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THE FOLIATED STRUCTURE OF CONTACT METRIC (κ,μ)-SPACES

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ABSTRACT. In this note, we study the foliated structure of a contact metric (κ,μ)-space. In particular, using the theory of Legendre foliations, we give a geometric interpretation of the Boeckx's classification of contact metric (κ,μ)-spaces and we find necessary conditions for a contact manifold to admit a compatible contact metric (κ,μ)-structure. Finally, we prove that any contact metric (κ,μ)-space M whose Boeckx invariant I_M is different from ± 1 admits a compatible Sasakian or Tanaka–Webster parallel structure according to the circumstance that $|I_M|>1$ or $|I_M|<1$, respectively.

1. Introduction

A contact metric manifold (M,φ,ξ,η,g) is called a contact metric (κ,μ)-manifold if the Reeb vector field ξ belongs to the (κ,μ)-nullity distribution, that is, the curvature tensor field satisfies, for all vector fields X and Y on M,

$$R_{XY}\xi=\kappa\big(\eta(Y)X-\eta(X)Y\big)+\mu\big(\eta(Y)hX-\eta(X)hY\big) \tag{1}$$

for some real numbers κ and μ; here $2h$ denotes the Lie derivative of φ in the direction of ξ. This definition was introduced by Blair, Koufogiorgos and Papantoniou [4] and can be regarded as a generalization both of the Sasakian condition $R_{XY}\xi=\eta(Y)X-\eta(X)Y$ and of those contact metric manifolds satisfying $R_{XY}\xi=0$ which were studied by Blair in [1].

Lately, contact metric (κ,μ)-manifolds have attracted the attention of many authors and various recent papers have appeared on this topic (e.g., [6], [12], [14]). In fact, there are many motivations for studying (κ,μ)-manifolds: the first is that, in the non-Sasakian case (that is for $\kappa\neq 1$), the condition (1) determines the curvature completely; moreover, while the values of κ and μ

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may change, the form of (1) is invariant under $\mathcal{D}$-homothetic deformations; finally, a complete classification of contact metric (κ,μ)-manifolds is known [5] and there are nontrivial examples of such manifolds, the most important being the tangent sphere bundle of a Riemannian manifold of constant sectional curvature with its usual contact metric structure.

One of the peculiarities of contact metric (κ,μ)-manifolds is that they give rise to three mutually orthogonal involutive distributions $\mathcal{D}(\lambda)$, $\mathcal{D}(-\lambda)$ and $\mathbb{R}\xi=\mathcal{D}(0)$, corresponding to the eigenspaces λ, $-\lambda$ and 0 of the operator h, where $\lambda=\sqrt{1-\kappa}$. In particular, $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ define two transverse Legendre foliations of M so that any contact metric (κ,μ)-manifold is canonically endowed with a bi-Legendrian structure. The study of the bi-Legendrian structure of a contact metric (κ,μ)-manifold was initiated in [9], where the following characterization of contact metric (κ,μ)-manifolds in terms of Legendre foliations was proven.

THEOREM 1 ([9]). *Let (M,φ,ξ,η,g) be a non-Sasakian contact metric manifold. Then M is a contact metric (κ,μ)-manifold if and only if it admits two mutually orthogonal Legendre distributions L and Q and a unique linear connection $\bar{\nabla}$ satisfying the following properties*:

(i) $\bar{\nabla}L\subset L$, $\bar{\nabla}Q\subset Q$,
(ii) $\bar{\nabla}\eta=0$, $\bar{\nabla}d\eta=0$, $\bar{\nabla}g=0$, $\bar{\nabla}\varphi=0$, $\bar{\nabla}h=0$,
(iii) $\bar{T}(X,Y)=2d\eta(X,Y)\xi$ *for all* $X,Y\in\Gamma(\mathcal{D})$, $\bar{T}(X,\xi)=[\xi,X_L]_Q+[\xi,X_Q]_L$ *for all* $X\in\Gamma(TM)$,

where $\bar{T}$ denotes the torsion tensor field of $\bar{\nabla}$ and X_L and X_Q are, respectively, the projections of X onto the subbundles L and Q of TM. Furthermore, L and Q are integrable and coincide with the eigenspaces $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ of the operator h, and $\bar{\nabla}$ coincides in fact with the bi-Legendrian connection ∇^{bl} associated to the bi-Legendrian structure (L,Q) (cf. [7], [8]).

Using the approach of Theorem 1, in [10] the authors recently were able to prove the strong result that any invariant submanifold of a non-Sasakian contact metric (κ,μ)-space is totally geodesic.

In this paper, the study of the foliated structure of a contact metric (κ,μ)-space is carried on. We start with the following question, which generalizes the well-known problem of finding conditions ensuring the existence of Sasakian structures compatible with a given contact form: let (M,η) be a contact manifold; then does (M,η) admit a compatible contact metric (κ,μ)-structure? As a matter of fact, the answer to this question involves the foliated nature of contact metric (κ,μ)-spaces. In particular, we find necessary conditions, in terms of bi-Legendrian structures, for a contact manifold (M,η) to admit a compatible contact metric (κ,μ)-structure (cf. Theorem 6 and Theorem 7).

Moreover, we interpret the Boeckx classification [5] of contact metric (κ,μ)-manifolds in terms of the Pang classification [16] of Legendre foliations, clarifying the geometric meaning of the invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$$

which was defined by Boeckx in [5] in a rather obscure way.

It follows that contact metric (κ,μ)-spaces divide into 5 main classes, according to the behavior of each Legendre foliation $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$. We prove that those classes of contact metric (κ,μ)-manifolds such that $|I_M| \neq 1$ admit a family $(\varphi_{a,b},\xi,\eta,g_{a,b})$ of compatible contact metric $(\kappa_{a,b},\mu_{a,b})$-structures, where the constants $\kappa_{a,b}$, $\mu_{a,b}$ are parameterized by the real numbers a and b satisfying the relation $ab=(2-\mu)^2-4(1-\kappa)$, namely,

$$\kappa_{a,b}=1-\frac{(a-b)^2}{16}, \qquad \mu_{a,b}=2-\frac{a+b}{2}.$$

In particular, we show that, in the case $|I_M|>1$, choosing $a=b$, the above contact metric $(\kappa_{a,b},\mu_{a,b})$-structures are in fact Sasakian. Thus, rather surprisingly, it follows that any contact metric (κ,μ)-manifold such that $|I_M|>1$ admits a compatible Sasakian structure and hence, under the assumption of compactness, for each $1\leq p\leq 2n$, the pth Betti number of M is even, where $2n+1$ is the dimension of the manifold. At the knowledge of the author, the last one is the first topological obstruction for contact metric (κ,μ)-manifolds known at the moment. Whereas, if $|I_M|<1$, choosing $a=-b$, we obtain a family of Tanaka–Webster parallel structures, that is, contact metric structures whose Tanaka–Webster connection preserves the Tanaka–Webster torsion and the Tanaka–Webster curvature [6].

Finally, we show that those contact metric manifolds with $|I_M|=1$ admit a family (φ_c,ξ,η,g_c) of compatible contact metric (κ_c,μ_c)-structures, with

$$\kappa_c=1-\frac{c^2}{16}, \qquad \mu_c=2\left(1-\frac{c}{4}\right),$$

where c varies in the interval $(0,4]$ in the case $I_M=1$ and $[-4,0)$ in the case $I_M=-1$.

2. Preliminaries

2.1. Contact geometry. A *contact manifold* is a $(2n+1)$-dimensional smooth manifold M which carries a 1-form η, called *contact form*, satisfying $\eta\wedge(d\eta)^n\neq 0$ everywhere on M. It is well known that given η there exists a unique vector field ξ, called *Reeb vector field*, such that $i_\xi\eta=1$ and $i_\xi d\eta=0$. In the sequel, we will denote by $\mathcal{D}$ the $2n$-dimensional distribution defined by $\ker(\eta)$, called the *contact distribution*. It is easy to see that the Reeb vector field is an infinitesimal automorphism with respect to the contact distribution and the tangent bundle of M splits as the direct sum $TM=\mathcal{D}\oplus\mathbb{R}\xi$.

It is well known that any contact manifold (M,η) admits a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\varphi^2 = -I + \eta\otimes\xi, \qquad d\eta(X,Y) = g(X,\varphi Y), \\ g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2}$$

for all $X,Y\in\Gamma(TM)$, from which it follows that $\varphi\xi = 0$, $\eta\circ\varphi = 0$ and $\eta = g(\cdot,\xi)$. The structure (φ,ξ,η,g) is called a *contact metric structure* and the manifold M endowed with such a structure is said to be a *contact metric manifold*. In a contact metric manifold M, the $(1,1)$-tensor field $h := \frac{1}{2}\mathcal{L}_\xi\varphi$ is symmetric and satisfies

$$h\ \xi = 0, \qquad \eta\circ h = 0, \qquad h\varphi + \varphi h = 0, \\ \nabla\xi = -\varphi - \varphi h, \qquad \mathrm{tr}(h) = \mathrm{tr}(\varphi h) = 0, \tag{3}$$

where ∇ is the Levi Civita connection of (M,g). The tensor field h vanishes identically if and only if the Reeb vector field is Killing, and in this case the contact metric manifold in question is said to be *K-contact*.

Moreover, in any contact metric manifold one can consider the tensor field N_φ, defined by

$$N_\varphi(X,Y) := \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] + 2d\eta(X,Y)\xi$$

for all $X,Y\in\Gamma(TM)$. The tensor field N_φ satisfies the following formula, which will turn out very useful in the sequel,

$$\varphi N_\varphi(X,Y) + N_\varphi(\varphi X,Y) = 2\eta(X)hY \tag{4}$$

for all $X,Y\in\Gamma(TM)$, from which, in particular, it follows that

$$\eta(N_\varphi(\varphi X,Y)) = 0. \tag{5}$$

A contact metric manifold such that N_φ vanishes identically is said to be *Sasakian*. In terms of the covariant derivative, the Sasakian condition can be expressed by the following formula

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{6}$$

whereas, in term of the curvature tensor field, the Sasakian condition is

$$R_{XY}\xi = \eta(Y)X - \eta(X)Y.$$

Any Sasakian manifold is K-contact, and in dimension 3 the converse also holds (see [2] for more details).

A recent generalization of Sasakian manifolds is the notion of *contact metric* (κ,μ)*-manifolds* [4]. Let (M,φ,ξ,η,g) be a contact metric manifold. If the curvature tensor field of the Levi Civita connection satisfies

$$R_{XY}\xi = \kappa\big(\eta(Y)X - \eta(X)Y\big) + \mu\big(\eta(Y)hX - \eta(X)hY\big) \tag{7}$$

for some $\kappa,\mu\in\mathbb{R}$, we say that (M,φ,ξ,η,g) is a *contact metric* (κ,μ)*-manifold* (or that ξ belongs to the (κ,μ)-nullity distribution). This definition was introduced and deeply studied by Blair, Koufogiorgos and Papantoniou in [4]. Among other things, the authors proved the following result.

THEOREM 2 ([4]). *Let (M,φ,ξ,η,g) be a contact metric (κ,μ)-manifold. Then necessarily $\kappa\leq 1$. Moreover, if $\kappa=1$ then $h=0$ and (M,φ,ξ,η,g) is Sasakian; if $\kappa<1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}(0)=\mathbb{R}\xi$, $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ corresponding to the eigenspaces of h, where $\lambda=\sqrt{1-\kappa}$.*

Given a non-Sasakian contact metric (κ,μ)-manifold M, Boeckx [5] proved that the number

$$I_M:=\frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}},$$

is an invariant of the contact metric (κ,μ)-structure, and he demonstrated that two non-Sasakian contact metric (κ,μ)-manifolds $(M_1,\varphi_1,\xi_1,\eta_1,g_1)$ and $(M_2,\varphi_2,\xi_2,\eta_2,g_2)$ are locally isometric as contact metric manifolds if and only if $I_{M_1}=I_{M_2}$. Then the invariant I_M has been used by Boeckx for giving a full classification of contact metric (κ,μ)-spaces.

The standard example of contact metric (κ,μ)-manifolds is given by the tangent sphere bundle T_1N of a manifold of constant curvature c endowed with its standard contact metric structure. In this case, $\kappa=c(2-c)$, $\mu=-2c$ and $I_{T_1N}=\frac{1+c}{|1-c|}$. Therefore, as c varies over the reals, I_{T_1N} takes on every value strictly greater than -1. Moreover, one can easily find that $I_{T_1N}<1$ if and only if $c<0$.

2.2. Legendre foliations. Let (M,η) be a $(2n+1)$-dimensional contact manifold. Notice that the condition $\eta\wedge(d\eta)^n\neq 0$ implies that the contact distribution is never integrable. One can prove that in fact the maximal dimension of an integrable subbundle of $\mathcal{D}$ is n. This motivates the following definition. A *Legendre distribution* on a contact manifold (M,η) is an n-dimensional subbundle L of the contact distribution such that $d\eta(X,X')=0$ for all $X,X'\in\Gamma(L)$. Then by a *Legendre foliations* of (M,η) we mean a foliation $\mathcal{F}$ of M whose tangent bundle $L=T\mathcal{F}$ is a Legendre distribution, according to the above definition.

Legendre foliations have been extensively investigated in recent years from various points of views. In particular, Pang [16] provided a classification of Legendre foliations by means of a bilinear symmetric form $\Pi_{\mathcal{F}}$ on the tangent bundle of the foliation $\mathcal{F}$, defined by

$$\Pi_{\mathcal{F}}(X,X')=-(\mathcal{L}_X\mathcal{L}_{X'}\eta)(\xi)=2d\eta([\xi,X],X').$$

He called a Legendre foliation *nondegenerate*, *degenerate* or *flat* according to the circumstance that the bilinear form $\Pi_{\mathcal{F}}$ is nondegenerate, degenerate or vanishes identically, respectively. In terms of Lie brackets, the flat condition is equivalent to the requirement that $[\xi, X] \in \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$. Two interesting subclasses of nondegenerate Legendre foliations are given by those for which $\Pi_{\mathcal{F}}$ is positive definite and negative definite; we then speak of *positive definite* and *negative definite Legendre foliations*, respectively.

For a nondegenerate Legendre foliation $\mathcal{F}$, Libermann [15] defined a linear map $\Lambda_{\mathcal{F}} : TM \longrightarrow T\mathcal{F}$, whose kernel is $T\mathcal{F} \oplus \mathbb{R}\xi$, such that

$$\Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}} Z, X) = d\eta(Z, X) \tag{8}$$

for any $Z \in \Gamma(TM)$, $X \in \Gamma(T\mathcal{F})$. The operator $\Lambda_{\mathcal{F}}$ is surjective, satisfies $(\Lambda_{\mathcal{F}})^2 = 0$ and

$$\Lambda_{\mathcal{F}}[\xi, X] = \frac{1}{2} X \tag{9}$$

for all $X \in \Gamma(T\mathcal{F})$. Then we can extend $\Pi_{\mathcal{F}}$ to a symmetric bilinear form on TM by putting

$$\overline{\Pi}_{\mathcal{F}}(Z, Z') := \begin{cases} \Pi_{\mathcal{F}}(Z, Z'), & \text{if } Z, Z' \in \Gamma(T\mathcal{F}), \\ \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}} Z, \Lambda_{\mathcal{F}} Z'), & \text{otherwise.} \end{cases}$$

If (M, η) admits two transversal Legendre distributions L_1 and L_2, we say that (M, η, L_1, L_2) is an *almost bi-Legendrian manifold.* Thus, in particular, the tangent bundle of M splits up as the direct sum $TM = L_1 \oplus L_2 \oplus \mathbb{R}\xi$. When both L_1 and L_2 are integrable we speak of *bi-Legendrian manifold.* An (almost) bi-Legendrian manifold is said to be flat, degenerate or nondegenerate if and only if both the Legendre distributions are flat, degenerate or nondegenerate, respectively. Any contact manifold (M, η) endowed with a Legendre distribution L admits a canonical almost bi-Legendrian structure. Indeed let (φ, ξ, η, g) be a compatible contact metric structure. Then by the relation $d\eta(\phi\cdot, \phi\cdot) = d\eta$ it easily follows that $Q := \phi L$ is a Legendre distribution on M which is g-orthogonal to L. Q is usually called the *conjugate Legendre distribution* of L and in general is not integrable, even if L is.

In [7] (see also [8]), a canonical connection, which plays an important role in the study of almost bi-Legendrian manifolds, has been introduced.

THEOREM 3 ([7]). *Let (M, η, L_1, L_2) be an almost bi-Legendrian manifold. There exists a unique connection ∇^{bl} such that*

(i) $\nabla^{bl} L_1 \subset L_1$, $\nabla^{bl} L_2 \subset L_2$,
(ii) $\nabla^{bl} \xi = 0$, $\nabla^{bl} d\eta = 0$,
(iii) $T^{bl}(X, Y) = 2d\eta(X, Y)\xi$ *for all* $X \in \Gamma(L_1)$, $Y \in \Gamma(L_2)$, $T^{bl}(X, \xi) = [\xi, X_{L_1}]_{L_2} + [\xi, X_{L_2}]_{L_1}$ *for all* $X \in \Gamma(TM)$,

where T^{bl} denotes the torsion tensor field of ∇^{bl} and X_{L_1} and X_{L_2} the projections of X onto the subbundles L_1 and L_2 of TM, respectively.

Such a connection is called the *bi-Legendrian connection* of the almost bi-Legendrian manifold (M,η,L_1,L_2). We recall also the complete expression of the torsion tensor field of ∇^{bl},

$$T^{bl}(X,Y) = -[X_{L_1},Y_{L_1}]_{L_2\oplus\mathbb{R}\xi} - [X_{L_2},Y_{L_2}]_{L_1\oplus\mathbb{R}\xi} + 2d\eta(X,Y)\xi + \eta(Y)([\xi,X_{L_1}]_{L_2} + [\xi,X_{L_2}]_{L_1}) - \eta(X)([\xi,Y_{L_1}]_{L_2} + [\xi,Y_{L_2}]_{L_1}). \tag{10}$$

3. The main results

By Theorem 2, it follows that any non-Sasakian contact metric (κ,μ)-manifold is endowed with a canonical bi-Legendrian structure given by the mutually orthogonal integrable distributions $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$. Therefore, we can classify non-Sasakian contact metric (κ,μ)-manifolds by using the aforementioned Pang's classification of Legendre foliations based on the behavior of the invariants $\Pi_{\mathcal{D}(\lambda)}$ and $\Pi_{\mathcal{D}(-\lambda)}$. The explicit expression of the invariants of the Legendre foliations defined by $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ was found in in [9]:

$$\Pi_{\mathcal{D}(\lambda)} = \frac{(\lambda+1)^2-\kappa-\mu\lambda}{\lambda}g\Big|_{\mathcal{D}(\lambda)\times\mathcal{D}(\lambda)} = \big(2\sqrt{1-\kappa}-\mu+2\big)g|_{\mathcal{D}(\lambda)\times\mathcal{D}(\lambda)}, \tag{11}$$

$$\Pi_{\mathcal{D}(-\lambda)} = \frac{-(\lambda-1)^2+\kappa-\mu\lambda}{\lambda}g\Big|_{\mathcal{D}(-\lambda)\times\mathcal{D}(-\lambda)} = \big(-2\sqrt{1-\kappa}-\mu+2\big)g|_{\mathcal{D}(-\lambda)\times\mathcal{D}(-\lambda)}. \tag{12}$$

Using (11)–(12) we can classify non-Sasakian contact metric (κ,μ)-manifolds as follows.

THEOREM 4. *Let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-manifold. Then the bi-Legendrian structure $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ associated to (M,φ,ξ,η,g) is nonflat. More precisely, only one among the following cases occurs*:

(I) *both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are positive definite*;
(II) *$\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is negative definite*;
(III) *both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are negative definite*;
(IV) *$\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is flat*;
(V) *$\mathcal{D}(\lambda)$ is flat and $\mathcal{D}(-\lambda)$ is negative definite.*

Furthermore, M belongs to the class (I), (II), (III), (IV), (V) *if and only if $I_M>1$, $-1<I_M<1$, $I_M<-1$, $I_M=1$, $I_M=-1$, respectively.*

Proof. By (11)–(12), we have immediately that $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are either positive definite or positive negative or flat, depending on the sign of the functions $f_1(\kappa,\mu)=2\sqrt{1-\kappa}-\mu+2$ and $f_2(\kappa,\mu)=-2\sqrt{1-\kappa}-\mu+2$. Since

$f_1(\kappa,\mu)$ and $f_2(\kappa,\mu)$ both vanish if and only if $\kappa=1$, the bi-Legendrian structure $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ turns out to be nonflat. Moreover, one easily finds that $f_1(\kappa,\mu)>0$ if and only if $I_M>-1$ and $f_2(\kappa,\mu)>0$ if and only if $I_M>1$. Consequently, taking into account (11)–(12), the cases $\Pi_{\mathcal{D}(\lambda)}$ negative definite and $\Pi_{\mathcal{D}(-\lambda)}$ positive definite, $\Pi_{\mathcal{D}(\lambda)}=0$ and $\Pi_{\mathcal{D}(-\lambda)}$ positive definite, $\Pi_{\mathcal{D}(\lambda)}$ negative definite and $\Pi_{\mathcal{D}(-\lambda)}=0$ cannot occur, and the remaining combinations of all possible signs of $f_1(\kappa,\mu)$ and of $f_2(\kappa,\mu)$ give the claimed assertion. □

Using Theorem 4, we are able to study the following interesting problem. It is a well-known question in contact geometry whether, given a contact manifold (M,η), there exists a Sasakian structure on M compatible with the contact form η. Now we generalize this problem and we ask whether, given a contact manifold (M,η), there exists a compatible contact metric structure (φ,ξ,η,g) such that (M,φ,ξ,η,g) is a contact metric (κ,μ)-manifold. In order to answer this question, we need to recall the following lemma proven in [8].

Lemma 5 ([8]). *Let (M,φ,ξ,η,g) be a contact metric manifold endowed with a Legendre distribution L. Let $Q:=\varphi L$ be the conjugate Legendre distribution of L and ∇^{bl} the bi-Legendrian connection associated to (L,Q). Then the following statements are equivalent*:

(i) $\nabla^{bl}g=0$.

(ii) $\nabla^{bl}\varphi=0$.

(iii) *For all $X,X'\in\Gamma(L)$ and $Y,Y'\in\Gamma(Q)$, $\nabla^{bl}_X X'=-(\varphi[X,\varphi X'])_L$ and $\nabla^{bl}_Y Y'=-(\varphi[Y,\varphi Y'])_Q$, and the tensor field h maps the subbundle L onto L and the subbundle Q onto Q.*

(iv) *g is a bundle-like metric with respect both to the distribution $L\oplus\mathbb{R}\xi$ and to the distribution $Q\oplus\mathbb{R}\xi$.*

Furthermore, assuming L and Q integrable, (i)–(iv) *are equivalent to the total geodesicity* (*with respect to the Levi Civita connection of g*) *of the Legendre foliations defined by L and Q.*

Theorem 6. *Let (M,η) be a contact manifold endowed with a bi-Legendrian structure $(\mathcal{F}_1,\mathcal{F}_2)$ such that $\nabla^{bl}\Pi_{\mathcal{F}_1}=\nabla^{bl}\Pi_{\mathcal{F}_2}=0$. Assume that one of the following conditions holds*

(I) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite and there exist two positive numbers a and b such that $\overline{\Pi}_{\mathcal{F}_1}=ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2}=ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$,*

(II) *$\mathcal{F}_1$ is positive definite, $\mathcal{F}_2$ is negative definite and there exist $a>0$ and $b<0$ such that $\overline{\Pi}_{\mathcal{F}_1}=ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2}=ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$,*

(III) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are negative definite and there exist two negative numbers a and b such that $\overline{\Pi}_{\mathcal{F}_1}=ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2}=ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$.*

Then (M,η) admits a compatible contact metric structure (φ,ξ,η,g) such that

(i) *if $a=b$, (M,φ,ξ,η,g) is a Sasakian manifold*;

(ii) *if* $a \neq b$, (M,φ,ξ,η,g) *is a contact metric* (κ,μ)*-manifold, whose associated bi-Legendrian structure is* $(\mathcal{F}_1,\mathcal{F}_2)$, *where*

$$\kappa = 1 - \frac{(a-b)^2}{16}, \qquad \mu = 2 - \frac{a+b}{2}. \tag{13}$$

Proof. We consider, for each Legendre foliation $\mathcal{F}_1$ and $\mathcal{F}_2$, the Libermann operators $\Lambda_{\mathcal{F}_1} : TM \longrightarrow T\mathcal{F}_1$ and $\Lambda_{\mathcal{F}_2} : TM \longrightarrow T\mathcal{F}_2$ defined by (8). Then we set

$$\begin{gathered} g|_{T\mathcal{F}_1\times T\mathcal{F}_1} := \tfrac{1}{a}\Pi_{\mathcal{F}_1}, \qquad g|_{T\mathcal{F}_2\times T\mathcal{F}_2} := \tfrac{1}{b}\Pi_{\mathcal{F}_2}, \\ g := \eta\otimes\eta \quad \text{elsewhere.} \end{gathered} \tag{14}$$

That g is a Riemannian metric follows from the fact that the bilinear maps $\Pi_{\mathcal{F}_1}$ and $\Pi_{\mathcal{F}_2}$ are symmetric and, by the assumptions (I)–(III), they are positive or negative definite according to the signs of a and b, respectively, in such a way that the bilinear forms $\frac{1}{a}\Pi_{\mathcal{F}_1}$ and $\frac{1}{b}\Pi_{\mathcal{F}_2}$ are always positive definite. In particular by (14), we have that $\mathcal{F}_1 = \mathcal{F}_2^{\perp}\cap\mathcal{D}$ and $\mathcal{F}_2 = \mathcal{F}_1^{\perp}\cap\mathcal{D}$. Next, let us define a tensor field φ by

$$\varphi Z := \begin{cases} -b\Lambda_{\mathcal{F}_2}Z, & \text{if } Z\in\Gamma(T\mathcal{F}_1), \\ -a\Lambda_{\mathcal{F}_1}Z, & \text{if } Z\in\Gamma(T\mathcal{F}_2), \\ 0, & \text{if } Z\in\Gamma(\mathbb{R}\xi). \end{cases} \tag{15}$$

Notice that, by definition, φ maps $T\mathcal{F}_1$ onto $T\mathcal{F}_2$ and $T\mathcal{F}_2$ onto $T\mathcal{F}_1$. Moreover, for any $X,X'\in\Gamma(T\mathcal{F}_1)$,

$$\begin{aligned} \Pi_{\mathcal{F}_1}(\varphi^2X,X') &= ab\Pi_{\mathcal{F}_1}(\Lambda_{\mathcal{F}_1}\Lambda_{\mathcal{F}_2}X,X') = abd\eta(\Lambda_{\mathcal{F}_2}X,X') \\ &= -abd\eta(X',\Lambda_{\mathcal{F}_2}X) = -ab\Pi_{\mathcal{F}_2}(\Lambda_{\mathcal{F}_2}X',\Lambda_{\mathcal{F}_2}X) \\ &= -ab\overline{\Pi}_{\mathcal{F}_2}(X,X') = -\frac{ab}{ab}\overline{\Pi}_{\mathcal{F}_1}(X,X') = -\Pi_{\mathcal{F}_1}(X,X') \end{aligned}$$

from which it follows that $\varphi^2X = -X$. Analogously, one can prove that $\varphi^2Y = -Y$ for all $Y\in\Gamma(T\mathcal{F}_2)$. Thus, $\varphi^2 = -I+\eta\otimes\xi$. We prove that (φ,ξ,η,g) is in fact a contact metric structure. Indeed, for all $X,X'\in\Gamma(T\mathcal{F}_1)$

$$\begin{aligned} g(\varphi X,\varphi X') &= b^2g(\Lambda_{\mathcal{F}_2}X,\Lambda_{\mathcal{F}_2}X') = b\Pi_{\mathcal{F}_2}(\Lambda_{\mathcal{F}_2}X,\Lambda_{\mathcal{F}_2}X') \\ &= b\overline{\Pi}_{\mathcal{F}_2}(X,X') = \frac{1}{a}\Pi_{\mathcal{F}_1}(X,X') = g(X,X'). \end{aligned}$$

Analogously, one has $g(\varphi Y,\varphi Y') = g(Y,Y')$ for all $Y,Y'\in\Gamma(T\mathcal{F}_2)$, so that we can conclude that $g(\varphi\cdot,\varphi\cdot) = g(\cdot,\cdot)-\eta\otimes\eta$. Furthermore, for all $X\in\Gamma(T\mathcal{F}_1)$ and $Y\in\Gamma(T\mathcal{F}_2)$ we have

$$\begin{aligned} g(X,\varphi Y) &= \frac{1}{a}\Pi_{\mathcal{F}_1}(X,\varphi Y) = -\Pi_{\mathcal{F}_1}(X,\Lambda_{\mathcal{F}_1}Y) \\ &= -\Pi_{\mathcal{F}_1}(\Lambda_{\mathcal{F}_1}Y,X) = -d\eta(Y,X) = d\eta(X,Y) \end{aligned}$$

and, in the same way, $g(Y,\varphi X)=d\eta(Y,X)$. Moreover, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually orthogonal with respect to g and they are Legendre foliations, we have $d\eta(X,X')=0=g(X,\varphi X')$ and $d\eta(Y,Y')=0=g(Y,\varphi Y')$ for all $X,X'\in\Gamma(T\mathcal{F}_1)$ and for all $Y,Y'\in\Gamma(T\mathcal{F}_2)$. Therefore, $d\eta=g(\cdot,\varphi\cdot)$ and (φ,ξ,η,g) is contact metric structure. Notice that $\mathcal{F}_1$ and $\mathcal{F}_2$ are conjugate Legendre foliations with respect to (φ,ξ,η,g), since $\varphi(T\mathcal{F}_1)=T\mathcal{F}_2$ and $\varphi(T\mathcal{F}_2)=T\mathcal{F}_1$. Now, since $\nabla^{bl}\Pi_{\mathcal{F}_1}=\nabla^{bl}\Pi_{\mathcal{F}_2}=0$, we have that the bi-Legendrian connection preserves the Riemannian metric g and this, by Lemma 5, implies that $\nabla^{bl}\varphi=0$ and $h:=\frac{1}{2}\mathcal{L}_\xi\varphi$ preserves the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. Then, as $\ker(\Lambda_{\mathcal{F}_1})=T\mathcal{F}_1\oplus\mathbb{R}\xi$ and by (9) we have, for any $X\in\Gamma(T\mathcal{F}_1)$, $\varphi([\xi,X]_{T\mathcal{F}_2})=-a\Lambda_{\mathcal{F}_1}([\xi,X]_{T\mathcal{F}_2})=-a\Lambda_{\mathcal{F}_1}[\xi,X]=-\frac{a}{2}X$, hence

$$\varphi X=\frac{2}{a}[\xi,X]_{T\mathcal{F}_2}. \tag{16}$$

Analogously, one can prove that

$$\varphi Y=\frac{2}{b}[\xi,Y]_{T\mathcal{F}_1} \tag{17}$$

for all $Y\in\Gamma(T\mathcal{F}_2)$. Thus, for any $X\in\Gamma(T\mathcal{F}_1)$, $2hX=[\xi,\varphi X]-\varphi[\xi,X]=[\xi,\varphi X]_{T\mathcal{F}_1}+[\xi,\varphi X]_{T\mathcal{F}_2}-\varphi([\xi,X]_{T\mathcal{F}_1})-\varphi([\xi,X]_{T\mathcal{F}_2})$, from which, as $h(T\mathcal{F}_1)\subset T\mathcal{F}_1$, it follows that $2hX-[\xi,\varphi X]_{T\mathcal{F}_1}+\varphi([\xi,X]_{T\mathcal{F}_2})=[\xi,\varphi X]_{T\mathcal{F}_2}-\varphi([\xi,X]_{T\mathcal{F}_1})=0$. Hence, using (16)–(17),

$$hX=\frac{1}{2}\big([\xi,\varphi X]_{T\mathcal{F}_1}-\varphi([\xi,X]_{T\mathcal{F}_2})\big)=\frac{1}{2}\left(-\frac{b}{2}+\frac{a}{2}\right)X=\frac{a-b}{4}X. \tag{18}$$

In the same way one has, for any $Y\in\Gamma(T\mathcal{F}_2)$,

$$hY=\frac{1}{2}\big([\xi,\varphi Y]_{T\mathcal{F}_2}-\varphi([\xi,Y]_{T\mathcal{F}_1})\big)=\frac{1}{2}\left(-\frac{a}{2}+\frac{b}{2}\right)Y=-\frac{a-b}{4}Y. \tag{19}$$

We then distinguish the cases $a\neq b$ and $a=b$. In the first case, assuming for instance $a>b$, the manifold is not K-contact and $\mathcal{F}_1$ and $\mathcal{F}_2$ are the eigenspaces of the operator h corresponding to the eigenvalues $\lambda=\frac{a-b}{4}$ and $-\lambda$, respectively. Therefore, $\nabla^{bl}h=0$ and so (M,φ,ξ,η,g) fulfils all the conditions required by Theorem 1 and we conclude that it is a contact metric (κ,μ)-manifold. Comparing (14) with (11)–(12), we obtain the linear system $2\lambda-\mu+2=a, -2\lambda-\mu+2=b$ which admits the unique solution $\lambda=\frac{a-b}{4}$, $\mu=2-\frac{a+b}{2}$. Hence, $\kappa=1-\lambda^2=1-\frac{(a-b)^2}{16}$. Now we consider the case $a=b$. By (18)–(19), we have that $h=0$. Due to (iii) of Lemma 5 and using (10), we have for all $X,X'\in\Gamma(T\mathcal{F})$

$$\begin{aligned}(N_\varphi(X,X'))_{T\mathcal{F}_1}&=-[X,X']-(\varphi[\varphi X,X'])_{T\mathcal{F}_1}-(\varphi[X,\varphi X'])_{T\mathcal{F}_1}\\&=-[X,X']-\nabla^{bl}_{X'}X+\nabla^{bl}_{X}X'\\&=T^{bl}(X,X')\\&=-[X,X']_{T\mathcal{F}_2\oplus\mathbb{R}\xi}=0,\end{aligned}$$

because of the integrability of $\mathcal{F}_1$. Analogously, $(N_\varphi(Y,Y'))_{T\mathcal{F}_2}=0$ for all $Y,Y'\in\Gamma(T\mathcal{F}_2)$. Now, for all $X,X'\in\Gamma(T\mathcal{F}_1)$,

$$\begin{aligned} N_\varphi(\varphi X,\varphi X') &= -[\varphi X,\varphi X'] + [\varphi^2 X,\varphi^2 X'] - \varphi[\varphi^2 X,\varphi X'] - \varphi[\varphi X,\varphi^2 X'] \\ &= -[\varphi X,\varphi X'] + [X,X'] + \varphi[X,\varphi X'] + \varphi[\varphi X,X'] \\ &= -N_\varphi(X,X'), \end{aligned}$$

hence $(N_\varphi(X,X'))_{T\mathcal{F}_2} = -(N_\varphi(\varphi X,\varphi X'))_{T\mathcal{F}_2}=0$. Since, by (5), $g(N_\varphi(X,X'),\xi)=\eta(N_\varphi(X,X'))=0$, $N_\varphi(X,X')$ has zero component also in the direction of ξ and we conclude that $N_\varphi(X,X')\equiv 0$. In the same way, one can show that $N_\varphi(Y,Y')\equiv 0$ for all $Y,Y'\in\Gamma(T\mathcal{F}_2)$. Moreover, (4) implies that $N_\varphi(X,Y)=0$ for all $X\in\Gamma(T\mathcal{F}_1)$ and $Y\in\Gamma(T\mathcal{F}_2)$. Finally, directly by the definition of N_φ we have $\eta(N_\varphi(Z,\xi))=0$ for all $Z\in\Gamma(\mathcal{D})$, and from (4) it follows that $\varphi N_\varphi(Z,\xi)=0$. Hence, $N_\varphi(Z,\xi)\in\ker(\eta)\cap\ker(\varphi)=\{0\}$. Thus, the tensor field N_φ vanishes identically and (M,φ,ξ,η,g) is a Sasakian manifold. □

The expressions of κ and μ in (13) should be compared with the example presented by Boeckx in his local classification of non-Sasakian contact metric (κ,μ)-manifolds with $I_M\leq -1$ (cf. Section 4 of [5]). Therefore, in some sense Theorem 6 may be regarded also as a generalization of the Boeckx construction for every value of the invariant I_M.

Furthermore, it should be remarked that the cases (I), (II) and (III) of Theorem 6 correspond, respectively, to the classes (I), (II) and (III) of Theorem 4. This is also clear by the computation of the invariant I_M. Indeed by (13), we get straightforwardly $I_M=\frac{a+b}{|a-b|}$, so that, according to the signs of a and b, I_M can assume values strictly greater than 1, strictly lower than -1, or in the interval $(-1,1)$. However an easy computation shows that $I_M=\pm 1$ if and only if $a=0$ or $b=0$, that's impossible because of the assumptions of Theorem 6. Now, we complete our results by proving the following theorem concerning the remaining classes (IV) and (V) of Theorem 4.

THEOREM 7. *Let (M,η) be a contact manifold endowed with a bi-Legendrian structure $(\mathcal{F}_1,\mathcal{F}_2)$ such that $\nabla^{bl}\Pi_{\mathcal{F}_1}=0$ (respectively, $\nabla^{bl}\Pi_{\mathcal{F}_2}=0$). Assume that $\mathcal{F}_1$ is positive definite (respectively, flat) and $\mathcal{F}_2$ is flat (respectively, negative definite). Then for each $0<c\leq 4$ (respectively, $-4\leq c<0$) (M,η) admits a compatible contact metric (κ,μ)-structure, whose associated bi-Legendrian structure is $(\mathcal{F}_1,\mathcal{F}_2)$, where*

$$\kappa = 1-\frac{c^2}{16}, \qquad \mu = 2\left(1-\frac{c}{4}\right). \tag{20}$$

Proof. Let us assume that $\mathcal{F}_1$ is positive definite and $\mathcal{F}_2$ is flat. Then $\mathcal{F}_1$ is, in particular, nondegenerate and we can consider the corresponding linear map $\Lambda_{\mathcal{F}_1}: TM\longrightarrow T\mathcal{F}_1$ defined by (8). Since the operator $\Lambda_{\mathcal{F}_1}$ is surjective and its kernel is $T\mathcal{F}_1 \oplus \mathbb{R}\xi$, we have that $\Lambda_{\mathcal{F}_1}|_{T\mathcal{F}_2} : T\mathcal{F}_2 \longrightarrow T\mathcal{F}_1$ is an isomorphism. Then for each $c \in (0,4]$ we define a tensor field φ of type $(1,1)$ by

$$\varphi|_{T\mathcal{F}_1} := \frac{1}{c}(\Lambda_{\mathcal{F}_1}|_{T\mathcal{F}_2})^{-1}, \qquad \varphi|_{T\mathcal{F}_2} := -c\Lambda_{\mathcal{F}_1}|_{T\mathcal{F}_2}, \qquad \varphi\xi = 0. \tag{21}$$

Moreover, we put

$$\begin{gathered} g|_{T\mathcal{F}_1 \times T\mathcal{F}_1} := \tfrac{1}{c}\Pi_{\mathcal{F}_1}, \qquad g|_{T\mathcal{F}_2 \times T\mathcal{F}_2} := c\overline{\Pi}_{\mathcal{F}_1}|_{T\mathcal{F}_2 \times T\mathcal{F}_2}\ , \\ g := \eta \otimes \eta \quad \text{elsewhere.} \end{gathered} \tag{22}$$

Notice that g defines a Riemannian metric since, by assumption, $\mathcal{F}_1$ is positive definite and $c > 0$. We prove that in fact (φ, ξ, η, g) is a contact metric structure. Indeed we have easily that $\varphi^2 = -I + \eta \otimes \xi$. Next, for all $X, X' \in \Gamma(T\mathcal{F}_1)$ we have $g(\varphi X, \varphi X') = c\Pi_{\mathcal{F}_1}(\Lambda_{\mathcal{F}_1}\varphi X, \Lambda_{\mathcal{F}_1}\varphi X') = (1/c)\Pi_{\mathcal{F}_1}(\Lambda_{\mathcal{F}_1}\Lambda_{\mathcal{F}_1}^{-1}X, \Lambda_{\mathcal{F}_1}\Lambda_{\mathcal{F}_1}^{-1}X') = (1/c)\Pi_{\mathcal{F}_1}(X, X') = g(X, X')$. In a similar way, one can prove that $g(\varphi Y, \varphi Y') = g(Y, Y')$ for all $Y, Y' \in \Gamma(T\mathcal{F}_2)$. Moreover, the same arguments used in the proof of Theorem 6 show that g is an associated metric, that is $d\eta = g(\cdot, \varphi\cdot)$. Thus, (φ, ξ, η, g) is a contact metric structure. Notice that, by construction, $\mathcal{F}_1$ and $\mathcal{F}_2$ are conjugate Legendre foliations with respect to (φ, ξ, η, g). Finally, the definition of g and the assumption $\nabla^{bl}\Pi_{\mathcal{F}_1} = 0$ imply that the bi-Legendrian connection is metric with respect to g. Hence, by Lemma 5, the tensor field φ is ∇^{bl}-parallel and the operator h preserves the Legendre foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. We are now able to compute the explicit expression of h. For any $X \in \Gamma(T\mathcal{F}_1)$, we have $2hX = [\xi, \varphi X] - \varphi[\xi, X] = [\xi, \varphi X]_{T\mathcal{F}_1} + [\xi, \varphi X]_{T\mathcal{F}_2} - \varphi([\xi, X]_{T\mathcal{F}_1}) - \varphi([\xi, X]_{T\mathcal{F}_2})$. The flatness of $\mathcal{F}_2$ yields $[\xi, \varphi X]_{T\mathcal{F}_1} = 0$. Thus,

$$2hX + \varphi([\xi, X]_{T\mathcal{F}_2}) = [\xi, \varphi X]_{T\mathcal{F}_2} - \varphi([\xi, X]_{T\mathcal{F}_1}). \tag{23}$$

Since h preserves the foliations, the right-hand side of (23) is a section of both $T\mathcal{F}_1$ and $T\mathcal{F}_2$, hence vanishes. Consequently, taking into account that $\ker(\Lambda_{\mathcal{F}_1}) = T\mathcal{F}_1 \oplus \mathbb{R}\xi$,

$$hX = -\frac{1}{2}\varphi([\xi, X]_{T\mathcal{F}_2}) = \frac{c}{2}\Lambda_{\mathcal{F}_1}([\xi, X]_{T\mathcal{F}_2}) = \frac{c}{2}\Lambda_{\mathcal{F}_1}([\xi, X]) = \frac{c}{4}X. \tag{24}$$

Moreover, let Y be a section of $T\mathcal{F}_2$. As $\varphi(T\mathcal{F}_1) = T\mathcal{F}_2$, $Y = \varphi X$ for some $X \in \Gamma(T\mathcal{F}_1)$. Then, by (24), $hY = h\varphi X = -\varphi hX = -\frac{c}{4}\varphi X = -\frac{c}{4}Y$. Thus, the bi-Legendrian structure $(\mathcal{F}_1, \mathcal{F}_2)$ coincides with that one determined by the eigendistributions of the operator h. In particular, this implies that $\nabla^{bl}h = 0$. Therefore, all the conditions in Theorem 1 are satisfied and we conclude that $(M, \varphi, \xi, \eta, g)$ is a contact metric (κ, μ)-manifold such that $\mathcal{D}(\lambda) = \mathcal{F}_1$ and $\mathcal{D}(-\lambda) = \mathcal{F}_2$. Finally, comparing (22) with (11) and taking into account that $\Pi_{\mathcal{F}_2} = 0$, we get $\kappa = 1 - (\frac{c}{4})^2$ and $\mu = 2(1 - \frac{c}{4})$. The case when $\mathcal{F}_1$ is flat and $\mathcal{F}_2$ is negative definite is analogous, the only difference being to use $\Lambda_{\mathcal{F}_2}$

setting

$$\varphi|_{T\mathcal{F}_1} := -c\Lambda_{\mathcal{F}_2}|_{T\mathcal{F}_1}, \qquad \varphi|_{T\mathcal{F}_2} := \frac{1}{c}(\Lambda_{\mathcal{F}_2}|_{T\mathcal{F}_1})^{-1}, \qquad \varphi\xi = 0,$$
$$g|_{T\mathcal{F}_1\times T\mathcal{F}_1} := c\overline{\Pi}_{\mathcal{F}_2}|_{T\mathcal{F}_1\times T\mathcal{F}_1}, \qquad g|_{T\mathcal{F}_2\times T\mathcal{F}_2} := \frac{1}{c}\Pi_{\mathcal{F}_2},$$
$$g := \eta\otimes\eta \quad \text{elsewhere.}$$

where $c\in[-4,0)$. Arguing as in the previous case, one can find that (φ,ξ,η,g) is a contact metric (κ,μ)-structure, where κ and μ are given by (20) and $\mathcal{D}(\lambda)=\mathcal{F}_1$, $\mathcal{D}(-\lambda)=\mathcal{F}_2$. □

REMARK 1. Notice that, as expected, by (20), we get $I_M=1$ if $c>0$ and $I_M=-1$ if $c<0$. Furthermore, it should be remarked that for no value of c one can obtain a Sasakian structure, since $\kappa=1$ if and only if $c=0$. Whereas, for $c=4$ one gets $\kappa=\mu=0$, that is $R_{XY}\xi=0$ for all $X,Y\in\Gamma(TM)$. Such contact metric manifolds were deeply studied by Blair in [1].

COROLLARY 8. *Let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-manifold. Then*

(i) *if $I_M\neq\pm1$, (M,η) admits a family of compatible contact metric $(\kappa_{a,b},\mu_{a,b})$-structures, where a and b are real numbers such that $ab=(2-\mu)^2-4(1-\kappa)$;*

(ii) *if $I_M=1$ (respectively, $I_M=-1$), (M,η) admits a family of compatible contact metric (κ_c,μ_c)-structures, where $0<c\leq4$ (respectively, $-4\leq c<0$).*

Furthermore, the above contact metric $(\kappa_{a,b},\mu_{a,b})$ and (κ_c,μ_c)-structures are of the same classification as (φ,ξ,η,g).

Proof. In order to prove the statements, it suffices to show that (M,φ,ξ,η,g) satisfies all the hypotheses of Theorem 6 for the case (i) and of Theorem 7 for the case (ii).

(i) By (11)–(12) and by Theorem 1, we have immediately that $\nabla^{bl}\Pi_{\mathcal{D}(\lambda)}=\nabla^{bl}\Pi_{\mathcal{D}(-\lambda)}=0$, where, as usual, ∇^{bl} denotes the bi-Legendrian connection associated to the bi-Legendrian structure $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ defined by the eigendistributions of the operator h. Next, we compute the explicit expression of the Libermann operators $\Lambda_{\mathcal{D}(\lambda)}:TM\longrightarrow\mathcal{D}(\lambda)$ and $\Lambda_{\mathcal{D}(-\lambda)}:TM\longrightarrow\mathcal{D}(-\lambda)$. For any $X\in\Gamma(\mathcal{D}(\lambda))$ and $Y\in\Gamma(\mathcal{D}(-\lambda))$ we have, by (11),

$$\Pi_{\mathcal{D}(\lambda)}\big(\Lambda_{\mathcal{D}(\lambda)}Y,X\big)=d\eta(Y,X)=g(Y,\varphi X)=-\frac{1}{2\sqrt{1-\kappa}-\mu+2}g(\varphi Y,X),$$

from which it follows that

$$\Lambda_{\mathcal{D}(\lambda)}=\begin{cases}0, & \text{on } \mathcal{D}(\lambda)\oplus\mathbb{R}\xi,\\ \frac{1}{\mu-2-2\sqrt{1-\kappa}}\varphi, & \text{on } \mathcal{D}(-\lambda).\end{cases} \tag{25}$$

Whereas, using (12), one can find

$$\Lambda_{\mathcal{D}(-\lambda)} = \begin{cases} \frac{1}{\mu-2+2\sqrt{1-\kappa}}\varphi, & \text{on } \mathcal{D}(\lambda), \\ 0, & \text{on } \mathcal{D}(-\lambda)\oplus\mathbb{R}\xi. \end{cases} \tag{26}$$

Notice that the denominators in (25) and (26) are different from zero just because of the assumption $I_M \neq \pm 1$. Next, for all $X, X' \in \Gamma(\mathcal{D}(-\lambda))$,

$$\begin{aligned}\overline{\Pi}_{\mathcal{D}(-\lambda)}(X,X') &= \Pi_{\mathcal{D}(-\lambda)}\big(\Lambda_{\mathcal{D}(-\lambda)}X, \Lambda_{\mathcal{D}(-\lambda)}X'\big) \\ &= \frac{1}{-2\sqrt{1-\kappa}-\mu+2} g(\varphi X, \varphi X') \\ &= \frac{1}{-2\sqrt{1-\kappa}-\mu+2} g(X, X') \\ &= \frac{1}{(2-\mu)^2-4(1-\kappa)} \Pi_{\mathcal{D}(\lambda)}(X,X').\end{aligned}$$

Thus, $\overline{\Pi}_{\mathcal{D}(\lambda)} = ((2-\mu)^2 - 4(1-\kappa))\overline{\Pi}_{\mathcal{D}(-\lambda)}$ on $\mathcal{D}(\lambda)$ and in a similar manner one can find that $\overline{\Pi}_{\mathcal{D}(-\lambda)} = ((2-\mu)^2 - 4(1-\kappa))\overline{\Pi}_{\mathcal{D}(\lambda)}$ on $\mathcal{D}(-\lambda)$. We distinguish the cases (I) $I_M > 1$, (II) $-1 < I_M < 1$ and (III) $I_M < -1$. By Theorem 4, in the first case both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are positive definite, in the second $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ negative definite and in the third one both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are negative definite. Then we take any two $a, b \in \mathbb{R}$ such that $ab = (2-\mu)^2 - 4(1-\kappa)$ and $a > 0$, $b > 0$ in the case (I), $a > 0$, $b < 0$ in the case (II) and $a < 0$, $b < 0$ in the case (III). Thus, in any case the hypotheses of Theorem 6 are satisfied and so the structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ defined by (14) and (15) is a contact metric $(\kappa_{a,b}, \mu_{a,b})$-structure on (M, η).

(ii) If $I_M = 1$ then, by Theorem 4, $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is flat. Moreover, again by (11) and Theorem 1 we have that $\nabla^{bl}\Pi_{\mathcal{D}(\lambda)} = 0$. Thus, all the assumptions of Theorem 7 are satisfied and it suffices to take any $c \in (0, 4]$ for obtaining a contact metric (κ_c, μ_c)-structure given by (21) and (22). The proof for the case $I_M = -1$ is similar to that for $I_M = 1$. □

Corollary 9. *Any contact metric (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ such that $|I_M| > 1$ admits a compatible Sasakian structure.*

Proof. If $(M, \varphi, \xi, \eta, g)$ is Sasakian then the assertion is trivial, so we can assume that the structure (φ, ξ, η, g) is non-Sasakian. Then we can apply Corollary 8. The assumption $|I_M| > 1$ implies by Theorem 4 that the Legendre foliations $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are either both positive definite or both negative definite. So it is sufficient to take $a = b = \sqrt{(1-\frac{\mu}{2})^2 - (1-\kappa)}$ in the case of positive definiteness and $a = b = -\sqrt{(1-\frac{\mu}{2})^2 - (1-\kappa)}$ in the case of negative definiteness, for obtaining, by Theorem 6, a Sasakian structure on M compatible with the contact form η. □

COROLLARY 10. *For each* $1 \le p \le 2n$, *the pth Betti number of a compact contact metric* (κ,μ)*-manifold* M^{2n+1} *such that* $|I_{M^{2n+1}}| > 1$ *is even.*

Proof. The assertion is a consequence of Corollary 9 and the results in [3] and [11]. □

Finally, applying twice Theorem 6 and Corollary 9 we get the following result.

COROLLARY 11. *Let* (M,η) *be a contact manifold endowed with two positive definite or negative definite Legendre foliations satisfying the conditions* (I) *or* (III) *of Theorem* 6, *respectively. Then* (M,η) *admits a compatible Sasakian structure.*

We conclude by recalling the definition of Tanaka–Webster parallel space, recently introduced by Boeckx and Cho [6]. A contact metric manifold is a *Tanaka–Webster parallel space* if its generalized Tanaka–Webster torsion tensor $\hat{T}$ and its curvature tensor $\hat{R}$ satisfy $\hat{\nabla}\hat{T} = 0$ and $\hat{\nabla}\hat{R} = 0$, that is the Tanaka–Webster connection $\hat{\nabla}$ is invariant by parallelism (in the sense of [13]). Boeckx and Cho have proven that a contact metric manifold M is a Tanaka–Webster parallel space if and only if M is a Sasakian locally φ-symmetric space or a non-Sasakian $(\kappa,2)$-space ([6, Theorem 12]). Thus, in particular, we deduce the following corollaries of Theorem 6 and of Corollary 8.

COROLLARY 12. *Any non-Sasakian contact* (κ,μ)*-manifold* (M,φ,ξ,η,g) *such that* $|I_M| < 1$ *admits a compatible Tanaka–Webster parallel structure.*

Proof. The assumption $|I_M| < 1$ implies by Theorem 4 that the Legendre foliation $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is negative definite. So it is sufficient to take $a = -b = \sqrt{(1-\frac{\mu}{2})^2 - (1-\kappa)}$ for obtaining, according to Theorem 6, a compatible contact metric $(\kappa_{a,b},\mu_{a,b})$-structure $(\varphi_{a,b},\xi,\eta,g_{a,b})$ on (M,η) such that $\kappa = 1 - \frac{a^2}{4}$ and $\mu = 2$. Thus, by applying the aforementioned result by Boeckx and Cho, we conclude that $(M,\varphi_{a,b},\xi,\eta,g_{a,b})$ is a Tanaka–Webster parallel space. □

COROLLARY 13. *Let* (M,η) *be a contact manifold endowed with a positive definite Legendre foliation* $\mathcal{F}_1$ *and negative definite Legedre foliation* $\mathcal{F}_1$ *satisfying the condition* (II) *of Theorem* 6. *Then* (M,η) *admits a compatible Tanaka–Webster parallel structure.*

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