq H_1' = \delta \in \mathcal{A}_R$. For $\phi \in \mathcal{D}$, we have $\int_{-\infty}^{\infty} f_n\phi = n\int_0^{1/n} \phi(x)\,dx - \phi(1/n) \to 0$ by continuity. Hence, $f_n \to 0$ weakly and Corollary 30 also gives the same conclusion.

The following theorem follows from the Hölder inequality. See also [31], [33].

THEOREM 32 (Uniform bounded variation). *Suppose* $\{f_n\} \subset \mathcal{A}_R$, $f \in \mathcal{A}_R$, $\{g_n\} \subset \mathcal{BV}$ *and* $g \in \mathcal{BV}$ *such that* $\|f_n - f\| \to 0$, $V(g_n - g) \to 0$ *and* $g_n(a) \to g(a)$ *for some* $a \in \mathbb{R}$. *Then* $\|g_n - g\|_\infty \to 0$ *and* $\int_{-\infty}^{\infty} f_n g_n \to \int_{-\infty}^{\infty} fg$.

Proof. First, note that

$$\begin{aligned}
|g_n(x) - g(x)| &\leq |g_n(a) - g(a)| + |[g_n(x) - g(x)] - [g_n(a) - g(a)]| \\
&\leq |g_n(a) - g(a)| + V(g_n - g) \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, $\|g_n - g\|_\infty \to 0$. Now use the Hölder inequality (Theorem 13) to write

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} f_n g_n - \int_{-\infty}^{\infty} fg\right| &= \left|\int_{-\infty}^{\infty} f_n(g_n - g) + (f_n - f)g\right| \\
&\leq \|f_n\|\|g_n - g\|_{\mathcal{BV}} + \|f_n - f\|\|g\|_{\mathcal{BV}} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$
□

COROLLARY 33. *Suppose* $\{f_n\} \subset \mathcal{A}_R$ *and* $f \in \mathcal{A}_R$ *such that* $\|f_n - f\| \to 0$. *If* $g \in \mathcal{BV}$, *then* $\int_{-\infty}^{\infty} f_n g \to \int_{-\infty}^{\infty} fg$.

COROLLARY 34. *Suppose* $f \in \mathcal{A}_R$, $\{g_n\} \subset \mathcal{BV}$ *and* $g \in \mathcal{BV}$ *such that* $V(g_n - g) \to 0$ *and* $g_n(a) \to g(a)$ *for some* $a \in \mathbb{R}$. *Then* $\int_{-\infty}^{\infty} fg_n \to \int_{-\infty}^{\infty} fg$.

Note that in Theorem 32 and the two corollaries we also get convergence on each subinterval of $\mathbb{R}$.

EXAMPLE 35 (Convolution). Suppose $f \in \mathcal{A}_R$ and $g \in AC$ such that both limits $\lim_{x\to\pm\infty} g(x)$ exist in $\mathbb{R}$. The convolution $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$ exists for all $x \in \mathbb{R}$. By the change of variables theorem, $f * g = g * f$. (Use $G(y) = x - y$ in Theorem 8.) By the Hölder inequality, $\|f * g\|_\infty \leq \|f\|(\|g\|_\infty + Vg)$. For each $x \in \mathbb{R}$, we have $g(x - \cdot) \in \mathcal{BV}$. For each $y, z \in \mathbb{R}$, we have $\lim_{x\to z} g(x - y) = g(z - y)$. And,

$$V\big(g(z-\cdot) - g(x-\cdot)\big) = \|g'(z-\cdot) - g'(x-\cdot)\|_1$$
$$\to 0 \quad \text{as } z \to x \text{ by continuity in the } L^1 \text{ norm.}$$

By Corollary 34, $f * g$ is uniformly continuous on $\mathbb{R}$.

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