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GOOD REDUCTION OF PERIODIC POINTS ON PROJECTIVE VARIETIES

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ABSTRACT. We consider the dynamical system created by iterating a morphism of a projective variety defined over the field of fractions of a discrete valuation ring. We study the primitive period of a periodic point in this field in relation to the primitive period of the reduced point in the residue field, the order of the action on the cotangent space, and the characteristic of the residue field.

1. Introduction

We consider dynamical systems arising from iterating a morphism of a projective variety defined over the field of fractions of a discrete valuation ring. Our goal is to obtain information about the dynamical system over the field of fractions by studying the dynamical system over the residue field. In particular, we aim to bound the possible primitive periods of a periodic point. This topic is discussed for $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ with many references in [12, Section 2.6].

Recall that given a set A and map $f : A \to A$ we can create a dynamical system by iterating the map f on the set A. We denote f^n as the nth iterate of f. An element $a \in A$ such that $f^n(a) = a$ for some positive integer n is called a *periodic point* and the least such n is called the *primitive period of* a. We will use the following notation unless otherwise specified:

- R is a discrete valuation ring complete with respect to a normalized valuation v.
- $\mathfrak{m}$ is the maximal ideal of R with uniformizer π.
- K is the field of fractions of R.
- $k = R/\mathfrak{m}$ is the finite residue field of characteristic p.
- $\overline{}$ denotes reduction mod π.

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- $|\cdot|_v$ is the associated non-Archimedean absolute value.

Note that these conditions imply that K is a local field.

In Section 2, we establish a notion of good reduction for a projective variety X/K and a morphism $\phi\colon X \to X$ defined over K so that we can study the dynamics of ϕ over K by examining the dynamics of the reduced map over k.

In Section 3, we describe the primitive period of $P \in X(K)$ with the following two theorems.

THEOREM 1. *Let $\mathcal{X}/R \subseteq \mathbb{P}^N_R$ be a smooth projective model of X/K, a non-singular irreducible projective variety of dimension d. Let $\phi_R : \mathcal{X}/R \to \mathcal{X}/R$ be an R-morphism and $P_R \in \mathcal{X}(R)$ be a periodic point of primitive period n for ϕ_R with $\overline{P}$ of primitive period m. Then*

$$n = m$$

or there exists a ϕ-stable subspace V of the cotangent space of $\mathcal{X} \times_R \operatorname{Spec} k$ such that

$$n = m r_V p^e \quad \text{with } e \geq 0,$$

where r_V is the order of the map induced by $\overline{\phi^m}$ on V. Furthermore, let $\mathcal{V} \subset \mathcal{X}$ be the scheme theoretic union of $\{P_R, \phi_R(P_R), \ldots, \phi_R^{n-1}(P_R)\}$. Then V is the cotangent space of $\mathcal{V} \times_R \operatorname{Spec} k$. Let $d' \leq d$ be the dimension of V, then

$$r_V \leq (N\pi)^{d'} - 1,$$

where $N\pi$ is the norm of π.

This theorem generalizes the known result for $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ [12, Theorem 2.21] and is similar to [4].

THEOREM 2. *Let $\mathcal{X}/R \subseteq \mathbb{P}^N_R$ be a smooth projective model of X/K, a non-singular irreducible projective variety of dimension d. Let $\phi_R : \mathcal{X}/R \to \mathcal{X}/R$ be an R-morphism and $P_R \in \mathcal{X}(R)$ be a periodic point of primitive period n for ϕ_R. Using the notation from Theorem 1, we have for $n = m r_V p^e$ that*

$$e \leq \begin{cases} 1 + \log_2(v(p)), & p \neq 2, \\ 1 + \log_\alpha\left(\frac{\sqrt{5}v(2) + \sqrt{5(v(2))^2 + 4}}{2}\right), & p = 2, \end{cases}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

This theorem is a generalization of results such as those found in [8], [9], [10], [13] and, in particular, implies that the primitive period of a K-rational periodic point is bounded. In the following corollary, we state what the bound is when working over $\mathbb{Q}$.

COROLLARY 3. *Let $X/\mathbb{Q}$ be a smooth irreducible projective variety of dimension d and $\phi : X \to X$ a morphism defined over $\mathbb{Q}$ with good reduction at* *a prime p. Let $P \in X(\mathbb{Q})$ be a periodic point with primitive period n. Then we have*

$$e \leq \begin{cases} 1, & p \neq 2, \\ 3, & p = 2 \end{cases}$$

and

$$n \leq \begin{cases} p^{d+1}(p^d - 1), & p \neq 2, \\ 2^{d+3}(2^d - 1), & p = 2. \end{cases}$$

By assuming that $\phi : X \to X$ is étale, we may remove the hypotheses of smooth and irreducible to obtain the following theorem.

THEOREM 4. *Let $\phi : X \to X$ be an étale morphism of a projective variety defined over K. Let $P \in X(K)$ be a smooth periodic point for ϕ with primitive period n. Let $Y \subseteq X_{\mathrm{smooth}}$ be the irreducible component containing P. Let l be the smallest integer such that ϕ^l is an étale endomorphism of Y. Assume that ϕ^l restricted to Y has good reduction. Then we have $n = lm$ or $n = lmr_V p^e$ where m, r_V, and p^e are as in Theorem 1 and Theorem 2.*

It is well known that for $X \subset \mathbb{P}^N$ a variety defined by homogeneous polynomials of degree at most d, the number of irreducible components of X is bounded by d^N. In particular, Theorem 4 implies that the primitive period of a smooth K-rational periodic point is bounded for an étale morphism of a projective variety.

2. Good reduction

In this section, we consider the more general situation of a scheme $X/\operatorname{Spec} K$ and a K-morphism $\phi : X \to X$, unless otherwise stated. Following [12, Section 2.5], we define a notion of good reduction.

DEFINITION 5. A scheme X/K has *good reduction* if there exists a smooth proper scheme $\mathcal{X}/R$ with generic fiber X/K. We call such an $\mathcal{X}/R$ a *smooth proper model* for X/K.

If X/K has good reduction, each point in $X(K)$ corresponds to a unique point in the proper scheme $\mathcal{X}(R)$ and, consequently, a unique point in the special fiber (denoted as $\overline{P}$). In addition to a notion of good reduction for a scheme X/K, we also need a notion of good reduction for a K-morphism $\phi : X \to X$.

DEFINITION 6. Let X/K be a scheme and $\phi : X \to X$ a K-morphism. We say that ϕ has *good reduction* if there exists a smooth proper model $\mathcal{X}/R$ of X/K and an R-morphism $\phi_R : \mathcal{X} \to \mathcal{X}$ extending ϕ. Denote the restriction of ϕ_R to the special fiber as $\overline{\phi}$.

REMARK. Let $X=\mathbb{P}^1_K$ and $\phi:\mathbb{P}^1_K\to\mathbb{P}^1_K$ be a morphism defined over K. Then good reduction as defined in Definition 6 is equivalent to good reduction as defined in [12, Theorem 2.15]. For $X=\mathbb{P}^N_K$ and $\phi: X\to X$, a morphism over K, we can again formulate a definition of good reduction using resultants that is equivalent to Definition 6; see, for example [7, Section 1.1].

REMARK. For a morphism $\phi:\mathbb{P}^1_K\to\mathbb{P}^1_K$, Hsia [6, Section 3] defines a notion of *mildly bad reduction* as the case where ϕ has bad reduction but there exists a projective scheme $\mathcal{X}$ of finite type over R and an R-morphism from $\mathcal{X}\to\mathbb{P}^1_R$ that is an isomorphism on the generic fiber such that ϕ extends to an R-morphism that maps the smooth part of $\mathcal{X}$ to itself.

The following theorem and corollary show that for morphisms with good reduction, the dynamics of ϕ are related to the dynamics of $\overline{\phi}$.

THEOREM 7. *Let $\mathcal{X}/R$ be a smooth proper scheme with generic fiber X/K. Let $\phi_R:\mathcal{X}/R\to\mathcal{X}/R$ be an R-morphism and let $\phi: X/K\to X/K$ be the restriction of ϕ_R to the generic fiber. Let $\overline{\phi}$ be the restriction of ϕ_R to the special fiber.*

(a) $\overline{\phi}(\overline{P})=\overline{\phi(P)}$, *for all $P\in X(K)$.*
(b) *Let $\psi_R:\mathcal{X}/R\to\mathcal{X}/R$ be another R-morphism and $\psi: X/K\to X/K$ be the restriction of ψ_R to the generic fiber. Then $\overline{\phi\circ\psi}=\overline{\phi}\circ\overline{\psi}$.*
(c) *Let $\phi_R^n:\mathcal{X}/R\to\mathcal{X}/R$ be the nth iterate of ϕ_R and let $\overline{\phi^n}$ be ϕ_R^n restricted to the generic fiber. Then*

$$\overline{\phi^n(P)}=\overline{\phi}^n(\overline{P})\quad \textit{for all } P\in X(K).$$

Proof.

(a) Let $P\in X(K)$, that is, $P\in \operatorname{Hom}_{\operatorname{Spec}K}(\operatorname{Spec}K, X)$. Since the scheme is proper, we have a unique associated

$$P_R\in \operatorname{Hom}_{\operatorname{Spec}R}(\operatorname{Spec}R,\mathcal{X}).$$

Using the universal property of fiber products, it is easy to see that

$$(\phi_R\circ P_R)|_{\operatorname{Spec}k}=(\phi_R)|_{\operatorname{Spec}k}\circ(P_R)|_{\operatorname{Spec}k}.$$

(b) A composition of morphisms is a morphism, so we have that $\phi_R\circ\psi_R$ is also a morphism of schemes. Using an argument with fiber products, we see that $\overline{\phi\circ\psi}=\overline{\phi}\circ\overline{\psi}$.
(c) To prove this statement, we proceed by induction on n by applying (b) to the maps ϕ_R and ϕ_R^{n-1}. □

REMARK. There are some interesting questions to be raised concerning good reduction of K-morphisms.

- If ϕ^2 has good reduction, does that necessarily imply ϕ has good reduction? For $N=1$, Benedetto [1, Theorem B] proves for $\phi\in K(x)$, a rational map of degree ≥ 2, and any positive integer n that ϕ has good reduction if and only if ϕ^n has good reduction.

- If ϕ and ψ are distinct and both have good reduction, then is it necessarily true that $\phi \circ \psi$ has good reduction? The complication is whether a smooth proper model $\mathcal{X}/R$ exists where both ϕ and ψ extend to R-morphisms ϕ_R and ψ_R. It is not clear if the good reduction of ϕ and ψ is enough to ensure the existence of such a smooth proper model.

Definition 8. Let X/K be a scheme, $\phi : X \to X$ be a K-morphism, and $P \in X(K)$.

- The point P is *periodic* if $\phi^n(P) = P$ for some $n \in \mathbb{N}$. The integer n is called a *period* of P.
- If P is periodic with period n and $\phi^m(P) \neq P$ for all $0 < m < n$, then n is called the *primitive period* of P.
- If $\phi^{m+n}(P) = \phi^m(P)$ for some $m, n \in \mathbb{N}$, then P is called *preperiodic.*

Corollary 9. *Let $\mathcal{X}/R$ be a smooth proper scheme with generic fiber X/K. Let $\phi_R : \mathcal{X}/R \to \mathcal{X}/R$ be an R-morphism and $\phi : X/K \to X/K$ the restriction of ϕ_R to the generic fiber. Let $\overline{\phi}$ be the restriction of ϕ_R to the special fiber. Then the reduction map sends periodic points to periodic points and preperiodic points to preperiodic points. Furthermore, if $P \in X(K)$ has primitive period n and $\overline{P}$ has primitive period m, then $m \mid n$.*

Proof. Let $P \in X(K)$ be a point of primitive period n and let m be the primitive period of $\overline{P}$. We are given that ϕ has good reduction, so by Theorem 7 we know

$$\overline{\phi^n(P)} = \overline{\phi}^n(\overline{P}) = \overline{P}. \tag{1}$$

From (1), we deduce that $m \leq n$ and $n \equiv 0 \mod m$, since the smallest period m is the greatest common divisor of all of the periods. ☐

Before we examine the primitive period, we state what it means for a scheme defined over a number field to have good reduction at a particular prime.

Definition 10. Let L be a number field with ring of integers A and X/L a scheme over L. Let $\mathfrak{p}$ be a prime of L and $A_{\mathfrak{p}}$ the localization of A at $\mathfrak{p}$. We say that X has *good reduction at* $\mathfrak{p}$ if there exists a smooth proper model $\mathcal{X}/A_{\mathfrak{p}}$ with generic fiber X/L.

Similarly, $\phi : X/L \to X/L$ has *good reduction at* $\mathfrak{p}$ if ϕ has good reduction over $A_{\mathfrak{p}}$.

For the rest of this article, we work with morphisms of projective varieties defined over K. We assume that $\mathcal{X}/R$ is, in fact, a smooth *projective* scheme with generic fiber X/K. Since projective implies proper, this is slightly more restrictive.

3. Description of the primitive period

3.1. Preliminary results.

Definition 11. Let $P \in \mathbb{P}^N_K$ be a point. We call a representation of P as $[P_0, \ldots, P_N]$ with $P_i \in R$ for $0 \leq i \leq N$ and at least one $P_i \in R^*$ a *normalization of* P. We define the *reduction of* P *modulo* π, denoted $\overline{P}$, by first choosing a normalization of P and then setting

$$\overline{P} = [\overline{P_0}, \ldots, \overline{P_N}] \in \mathbb{P}^N_k.$$

Note that $\overline{P}$ is independent of the choice of normalization.

We recall some standard facts about projective space.

Proposition 12. *Let q be the number of elements of k. Then*

(a) *Given any $N+2$ points $P_i \in \mathbb{P}^N_R$ for which the images $P_i(\operatorname{Spec} K)$ and their reductions [the images $P_i(\operatorname{Spec} k)$] satisfy that no $N+1$ of them are coplanar, there exists a transformation in $\mathrm{PGL}_{N+1}(R)$ that maps them to any other $N+2$ points Q_i in $\mathbb{P}^N_R$ for which the images $Q_i(\operatorname{Spec} K)$ and their reductions satisfy that no $N+1$ of them are coplanar.*
(b) *The number of hyperplanes of $\mathbb{P}^N_k$ is $q^N + q^{N-1} + \cdots + q + 1$.*
(c) *The number of hyperplanes of $\mathbb{P}^N_k$ through a point of $\mathbb{P}^N_k$ is $q^{N-1} + \cdots + q + 1$.*

Proof.

(a) It is a standard argument to show that given $N+2$ points in $\mathbb{P}^{N+1}_K$ with no $N+1$ of them coplanar, that we can find a element of $\mathrm{PGL}_{N+1}(K)$ that takes them to any other set of $N+2$ points in $\mathbb{P}^{N+1}_K$ with no $N+1$ of them coplanar. Using the same argument and the fact that their reductions [points in $\mathbb{P}^{N+1}(k)$] also satisfy that no $N+1$ of them are coplanar. We can find an element in $\mathrm{PGL}_{N+1}(R)$ that takes them to any other set of $N+2$ points in $\mathbb{P}^{N+1}_R$ with no $N+1$ of them coplanar.
(b) A d dimensional subspace of $\mathbb{P}^N_k$ is isomorphic to $\mathbb{P}^d_k$ which has $q^d + q^{d-1} + \cdots + q + 1$ points.
(c) Let $P \in \mathbb{P}^N_k$ and H a hyperplane of $\mathbb{P}^N_k$ not containing P. Any hyperplane of $\mathbb{P}^N_k$ through P meets H in a hyperplane of H. By (b), there are $q^{N-1} + \cdots + q + 1$ hyperplanes of H. $\square$

For the proof of Theorem 1, we need a moving lemma for the orbit of a point whose reduction is a fixed point. Let $\mathbb{A}^N_i$ be the standard affine open sets in $\mathbb{P}^N$ obtained by sending

$$[t_0, \ldots, t_N] \to \left(\frac{t_0}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_N}{t_i}\right).$$

Lemma 13. *Let $N \geq 2$. Given any finite set of points $\mathcal{P} = \{P_\alpha\}_{\alpha \in I} \subset \mathbb{P}^N_R$ whose image in the special fiber is a single point and any fixed i, $0 \leq i \leq N$, we can find a transformation $f \in \mathrm{PGL}_{N+1}(R)$ such that $f(\mathcal{P}) \subset (\mathbb{A}^N_i)_R$.*

Proof. Let q be the number of elements of the residue field k.

Without loss of generality, consider $(\mathbb{A}_0^N)_R$. We need an $f \in \mathrm{PGL}_{N+1}(R)$ that sends the points of $\mathcal{P}$ to points not in the hyperplane $t_0 = 0$. So we need to find a hyperplane H in $\mathbb{P}_R^N$ that does not contain any of the points of $\mathcal{P}$ and send it to the hyperplane $t_0 = 0$.

The image in the special fiber of $\mathcal{P}$ is a single point, denoted $\overline{\mathcal{P}}$. By Proposition 12(b) and (c), there are

$$(q^N + \cdots + q + 1) - (q^{N-1} + \cdots + q + 1) = q^N$$

hyperplanes of $\mathbb{P}_k^N$ which do not go through $\overline{\mathcal{P}}$. Let $\overline{H}$ be any hyperplane in $\mathbb{P}_k^N$ that does not contain $\overline{\mathcal{P}}$. Choose any N points on $\overline{H}$. We need to find two additional points in $\mathbb{P}_k^N$ which, when combined with the already chosen N points, satisfy that no $N+1$ of them are coplanar in $\mathbb{P}_k^N$. Let one such point be $\overline{\mathcal{P}}$, which is not on $\overline{H}$ by construction.

There are $\binom{N+1}{N} = N+1$ subsets of N points of the $N+1$ chosen points. We need to choose an additional point which does not lie on a hyperplane containing any of those $N+1$ subsets. Each subset defines a unique hyperplane, so we must choose a point not on $N+1$ hyperplanes. By Proposition 12(b), there are

$$q^N + \cdots + q + 1$$

total hyperplanes. So we need

$$q^N + \cdots + q + 1 > N + 1$$

to be able to find such a point. We know that $q \geq 2$ and $N \geq 2$, so we can always find such a point.

Having found the necessary points on the special fiber, we can lift them to (not necessarily unique) points on the generic fiber. Since $\mathbb{P}_R^N$ is proper, these points on the generic fiber correspond to unique points of the scheme. We have found $N+2$ points that satisfy the hypothesis of Proposition 12(a); hence, there exists an element f of $\mathrm{PGL}_{N+1}(R)$ as desired. □

To be able to apply Lemma 13, we need to be certain that we do not change the dynamics.

DEFINITION 14. Let $X \subset \mathbb{P}_K^N$ be a projective variety, $\phi : X \to X$ a morphism, and $f \in \mathrm{PGL}_{N+1}(R)$. Define $\phi^f = f^{-1} \circ \phi \circ f$.

PROPOSITION 15. *Let $\mathcal{X}/R \subset \mathbb{P}_R^N$ be a smooth proper model for X, a projective variety defined over K. Let $\phi_R : \mathcal{X} \to \mathcal{X}$ be an R-morphism extending $\phi : X/K \to X/K$. Let $f \in \mathrm{PGL}_{N+1}(R)$ and let $P \in X(K)$ be a periodic point of primitive period n whose reduction modulo π has primitive period m. Then,*

(a) *$\phi^f : f^{-1}(X) \to f^{-1}(X)$ has good reduction.*
(b) *$f^{-1}(P)$ has primitive period n for ϕ^f.*

(c) $\overline{f^{-1}(P)}$ *has primitive period* m *for* $\overline{\phi^f}$.

Proof. Note that the embedding $\mathcal{X}/R \subset \mathbb{P}^N_R$ induces an embedding $X/K \subset \mathbb{P}^N_K$, so the action of f on X is defined. We need to show that there exists a smooth projective model $\mathcal{Y}/R$ and an R-morphism $\psi_R : \mathcal{Y} \to \mathcal{Y}$ such that the following diagram commutes and the vertical maps are isomorphisms:

$$\begin{tikzcd} \mathcal{Y}/R \arrow[r, "\psi_R"] \arrow[d, "f_R"'] & \mathcal{Y}/R \arrow[d, "f_R"] \\ \mathcal{X}/R \arrow[r, "\phi_R"] & \mathcal{X}/R\,. \end{tikzcd} \tag{2}$$

We know that $f \in \mathrm{PGL}_{N+1}(R)$ is an automorphism of $\mathbb{P}^N_K$ that has good reduction with smooth projective model $\mathbb{P}^N_R$. Let $f_R : \mathbb{P}^N_R \to \mathbb{P}^N_R$ be the morphism extending f. Then f_R is an automorphism of $\mathbb{P}^N_R$ and

$$f_R^{-1} : \mathcal{X}/R \to f_R^{-1}(\mathcal{X}/R)$$

is an isomorphism. Let $\mathcal{Y}/R$ be defined as $f_R^{-1}(\mathcal{X}/R)$, thus the vertical maps are isomorphisms. Let

$$\psi_R = \phi_R^{f_R} = f_R^{-1} \circ \phi_R \circ f_R.$$

Diagram (2) now clearly commutes.

To prove (a), the smooth projective model is $\mathcal{Y}/R$ and the morphism is ψ_R. Since the vertical maps are isomorphisms and

$$(f^{-1} \circ \phi \circ f)^s = f^{-1} \circ \phi^s \circ f \quad \text{for all } s \geq 1,$$

(b) and (c) follow immediately from the diagram (2). □

To prove Theorem 1, we need a π-adic version of the Implicit Function theorem that includes an explicit bound on the size of the neighborhood of convergence. To do this, we recall without proof a version of Hensel's lemma for systems of power series. This version of Hensel's lemma comes from Greenberg [5, Chapter 5], and we adopt his terminology.

Definition 16. We say $\mathbf{f}$ is a *system of formal power series* over R if $\mathbf{f}$ is a vector where each component is a formal power series with coefficients in R having zero constant term.

Proposition 17 (Hensel's lemma). *Let*

$$F_1, \dots, F_N \in R[x_1, \dots, x_N] \quad \textit{and} \quad a_1, \dots, a_N \in R$$

with

$$|F_i(a_1, \dots, a_N)|_v < \left| \det\left(\left(\frac{\partial F_i}{\partial x_j} \right)_{i,j} (a_1, \dots, a_N) \right) \right|_v^2 \leq 1$$

for $1 \le i \le N$. Then there exist $b_1, \dots, b_N \in R$ with

$$F_i(b_1, \dots, b_N) = 0 \quad \textit{and} \quad |b_i - a_i|_v < \left| \det\left(\left(\frac{\partial F_i}{\partial x_j} \right)_{i,j} (a_1, \dots, a_N) \right) \right|_v$$

for $1 \le i \le N$. Furthermore, the system of power series taking $(a_1, \dots, a_N)$ to the solution $(b_1, \dots, b_N)$ is defined over R.

Recall that a function $\mathbf{F} : \mathbb{A}^{N+M}(K) \to \mathbb{A}^M(K)$ is *smooth* at a point $P \in \mathbb{A}^{N+M}(K)$ if its Jacobian matrix has rank M at P.

PROPOSITION 18 (π-adic Implicit Function theorem). *Let*

$$\mathbf{F} = [F_1, \dots, F_M] : \mathbb{A}^{N+M}(K) \to \mathbb{A}^M(K)$$

be a smooth function with each F_i defined over R. Label the coordinates of $\mathbb{A}^{N+M}$ as $(x_1, \dots, x_N, y_1, \dots, y_M)$. Let $(\mathbf{a}_0, \mathbf{b}_0)$ be a point in $\mathbb{A}^{N+M}(K)$ such that

(a) $F(\mathbf{a}_0, \mathbf{b}_0) = 0$,

(b) *the minor*

$$\left(\frac{\partial F_i}{\partial y_j} \right)_{\substack{i=1,\dots,M \\ j=1,\dots,M}} (\mathbf{a}_0, \mathbf{b}_0)$$

is full rank, and

(c) $|F_i(\mathbf{a}_0, \mathbf{b}_0)|_v < \delta^2$ *for* $1 \le i \le M$

with

$$\delta = \left| \det\left(\left(\frac{\partial F_i}{\partial y_j} \right)_{\substack{i=1,\dots,M \\ j=1,\dots,M}} (\mathbf{a}_0, \mathbf{b}_0) \right) \right|_v \le 1.$$

Let B_ε denote the π-adic open ball of radius ε around the origin. Then for all $\mathbf{a}$ *such that*

$$|\mathbf{a} - \mathbf{a}_0|_v < \delta,$$

there exists a unique system over R

$$\mathbf{g} : B_{\delta^2} \to B_\delta \subset \mathbb{A}^M$$

such that

$$\mathbf{g}(\mathbf{a}_0) = \mathbf{b}_0 \quad \textit{and} \quad \mathbf{F}(\mathbf{a}, g(\mathbf{a})) = 0.$$

Proof. By assumption, we have that

$$|F_i(\mathbf{a}_0, \mathbf{b}_0)|_v < \delta^2 \quad \text{for } 1 \le i \le M.$$

After a translation, we may assume $\mathbf{a}_0 = (0, \dots, 0)$.

For any $\mathbf{a}$ and i, $1 \le i \le M$, $F_i(\mathbf{a}, \mathbf{b}_0)$ has a zero constant term in $x_1, \dots, x_N$; so for

$$|a_i|_v < \delta^2 \quad \text{for } 1 \le i \le N$$

we have

$$|F_i(a_1, \dots, a_N, \mathbf{b}_0)|_v < \delta^2 \quad \text{for } 1 \le i \le M.$$

We may apply Hensel's lemma (Proposition 17), which concludes that there exists a system of power series $\mathbf{g}$ defined over R which takes a point $(a_1,\dots,a_N)$ and sends it to the unique point $\mathbf{g}(a_1,\dots,a_N) = (b_1,\dots,b_M)$, satisfying

(a) $b_1,\dots,b_M \in R$,
(b) $\mathbf{F}(a_1,\dots,a_N,b_1,\dots,b_M) = 0$, and
(c) $|b_i - a_i|_v < \delta$ for $1 \le i \le M$.

Note that since $|a_i|_v < \delta^2$ for $1 \le i \le N$ and $|b_i - a_i|_v < \delta$ for $1 \le i \le M$, we have

$$\mathbf{g} : B_{\delta^2} \to B_\delta. \qquad \square$$

3.2. Proof of Theorem 1. Let $\mathcal{X}/R$ be a smooth projective R-scheme of dimension d whose generic fiber is a nonsingular irreducible projective variety X/K. Let $\phi_R : \mathcal{X}/R \to \mathcal{X}/R$ be an R-morphism and denote the restrictions to the generic fiber and the special fiber as ϕ and $\overline{\phi}$, respectively. Let $P_R \in \mathcal{X}(R)$ be a periodic point of primitive period n for ϕ_R and let $P = P_R(\operatorname{Spec} K)$ and $\overline{P} = P_R(\operatorname{Spec} k)$ with $\overline{P}$ of primitive period m for $\overline{\phi}$. There are three main steps in analyzing the primitive period of the reduction of a periodic point.

- Use good reduction to show that there is an open π-adic neighborhood $\mathcal{U}$ and a system of functions $\mathbf{f}$ regular on $\mathcal{U}$ and defined over K such that ϕ_R is represented by $\mathbf{f}$ as a regular map on $\mathcal{U}$ and all of the iterates of P_R by ϕ_R are contained in $\mathcal{U}$.
- Show that $\mathbf{f}$ has a local power series representation on $\mathcal{U}$ with coefficients in R.
- Noticing that ϕ acts as the cyclic group of order n on the scheme theoretic union $\mathcal{V}$ of the finite set of points $\{P_R, \phi_R(P_R), \dots, \phi_R^{n-1}(P_R)\}$, iterate a local power series representation of $\mathbf{f}$ to obtain information about n.

Recall that we know $m \mid n$ from Corollary 9. Replacing ϕ_R by ϕ_R^m and n by n/m, we may assume that $m = 1$. We resolve the first two steps in the following lemma.

LEMMA 19. *Let $\mathcal{X}/R \subseteq \mathbb{P}_R^N$ be a smooth projective model of X/K a nonsingular irreducible projective variety of dimension d. Let $\phi_R : \mathcal{X}/R \to \mathcal{X}/R$ be an R-morphism and let $P_R \in \mathcal{X}(R)$ be a periodic point of primitive period n for ϕ_R such that $\overline{P}$ is fixed by $\overline{\phi}$. Let $\mathcal{V} \subset \mathcal{X}$ be the scheme theoretic union of $\{P_R, \phi_R(P_R), \dots, \phi^{n-1}(P_R)\}$. Then*

(a) *there exists a $g \in \mathrm{PGL}_{N+1}(R)$ such that*

$$g^{-1}(\{P_R, \phi_R(P_R), \dots, \phi^{n-1}(P_R)\}) \subset (\mathbb{A}_0^N)_R$$

and

$$g^{-1}(P_R) = [1, 0, \dots, 0],$$

(b) *there exists a π-adic neighborhood $\mathcal{U}$ and regular maps $\mathbf{f} = [f_i]$ defined over R on $\mathcal{U}$ such that*

(1) *$\phi_R(\rho)=\mathbf{f}(\rho)$ for all $\rho\in\mathcal{U}$ and all of the iterates of P_R by ϕ_R are contained in $\mathcal{U}$,*

(2) *each $\mathbf{f}_i$ has a power series representation over R on $\mathcal{U}\cap(\mathbb{A}_0^N)_R$ that converges for values in $\mathfrak{m}$ and that we can iterate at P_R,*

(3) *the local system of parameters for $\mathcal{O}_{\mathcal{V},P_R}$ is a subset of the local system of parameters for $\mathcal{O}_{\mathcal{X},P_R}$. We can write $\phi_R|_{\mathcal{V}}=[f_i']$ for regular maps $\mathbf{f}'=[f_i']$ defined over R, where each f_i' has a local power series representation over R on $\mathcal{V}$ in the local system of parameters for $\mathcal{O}_{\mathcal{V},P_R}$ that converges for values in $\mathfrak{m}$ and that we can iterate at P_R.*

Proof. The embedding $\mathcal{X}\subseteq\mathbb{P}_R^N$ induces an embedding $X\subset\mathbb{P}_K^N$. Let $[t_0,\dots,t_N]$ be the coordinates of $\mathbb{P}_R^N$. Let

$$\mathcal{P}=\{P_R,\phi_R(P_R),\dots,\phi_R^{n-1}(P_R)\}.$$

This is a finite set of points in $\mathcal{X}(R)$ whose images on the special fiber are a single point $\overline{P}$. By Lemma 13, we can find a $g\in\mathrm{PGL}_{N+1}(R)$ such that

$$g^{-1}(\mathcal{P})\subset(\mathbb{A}_0^N)_R \quad\text{and}\quad g^{-1}(P_R(\operatorname{Spec}R))=[1,0,\dots,0].$$

Replace ϕ_R by $\phi_R^g=g^{-1}\circ\phi_R\circ g$ and P_R by $g^{-1}(P_R)$. We are now working over $g^{-1}(\mathcal{X})$, so replace $\mathcal{X}$ by $g^{-1}(\mathcal{X})$ and hence $\mathcal{V}$ by $g^{-1}(\mathcal{V})$. Note that $\mathcal{V}\subset(\mathbb{A}_0^N)_R$. Dehomogenize with respect to t_0 and label the affine coordinates

$$T_i=\frac{t_i}{t_0}\quad\text{for } 1\le i\le N.$$

Let $\mathcal{Y}/R=(\mathcal{X}/R)\cap(\mathbb{A}_0^N)_R$. Note that $\mathcal{Y}$ is a smooth irreducible affine scheme and, consequently, integral. We know that $\mathcal{V}\subset\mathcal{Y}$ is a closed subscheme. In particular, we have that

$$\mathfrak{m}_{\mathcal{V},P_R}/\mathfrak{m}_{\mathcal{V},P_R}^2\subset\mathfrak{m}_{\mathcal{Y},P_R}/\mathfrak{m}_{\mathcal{Y},P_R}^2$$

as vector spaces. Hence, we can choose a basis for $\mathfrak{m}_{\mathcal{Y},P_R}/\mathfrak{m}_{\mathcal{Y},P_R}^2$ that contains a basis for $\mathfrak{m}_{\mathcal{V},P_R}/\mathfrak{m}_{\mathcal{V},P_R}^2$ as a subset. Since we are working in $(\mathbb{A}_0^N)_R$, this change of basis comes from an element of $\mathrm{GL}_N(R)$ and, hence, an element of $\mathrm{PGL}_{N+1}(R)$ acting on $\mathcal{X}$. Continue to label the coordinates as $\{T_1,\dots,T_N\}$. Let $g_1,\dots,g_s\in R[T_1,\dots,T_N]$ be equations that define $\mathcal{Y}$, i.e.,

$$\mathcal{Y}=\operatorname{Spec}R[T_1,\dots,T_N]/(g_1,\dots,g_s).$$

Identifying the points $\mathcal{Y}(R)=\operatorname{Hom}(\operatorname{Spec}R,\mathcal{Y})$ with the solutions to $g_1=\cdots=g_s=0$ in $\mathbb{A}_R^N$, we have the resulting map and periodic point on $\mathcal{Y}$:

$$\phi_R:\mathcal{Y}\to\mathcal{X}\quad\text{with } \phi_R:\mathcal{P}_R\to\mathcal{P}_R,\quad\text{and}\quad P_R=(0,\dots,0).$$

Since the scheme $\mathcal{Y}$ is smooth, the determinants of the $(N-d)\times(N-d)$ minors of the Jacobian matrix

$$\left(\frac{\partial g_i}{\partial T_j}\right)_{\substack{i=1,\dots,s\\ j=1,\dots,N}}$$

generate the unit ideal in $R[T_1,\dots,T_N]/(g_1,\dots,g_s)$ (see, for example [11, Section IV.2]). In particular, since R is a discrete valuation ring, P_R a nonsingular point of $\mathcal{Y}$ implies that the determinant of one of the $(N-d)\times(N-d)$ minors of the matrix

$$\left(\frac{\partial g_i}{\partial T_j}(P_R)\right)_{\substack{i=1,\dots,s\\ j=1,\dots,N}}$$

is a unit in R. Relabeling, let

$$\left(\frac{\partial g_i}{\partial T_j}(P_R)\right)_{\substack{i=1,\dots,N-d\\ j=d+1,\dots,N}}$$

be such a minor and, hence,

$$\delta = \left|\det\left(\left(\frac{\partial g_i}{\partial T_j}(P_R)\right)_{\substack{i=1,\dots,N-d\\ j=d+1,\dots,N}}\right)\right|_v = 1.$$

Then the restrictions $T_1,\dots,T_d$ to $\mathcal{Y}$ of the first d coordinates form a system of local parameters of $\mathcal{Y}$ at P_R. Let $\mathcal{Y}'$ be the union of all components that pass through P_R of the scheme defined by the equations $g_1=\cdots=g_{N-d}=0$. Since $\delta\neq 0$, the dimension of the tangent space Θ' to $\mathcal{Y}'$ at P_R is d. Since $\dim_P \mathcal{Y}'\geq d$ and $\dim\Theta'\geq \dim_P\mathcal{Y}'$, we have that $\dim\mathcal{Y}'=d$ and P_R is a non-singular point of $\mathcal{Y}'$. The scheme $\mathcal{Y}'$ is irreducible and reduced, so then $\mathcal{Y}'=\mathcal{Y}$. We have that $\mathcal{Y}$ is defined locally by $N-d$ equations (i.e., $\mathcal{Y}$ is a local complete intersection) and those equations satisfy $\delta=1$.

Since the ring of formal power series is complete, we apply the Implicit Function theorem (Proposition 18) to deduce that there exists a system of power series $F_1,\dots,F_{N-d}$ over R in d variables $T_1,\dots,T_d$ such that $F_i(T_1,\dots,T_d)$ converges for all $|T_j|_v<1=\delta^2$ and

$$g_i(T_1,\dots,T_d,F_1(T_1,\dots,T_d),\dots,F_{N-d}(T_1,\dots,T_d))=0 \tag{3}$$

for $1\leq i\leq N-d$. Moreover, the coefficients of the power series F_i are uniquely determined by (3). Let $\tau:\mathcal{O}_{\mathcal{Y},P_R}\to K[[T]]$ be the uniquely determined map that takes each function to its Taylor series. Assuming that $T_1,\dots,T_d$ are chosen as local parameters, the formal power series

$$\tau(T_{d+1}),\dots,\tau(T_N)$$

also satisfy (3), and, hence, must coincide with $F_1,\dots,F_{N-d}$. It follows that

$$\tau(T_{d+1}),\dots,\tau(T_N)$$

converge for

$$|T_i|_v<1=\delta^2 \quad \text{for } 1\leq i\leq d.$$

We have $P_R = (0,\dots,0)$ in $\mathbb{A}_R^N$ and $\overline{\phi_R(P_R)} = (\overline{0},\dots,\overline{0})$ in $\mathbb{A}_k^N$. Hence, we have

$$\overline{\phi_R^s(P_R)} = (\overline{0},\dots,\overline{0}) \quad \text{for all } s \geq 0.$$

Equivalently,

$$\phi_R^s(P_R) \in \mathfrak{m} \quad \text{for all } s \geq 0.$$

So we have that $\mathcal{P}$ is mapped to the affine neighborhood where $\mathcal{O}_{\mathcal{Y},P_R}$ has its maximal ideal generated by $T_1,\dots,T_d$. Note that we also have that the generators of the maximal ideal of $\mathcal{O}_{\mathcal{V},P_R}$ is a subset of $\{T_1,\dots,T_d\}$. Choose a neighborhood $\mathcal{U} \subset \mathcal{Y}$ such that $\phi_R(P_R) \in \mathcal{U}$.

The morphism ϕ_R can be written locally as a vector of homogeneous polynomials of the same degree D with coefficients in R and at least one coefficient in R^*. Denote $\overline{\phi}$ the restriction of ϕ_R to the special fiber. Recall that $[1,0,\dots,0]$ is a fixed point of the morphism $\overline{\phi}$; hence

$$\overline{\phi([1,0,\dots,0])} = \overline{\phi}(\overline{[1,0,\dots,0]}) = \overline{[1,0,\dots,0]}. \tag{4}$$

Therefore, there is some normalized representation $[\Phi_0,\dots,\Phi_N]$ of ϕ_R near $[1,0,\dots,0]$ for which $\overline{\phi} = [\overline{\Phi_0},\dots,\overline{\Phi_N}]$ is a morphism. Label such a representation as

$$\Phi_i = \sum_{|I|=D} \alpha_{i,I}\mathbf{T}^I,$$

where I is a multi-index. By (4), we have

$$\overline{[\alpha_{0,(D,0,\dots,0)},\dots,\alpha_{N,(D,0,\dots,0)}]} = \overline{[1,0,\dots,0]}. \tag{5}$$

We will now check that $\alpha_{i,(D,0,\dots,0)} \in R^*$ for some $0 \leq i \leq N$. Assume that

$$\alpha_{i,(D,0,\dots,0)} \equiv 0 \mod \pi \quad \text{for } 0 \leq i \leq N.$$

Since $\overline{\phi}(\overline{[1,0,\dots,0]}) = \overline{[1,0,\dots,0]}$ and by assumption every monomial contains one of $T_1,\dots,T_N$, we must have some monomial in $T_1,\dots,T_N$ dividing every $\overline{\Phi_i}$ for $0 \leq i \leq N$. This contradicts the fact that $\overline{\phi}$ is a morphism represented by $[\overline{\Phi_0},\dots,\overline{\Phi_N}]$ at $\overline{[1,0,\dots,0]}$. Consequently, by (5), we have

$$\begin{aligned} \alpha_{i,(D,0,\dots,0)} &\equiv 0 \mod \pi \quad \text{for } 1 \leq i \leq N,\\ \alpha_{0,(D,0,\dots,0)} &\not\equiv 0 \mod \pi \quad \text{i.e., } \alpha_{0,(D,0,\dots,0)} \in R^*. \end{aligned}$$

On the affine neighborhood $\mathcal{U}$, we have that $T_1,\dots,T_d$ form a system of local parameters of $\mathcal{O}_{\mathcal{Y},P}$ so we can dehomogenize and write

$$\left(\frac{\Phi_1}{\Phi_0}(\rho),\dots,\frac{\Phi_d}{\Phi_0}(\rho)\right) = \mathbf{f}(\rho) = (f_1(\rho),\dots,f_d(\rho)) = \phi(\rho) \quad \text{for all } \rho \in \mathcal{U},$$

yielding each f_i as the quotient of polynomials with coefficients in R:

$$f_i = \frac{\sum_{|I|\leq D} \alpha_{i,I}\mathbf{T}^I}{\sum_{|I|\leq D} \alpha_{0,I}\mathbf{T}^I} \quad \text{for } 1 \leq i \leq d.$$

Since $\alpha_{0,(D,0,\ldots,0)} \in R^*$, we can divide the numerators and denominators by $\alpha^{-1}_{0,(D,0,\ldots,0)}$ to get

$$f_i = \frac{\sum_{|I|\leq D} \alpha'_{i,I}\mathbf{T}^I}{1+\sum_{|I|\leq D} \alpha'_{0,I}\mathbf{T}^I}.$$

The denominators are series (in fact polynomials) with coefficients in R and, hence, converge π-adically for $T_i \in \mathfrak{m}$ and $0 \leq i \leq d$, so we can find a multiplicative inverse. We can write

$$f_i = \omega_i + \Lambda_i \mathbf{T} + \cdots,$$

where ω_i and Λ_i are vectors. These power series converge for values in $\mathfrak{m}$ and have coefficients in R, and every iterate of P_R has coordinates in $\mathfrak{m}$. Hence, we can iterate $\mathbf{f}$ at P_R.

We now have near P_R on $\mathcal{Y}$ that

$$\phi_R = \mathbf{f} = \omega + d\mathbf{f}_P\mathbf{T} + (\text{higher degree terms}),$$

where $d\mathbf{f}_P$ is the Jacobian matrix of $\mathbf{f}$ at P_R.

To get the representation of $\phi_R|_{\mathcal{V}}$, we project $\mathbf{f}$ onto $\mathcal{O}_{\mathcal{V},P_R} \subset \mathcal{O}_{\mathcal{Y},P_R}$ by sending $T_{d'+1} = \cdots = T_d = 0$. This gives us a representation $\mathbf{f}'$ for $\phi_R|_{\mathcal{V}}$ with the appropriate properties. □

Proof of Theorem 1. Recall that we know $m \mid n$ from Corollary 9. If $m = n$ we are done; otherwise, replace ϕ_R by ϕ_R^m and n by n/m, so $m = 1$. From Lemma 19, after dehomogenizing, we have $P_R = (0,\ldots,0)$ and on $\mathcal{V}$ we have

$$\phi_R|_{\mathcal{V}} = \mathbf{f} = \omega + d\mathbf{f}_P\mathbf{T} + (\text{higher degree terms}),$$

where $d\mathbf{f}_P$ is the Jacobian of $\mathbf{f}$ at P_R, and we can iterate $\mathbf{f}$ at P_R. The morphism ϕ_R acts on $\mathcal{V}$ as a cyclic group of order n, thus $(\phi_R^n)_{|\mathcal{V}}$ is the identity map; in other words, $d\phi_P^n$ fixes $T_1,\ldots,T_{d'}$. Let r_V be the order of $d\overline{\phi}_P$ on V and note that $r_V \mid n$ since ϕ_R^n fixes $\mathcal{V}$. Replace n by n/r_V, ϕ_R by $\phi_R^{r_V}$, and P_R by $\phi_R^{r_V}(P_R)$. We will continue to write

$$\phi_R|_{\mathcal{V}} = \mathbf{f} = \omega + d\mathbf{f}_P\mathbf{T} + (\text{higher degree terms}).$$

If $n = 1$ we are done, so assume $n \neq 1$. We know that $d\overline{\phi}_P$ fixes $T_1,\ldots,T_{d'}$. In other words, $d\mathbf{f}_P$ is the identity modulo π. Labeling $df_P = I + \pi^c A$ where I is the identity matrix and A has at least one entry in R^*. If $df_P = I$, then set $c = \infty$. Iterating modulo $(\omega)^2$, we have

$$\mathbf{f}^n(0) \equiv \left(nI + \sum_{i=1}^{n} \binom{i}{1} \pi^c A + \cdots + \sum_{i=1}^{n} \binom{i}{n} \pi^{nc}A^n \right) \omega \equiv 0 \mod (\omega)^2.$$

Since ω_j generates (ω) for some $1 \leq j \leq d'$, we must have

$$n \equiv 0 \pmod{\pi}.$$

Replace n by n/p and ϕ_R by ϕ_R^p. If $n=1$ we are done; if not we can repeat the above argument to see that

$$n = mr_V p^e \quad \text{for } e \geq 0.$$

Note that r_V is independent of the choice of local parameters and power series representation for ϕ_R near P_R.

We have that $d\overline{\phi^m}_P$ restricted to $\mathcal{V}$ is an element of $\mathrm{GL}_{d'}(\mathbb{F}_{N\pi})$; consequently, its multiplicative order is bounded by $((N\pi)^{d'}-1)$ [3, Corollary 2]. $\square$

Remark. We can use Theorem 1 to bound the primitive period of a periodic point using good reduction information as in [12, Corollary B], but such a result is not stated here since the proof is identical to the one-dimensional case. Furthermore, these types of bounds, using only information at the primes of good reduction, have been superseded in the one-dimensional case by Benedetto [2, main theorem].

Theorem 20. *Let K be a number field and $X \subseteq \mathbb{P}_K^N$ a projective variety defined over K. Let $\phi : X \to X$ be a morphism defined over K and $P \in X(K)$ a periodic point of primitive period n. For a prime of good reduction $\mathfrak{p} \in K$, let $m_{\mathfrak{p}}$ be the primitive period of $\overline{P}$. Then there are only finitely many primes $\mathfrak{p} \in K$ where $n \neq m_{\mathfrak{p}}$.*

Proof. Let $[t_0, \dots, t_N]$ be coordinates for $\mathbb{P}_K^N$. Denote the ith coordinate of a point $Q \in \mathbb{P}_K^N$ as Q_i. Then we know that, after normalizing the points so that the first non-zero coordinate is 1, we have

$$\phi^n(P)_i - P_i = 0$$

for each coordinate $0 \leq i \leq N$. We know that there are only finitely many divisors of $n \in \mathbb{Z}$, so we have finitely many normalized expressions

$$\phi^s(P)_i - P_i$$

with $s \mid n$ and $s < n$. We have

$$\phi^s(P)_i - P_i \neq 0 \quad \text{for } 0 \leq i \leq N$$

since n is the primitive period of P. Hence, $\phi^s(P)_i - P_i$ is some value in K, which can be factored into finitely many primes. We can do this for each of the finitely many $0 \leq i \leq N$. So there are at most finitely many primes where there is some $s \mid n$ with $s < n$ such that $\overline{\phi^s(P)} = \overline{P}$. Hence, there are only finitely many primes $\mathfrak{p}$ such that $n \neq m_{\mathfrak{p}}$. $\square$

3.3. Proof of Theorem 2. We now bound the exponent e in Theorem 1. We replace ϕ_R by $\phi_R^{mr_V}$ and P_R by $\phi_R^{mr_V}(P_R)$. From Lemma 19, we can reduce to the case of considering a local power series representation of $\phi_R|_{\mathcal{V}}$

at the origin. We write

$$\phi_R|_{\mathcal{V}} = \mathbf{f} = \omega + d\mathbf{f}_0\mathbf{T} + (\text{higher degree terms}),$$

where $d\mathbf{f}_0$ is the identity modulo π. We will use the notation $x_t(P)$ to denote the tth coordinate of a point P. The method of proof is similar to [10].

Definition 21.

- Define $F_k(x_1,\dots,x_n) = \mathbf{f}^{p^k}(x_1,\dots,x_n)$.
- Define $\mathbf{x}_0 = (0,\dots,0)$ and $\mathbf{x}_k = \mathbf{f}^{p^k}(\mathbf{x}_0) = F_k(\mathbf{x}_0) = \mathbf{f}^p(\mathbf{x}_{k-1}) = F_{k-1}^p(\mathbf{x}_0)$.
- Define $b_k = \min_{i=1,\dots,d'} v(x_i(\mathbf{x}_k))$.
- Write $d\mathbf{f}_0^{p^k} = I + \pi^{c_k}A_k$ where I is the identity matrix and A_k has at least one entry in R^*. If $d\mathbf{f}_0^{p^k} = I$, then we set $c_k = \infty$.
- Assume that $(0,\dots,0)$ has primitive period p^e.

Lemma 22. *For $k = 1,\dots,e-1$, we have*

$$b_{k+1} \ge \min\bigl(2b_k, b_k + (p-1)c_k, b_k + v(p)\bigr),$$
$$c_{k+1} \ge \min\bigl(b_k, c_k + v(p), pc_k\bigr).$$

Furthermore,

$$\min(b_k, c_k) \le v(p)$$

and if $b_k > v(p)$, then $(p-1)c_k \le v(p)$.

Proof. We have

$$\begin{aligned}
\mathbf{x}_{k+1} &\equiv F_k^p(\mathbf{x}_0) \\
&\equiv \bigl(I + (I+\pi^{c_k}A_k) + \cdots + (I+\pi^{c_k}A_k)^{p-1}\bigr)\mathbf{x}_k \pmod{\pi^{2b_k}} \\
&\equiv \left(pI + \sum_{i=1}^{p-1}\binom{i}{1}\pi^{c_k}A_k + \cdots \right.\\
&\qquad \left. + \sum_{i=1}^{p-1}\binom{i}{p-1}\pi^{(p-1)c_k}A_k^{p-1}\right)\mathbf{x}_k \pmod{\pi^{2b_k}}.
\end{aligned} \tag{6}$$

Using the identity

$$\sum_{j=1}^{n}\binom{j}{k} = \frac{n+1}{n-k+1}\binom{n}{k} \tag{7}$$

we see that each intermediary sum in (6) is divisible by p. So we have

$$b_{k+1} \ge \min\bigl(2b_k, b_k + (p-1)c_k, b_k + v(p)\bigr).$$

By the chain rule, we have

$$\begin{aligned}
df_0^{k+1} &\equiv (df_0^k)^p \equiv (I+\pi^{c_k}A_k)^p \pmod{\pi^{b_k}} \\
&\equiv I + \binom{p}{1}\pi^{c_k}A_k + \binom{p}{2}\pi^{2c_k}A_k^2 + \cdots + \pi^{pc_k}A_k^p \pmod{\pi^{b_k}}.
\end{aligned}$$

Therefore,

$$c_{k+1} \geq \min\bigl(b_k, c_k + v(p), pc_k\bigr).$$

To prove the final statement consider (6) with $k = e - 1$:

$$\mathbf{x}_e \equiv \bigl(pI + \cdots + \pi^{(p-1)c_{e-1}} A_{e-1}^{p-1}\bigr)\mathbf{x}_{e-1} \pmod{\pi^{2b_{e-1}}}.$$

By (7), each intermediary term is divisible by $p\pi^{c_{e-1}}$. If $b_{e-1} > v(p)$, then we must have $(p-1)c_{e-1} \leq v(p)$. If the assertion fails for some k, then $\min(b_{e-1}, c_{e-1}) > v(p)$. □

LEMMA 23. *For $k = 1, \ldots, e - 1$ and $p \neq 2$ we have*

$$b_k \geq 2^k \quad \textit{for } b_k \leq v(p),$$
$$c_k \geq 2^{k-1}$$

and for $p = 2$, we have

$$b_k \geq G_{k+1} \quad \textit{for } b_k \leq v(p),$$
$$c_k \geq G_k,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $G_k = \frac{\alpha^k - (-1/\alpha)^k}{\sqrt{5}}$ is the kth Fibonacci number.

Proof. Consider first $p \neq 2$. Since $b_1 \geq 1$ and $c_1 \geq 1$ we have established $k = 1$. Assume $b_k \geq 2^k$ and $c_k \geq 2^{k-1}$ for some k. If $b_k \leq v(p)$, then from Lemma 22 we have $b_{k+1} \geq 2^{k+1}$ and $c_{k+1} \geq 2^k$. If $b_k > v(p)$, then from Lemma 22 we have $c_{k+1} \geq pc_k \geq 3 \cdot 2^{k-1}$, establishing the lemma for $p \neq 2$.

If $p = 2$, we have $b_1 \geq 1$ and $c_1 \geq 1$ establishing $k = 1$. So assume $b_k \geq G_{k+1}$ and $c_k \geq G_k$ for some k. If $b_k \leq v(p)$, then by Lemma 22 we have $b_{k+1} \geq G_{k+2}$ and $c_{k+1} \geq G_{k+1}$ since $G_{i+1} \leq 2 \cdot G_i$ for each $i > 1$. Similarly, if $b_k > v(p)$, then $c_{k+1} \geq 2c_k \geq G_{k+1}$. □

Proof of Theorem 2. From Lemma 22, we have $\min(c_{e-1}, b_{e-1}) \leq v(p)$, and if $b_{e-1} > v(p)$, then $(p-1)c_{e-1} \leq v(p)$. Using Lemma 23, for $p \neq 2$ the exponent e must satisfy

$$2^{e-1} \leq v(p)$$

and for $p = 2$ the exponent e must satisfy

$$F_{e-1} = \frac{\alpha^{e-1} - (-1/\alpha)^{e-1}}{\sqrt{5}} \leq v(2).$$

The formulas stated are now clear. □

Proof of Corollary 3. Apply the bounds from Theorem 1 and Theorem 2 with $v(p) = 1$. To bound m we note that there are at most p choices for each coordinate of $\overline{P}$, so we have $m \leq p^d$. □

4. Proof of Theorem 4

Proof. Since ϕ is étale it sends smooth points to smooth points and permutes the irreducible components of X. Therefore, there exists a finite integer l such that $\phi^l : Y \to Y$. Since Y is smooth and irreducible and the restriction of ϕ^l to Y has good reduction, we may apply Theorem 1 and Theorem 2. $\square$

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