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IDEMPOTENT SUBQUOTIENTS OF SYMMETRIC QUASI-HEREDITARY ALGEBRAS

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ABSTRACT. We show how any finite-dimensional algebra can be realized as an idempotent subquotient of some symmetric quasi-hereditary algebra. In the special case of rigid symmetric algebras, we show that they can be realized as centralizer subalgebras of symmetric quasi-hereditary algebras. We also show that the infinite-dimensional symmetric quasi-hereditary algebras we construct admit quasi-hereditary structures with respect to two opposite orders, that they have strong exact Borel and Δ-subalgebras and the corresponding triangular decompositions.

1. Introduction

A classical result of Dlab and Ringel (see [DR2]) says that every finite-dimensional algebra can be realized as a centralizer subalgebra of some quasi-hereditary algebra. Motivated by the discovery of (infinite-dimensional) symmetric quasi-hereditary algebras in [Pe] (see also [CT], [MT1], [MT2], [BS]), we address the question whether every symmetric finite-dimensional algebra can be realized as a centralizer subalgebra of some symmetric quasi-hereditary algebra. Note that a symmetric quasi-hereditary algebra is either semisimple or infinite-dimensional.

In the present paper, we generalize the construction from [DR2] and show how one can realize finite-dimensional algebras as centralizer subalgebras of certain infinite-dimensional quasi-hereditary algebras. Under some natural assumptions on the original algebra (for example, if the original algebra is symmetric and rigid), we obtain that the resulting infinite-dimensional quasi-hereditary algebra is symmetric as well. In the general case, we show that

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every finite-dimensional algebra can be realized as an idempotent subquotient of a symmetric quasi-hereditary algebra. Our construction produces many new examples of symmetric quasi-hereditary algebras. Note that, in the case when the original algebra was not weakly symmetric, it of course cannot be realized as a centralizer subalgebra of any symmetric algebra. However, we do not know whether all symmetric finite-dimensional algebras can be realized as centralizer subalgebras of some symmetric quasi-hereditary algebras (an additional assumption in Theorem 5 is essential for our arguments).

The infinite-dimensional (symmetric) quasi-hereditary algebras, which we construct, have many interesting properties. To start with, all these algebras are quasi-hereditary with respect to two natural orders (one of them being the opposite of the other one). The standard and costandard modules for these structures have a natural description in terms of the original algebra. We also show that all these algebras have Δ-subalgebras in the sense of König ([Ko1], [Ko2]). Assuming that the original algebra is graded, we show that our algebras have a strong exact Borel subalgebra in the sense of König ([Ko1], [Ko2]), as well as the corresponding triangular decomposition.

The paper is organized as follows: In Section 2, we extend the construction from [DR2] and realize finite-dimensional algebras as centralizer subalgebras of some infinite-dimensional algebras and show that these infinite-dimensional algebras are quasi-hereditary with respect to two natural opposite orders. In Section 3, we prove that for symmetric rigid finite-dimensional algebras the infinite-dimensional quasi-hereditary algebras constructed in Section 2 are symmetric as well. For arbitrary algebras, we show how the construction can be generalized to realize every finite-dimensional algebra as an idempotent subquotient of some symmetric quasi-hereditary algebra. In Section 4, we describe strong exact Borel and Δ-subalgebras and the corresponding triangular decompositions for our infinite-dimensional quasi-hereditary algebras. Finally, in Section 5, we discuss some examples, in particular those coming from Schur algebras and the BGG category $\mathcal{O}$.

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers and $\Bbbk$ be an algebraically closed field. Consider a basic $\Bbbk$-linear category $\mathcal{A}$ which satisfies the following assumptions:

(I) $\mathcal{A}$ has finitely or countably many objects;
(II) for any $x, y \in \mathcal{A}$, the $\Bbbk$-vector space $\mathcal{A}(x, y)$ is finite dimensional;
(III) for any $x \in \mathcal{A}$, there exist only finitely many $y \in \mathcal{A}$ such that $\mathcal{A}(x, y) \neq 0$;
(IV) for any $x \in \mathcal{A}$, there exist only finitely many $y \in \mathcal{A}$ such that $\mathcal{A}(y, x) \neq 0$;
(V) for any $x \in \mathcal{A}$, the endomorphism algebra $\mathcal{A}(x, x)$ is local.

Under these assumptions, all indecomposable projective $\mathcal{A}$-modules $\mathcal{A}(x, \_)$ are finite-dimensional. As, clearly, the opposite category $\mathcal{A}^{\mathrm{op}}$ also satisfies all these assumptions, we obtain that all indecomposable injective $\mathcal{A}$-modules $\mathcal{A}^{\mathrm{op}}(\_,x)^* = \mathrm{Hom}_{\Bbbk}(\mathcal{A}^{\mathrm{op}}(\_,x),\Bbbk)$ are finite-dimensional as well. We refer the reader to [MOS] for generalities on modules over $\Bbbk$-linear categories. We will often loosely call such categories "algebras" (as they can be realized using infinite-dimensional associative quiver algebras which do not have a unit element in the general case) and use for them the standard matrix notation with infinite matrices. As in [CT], [MT1], we will call such algebras *quasi-hereditary* if their module categories are highest weight categories [CPS]. For $x \in \mathcal{A}$, we denote by $\mathtt{e}_x$ the identity element in $\mathcal{A}(x,x)$.

In this paper, we will study the category of finite-dimensional modules over a category, satisfying conditions (I)–(V). This category is obviously an Abelian Krull–Schmidt category having enough projectives and injectives.

Assume that for some $N \in \mathbb{N}$ we have a (fixed) finite filtration of $\mathcal{A}$ by two-sided ideals as follows:

$$\mathcal{A} = \mathcal{I}_0 \supsetneq \mathcal{I}_1 \supsetneq \mathcal{I}_2 \supsetneq \cdots \supsetneq \mathcal{I}_N = 0. \tag{1}$$

Assume further that $\mathcal{I}_i\mathcal{I}_j \subset \mathcal{I}_{i+j}$ and that $\mathcal{I}_i/\mathcal{I}_{i+1}$ are semi-simple as $\mathcal{A}$-bimodules.

Consider the new category $\mathfrak{A}$, whose objects are $x[i]$, $x \in \mathcal{A}$, $i \in \mathbb{Z}$. For $x,y \in \mathcal{A}$ and $i,j \in \mathbb{Z}$ set $\mathfrak{A}(x[i],y[j]) = \mathcal{A}(x,y)$. Then the multiplication in $\mathcal{A}$ induces a multiplication in $\mathfrak{A}$, which makes $\mathfrak{A}$ into a category. The category $\mathfrak{A}$ comes together with the natural action of $\mathbb{Z}$ by autoequivalences via shifts $[i]$, $i \in \mathbb{Z}$ (here $[1]$ means "shift by one to the right"). The category $\mathfrak{A}$ is equivalent to the category $\mathcal{A}$, moreover, every object from $\mathcal{A}$ has countably many isomorphic copies in $\mathfrak{A}$. We shall think of $\mathfrak{A}$ also as of infinite matrices of the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} & \cdots \\ \cdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} & \cdots \\ \cdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} & \cdots \\ \cdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Denote by $\mathfrak{B}$ the subcategory of $\mathfrak{A}$, which contains all objects but only the following morphisms: For $x,y \in \mathcal{A}$ and $i,j \in \mathbb{Z}$, set

$$\mathfrak{B}(x[i],y[j]) = \begin{cases} \mathfrak{A}(x[i],y[j]), & i \geq j; \\ \mathcal{I}_{j-i}(x,y), & \text{otherwise.} \end{cases}$$

One can think of $\mathfrak{B}$ also as of infinite matrices of the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} & \cdots \\ \cdots & \mathcal{I}_1 & \mathcal{A} & \mathcal{A} & \mathcal{A} & \cdots \\ \cdots & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \mathcal{A} & \cdots \\ \cdots & \mathcal{I}_3 & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consider the subset $\mathfrak{I}$ of $\mathfrak{B}$ with the same set of objects and morphisms given by

$$\mathfrak{I}(x[i],y[j]) = \begin{cases} \mathcal{I}_{N-(i-j)}(x,y), & 0<i-j<N; \\ \mathfrak{B}(x,y), & N \leq i-j; \\ 0, & \text{otherwise}. \end{cases}$$

The set $\mathfrak{I}$ is not a subcategory as it does not contain identity morphisms on objects. One can think of $\mathfrak{I}$ also as of infinite matrices of the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & \mathcal{I}_{N-1} & \mathcal{I}_{N-2} & \mathcal{I}_{N-3} & \cdots \\ \cdots & 0 & 0 & \mathcal{I}_{N-1} & \mathcal{I}_{N-2} & \cdots \\ \cdots & 0 & 0 & 0 & \mathcal{I}_{N-1} & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that $\mathfrak{I}$ is an ideal of $\mathfrak{B}$. Define the category $\mathfrak{C} = \mathfrak{C}(\mathcal{A}) = \mathfrak{B}/\mathfrak{I}$. One can think of $\mathfrak{C}$ as of infinite matrices of the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \mathcal{A}/\mathcal{I}_{N-2} & \mathcal{A}/\mathcal{I}_{N-3} & \cdots \\ \cdots & \mathcal{I}_1 & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \mathcal{A}/\mathcal{I}_{N-2} & \cdots \\ \cdots & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \cdots \\ \cdots & \mathcal{I}_3 & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that, given $x \in \mathcal{I}_i$ and some class $a+\mathcal{I}_j \in \mathcal{A}/\mathcal{I}_j$ we have $x(a+\mathcal{I}_j) \subset xa+\mathcal{I}_{i+j}$ due to our assumption that $\mathcal{I}_i\mathcal{I}_j \subseteq \mathcal{I}_{i+j}$, so multiplication of these matrices is well-defined. Note that, using the matrix notation, left modules are columns, while right modules are rows.

LEMMA 1. *The category $\mathfrak{C}$ satisfies conditions* (I)–(V).

Proof. The conditions (I), (II) and (V) follow directly from the definitions. To prove the condition (II), we observe that from the definition it follows that for $x,y \in \mathcal{A}$ and $i,j \in \mathbb{Z}$ from $\mathfrak{C}(x[i],y[j]) \neq 0$ we necessarily have $\mathcal{A}(x,y) \neq 0$

and $|i-j| \leq N$. This implies the condition (III) and the condition (IV) is checked similarly. This completes the proof. □

We consider two natural linear orders on $\mathbb{Z}$, we call the order where $i < i+1$ the first order, and the one where $i > i+1$ the second order. These orders induce partial orders on the equivalence classes of primitive idempotents in $\mathfrak{C}(\mathcal{A})$, which we will also call the first and the second orders, respectively. From Lemma 1, we have that all indecomposable projective and injective modules over $\mathfrak{C}$ are finite-dimensional. Hence, we can define both standard and costandard modules with respect to both orders defined above in the same way as it is done for finite-dimensional quasi-hereditary algebras (see [DR1], [CT], [MT1]). The following statement is a generalization of the main construction from [DR2].

PROPOSITION 2. (i) *Left standard modules in the first order are given by direct summands of the following modules*:

$$\Delta_{\mathfrak{C}}^{1,l} = \begin{pmatrix} \vdots \\ \mathcal{A}/\mathcal{I}_1 \\ \mathcal{A}/\mathcal{I}_1 \\ \mathcal{A}/\mathcal{I}_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

(ii) *Left standard modules in the second order are given by direct summands of the following module*:

$$\Delta_{\mathfrak{C}}^{2,l} = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ \mathcal{A}/\mathcal{I}_1 \\ \mathcal{I}_1/\mathcal{I}_2 \\ \mathcal{I}_2/\mathcal{I}_3 \\ \vdots \end{pmatrix}.$$

(iii) *Right standard modules for the first order are given by direct summands of the following module*:

$$\Delta_{\mathfrak{C}}^{1,r} = \begin{pmatrix} \cdots & \mathcal{I}_2/\mathcal{I}_3 & \mathcal{I}_1/\mathcal{I}_2 & \mathcal{A}/\mathcal{I}_1 & 0 & 0 & \cdots \end{pmatrix}$$

(iv) *Right standard modules for the second order are given by direct summands of the following module*:

$$\Delta_{\mathfrak{C}}^{2,r} = \begin{pmatrix} \cdots & 0 & 0 & \mathcal{A}/\mathcal{I}_1 & \mathcal{A}/\mathcal{I}_1 & \mathcal{A}/\mathcal{I}_1 & \cdots \end{pmatrix}$$

(v) *The category $\mathfrak{C}$ is quasi-hereditary with respect to both orders.*

Proof. Let $i \in \mathbb{Z}$. For the first order, the quotient of $\mathfrak{C}$ modulo the two-sided ideal, generated by all idempotents $\mathtt{e}_x[j]$, $x \in \mathcal{A}$, $j \in \mathbb{Z}$, $j > i$, looks as follows:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & * & * & \mathcal{A}/\mathcal{I}_1 & 0 & \cdots \\ \cdots & * & * & \mathcal{A}/\mathcal{I}_1 & 0 & \cdots \\ \cdots & \mathcal{I}_2/\mathcal{I}_3 & \mathcal{I}_1/\mathcal{I}_2 & \mathcal{A}/\mathcal{I}_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(here we do not care about the asterisks).

Similarly, for the second order, the quotient of $\mathfrak{C}$ modulo the two-sided ideal, generated by all idempotents $\mathtt{e}_x[j]$, $x \in \mathcal{A}$, $j \in \mathbb{Z}$, $j < i$, looks as follows:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \mathcal{A}/\mathcal{I}_1 & \mathcal{A}/\mathcal{I}_1 & \mathcal{A}/\mathcal{I}_1 & \cdots \\ \cdots & 0 & \mathcal{I}_1/\mathcal{I}_2 & * & * & \cdots \\ \cdots & 0 & \mathcal{I}_2/\mathcal{I}_3 & * & * & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As left modules are columns and right modules are rows, the claims (i)–(iv) follow.

The indecomposable right projective $\mathfrak{C}$-module, generated by $\mathtt{e}_x[i]$, $x \in \mathcal{A}$, is a direct summands of the following module P:

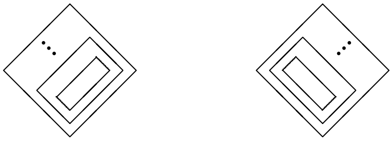
$$\big(\cdots \quad 0 \quad \mathcal{I}_{N-1} \quad \mathcal{I}_{N-2} \quad \cdots \quad \mathcal{I}_1 \quad \mathcal{A} \quad \mathcal{A}/\mathcal{I}_{N-1} \quad \cdots \quad \mathcal{A}/\mathcal{I}_1 \quad 0 \quad \cdots\big).$$

The filtration (1) induces a filtration on every component of P, whose subquotients could be organized into the following rhombal picture:

$$\begin{array}{ccccccccc}
\hline
\mathcal{I}_{N-1} & \cdots & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \mathcal{A}/\mathcal{I}_{N-2} & \cdots & \mathcal{A}/\mathcal{I}_1 \\
\hline
 & & & & \mathcal{I}_0/\mathcal{I}_1 & & & & \\
 & & & \mathcal{I}_1/\mathcal{I}_2 & & \mathcal{I}_0/\mathcal{I}_1 & & & \\
 & & \mathcal{I}_2/\mathcal{I}_3 & & \mathcal{I}_1/\mathcal{I}_2 & & \mathcal{I}_0/\mathcal{I}_1 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\mathcal{I}_{N-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \mathcal{I}_0/\mathcal{I}_1 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & \mathcal{I}_{N-1} & & \mathcal{I}_{N-2}/\mathcal{I}_{N-1} & & \mathcal{I}_{N-3}/\mathcal{I}_{N-2} & & \\
 & & & \mathcal{I}_{N-1} & & \mathcal{I}_{N-2}/\mathcal{I}_{N-1} & & & \\
 & & & & \mathcal{I}_{N-1} & & & & \\
\hline
\end{array} \tag{2}$$

Organizing these subquotients into a filtration of P as shown on the following pictures:

(3)



we obtain a filtration of P by direct summands of the module $\Delta_{\mathfrak{C}}^{1,r}$ and $\Delta_{\mathfrak{C}}^{2,r}$, respectively. This means that right $\mathfrak{C}$-projectives are filtered by standard modules for both orders. The claim (v) follows and the proof is complete. □

Corollary 3. (i) *Left costandard modules for the first order are given by direct summands of the following module*:

$$\nabla_{\mathfrak{C}}^{1,l} = \begin{pmatrix} \vdots \\ (\mathcal{I}_2/\mathcal{I}_3)^* \\ (\mathcal{I}_1/\mathcal{I}_2)^* \\ (\mathcal{A}/\mathcal{I}_1)^* \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

(ii) *Left costandard modules for the second order are given by direct summands of the following module*:

$$\nabla_{\mathfrak{C}}^{2,l} = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ (\mathcal{A}/\mathcal{I}_1)^* \\ (\mathcal{A}/\mathcal{I}_1)^* \\ (\mathcal{A}/\mathcal{I}_1)^* \\ \vdots \end{pmatrix}.$$

(iii) *Right costandard modules for the first order are given by direct summands of the following module*:

$$\nabla_{\mathfrak{C}}^{1,r} = \begin{pmatrix} \cdots & (\mathcal{A}/\mathcal{I}_1)^* & (\mathcal{A}/\mathcal{I}_1)^* & (\mathcal{A}/\mathcal{I}_1)^* & 0 & 0 & \cdots \end{pmatrix}.$$

(iv) *Right costandard modules for the second order are given by direct summands of the following module*:

$$\nabla_{\mathfrak{C}}^{2,r} = \begin{pmatrix} \cdots & 0 & 0 & (\mathcal{A}/\mathcal{I}_1)^* & (\mathcal{I}_1/\mathcal{I}_2)^* & (\mathcal{I}_2/\mathcal{I}_3)^* & \cdots \end{pmatrix}.$$

Proof. This follows from Proposition 2 applying duality. □

COROLLARY 4. *For every $x \in \mathcal{A}$ and every $i \in \mathbb{Z}$, there is an isomorphism* $\nabla^{2,l}_{\mathfrak{C}}(x,i) \cong \Delta^{1,l}_{\mathfrak{C}}(x,i+N)$.

Proof. Since the $\mathcal{A}$-module $\mathcal{A}/\mathcal{I}_1$ is semi-simple by our assumptions, the claim follows directly from Proposition 2(i) and Corollary 3(ii). □

Note that, by construction, the original category $\mathcal{A}$ is a centralizer subcategory of the category $\mathfrak{C}$.

3. Algebras as idempotent subquotients of symmetric quasi-hereditary algebras

From now on, we assume that $\mathcal{A}$ has finitely many objects. Let $A = \bigoplus_{x,y\in\mathcal{A}} \mathcal{A}(x,y)$ be the associative algebra of $\mathcal{A}$ with the natural multiplication. Then A is a finite-dimensional algebra and we may assume that it is given by a quiver Q with set of vertices $\{1,\dots,n\}$ and relations R. As in the previous section, we fix a filtration of A by two-sided ideals

$$A = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_N = 0 \tag{4}$$

with semisimple subquotients and such that $I_i I_j \subset I_{i+j}$. For example, we can take (4) to be the radical filtration of A. For $k \in \{1,\dots,n\}$, we denote by e_k the idempotent corresponding to the vertex k in A, and we denote the corresponding idempotent of $A[i]$ (that is in the (i,i)-th matrix position) by $e_{k,i}$. Set $\mathfrak{C} := \mathfrak{C}(A)$.

THEOREM 5. *Assume that A is symmetric with the symmetric trace form $(\cdot,\cdot)$ and that $(\cdot,\cdot)$ induces a nondegenerate pairing between A/I_j and I_{N-j} for every j. Then the algebra $\mathfrak{C}$ is symmetric.*

Proof. Define a bilinear form $(\cdot,\cdot)_{\mathfrak{C}}$ on $\mathfrak{C}$, by setting

$$(a_{i,j}, b_{k,l})_{\mathfrak{C}} := \delta_{j,k}\delta_{i,l}(a,b),$$

where $a,b \in A$ (in a suitable ideal if $i > j$ resp. $k > l$), $i,j,k,l \in \mathbb{Z}$, and $a_{i,j}$ means the element a in matrix position (i,j).

The form $(\cdot,\cdot)_{\mathfrak{C}}$ is bilinear, symmetric and associative by construction. Again, by construction, the form $(\cdot,\cdot)_{\mathfrak{C}}$ pairs matrix positions (i,j) and (j,i). By the definition of $\mathfrak{C}$, the corresponding components in these positions are A/I_s and I_{N-s} for some s. By our assumption, the form $(\cdot,\cdot)$ induces a nondegenerate pairing of A/I_s and I_{N-s}. This yields that $(\cdot,\cdot)_{\mathfrak{C}}$ is nondegenerate as well, completing the proof. □

COROLLARY 6. *Assume that A is symmetric and that* (4) *is both the radical and the socle filtration of ${}_AA$ (i.e., ${}_AA$ is rigid). Then $\mathfrak{C}$ is symmetric.*

Proof. By our assumptions, the filtration (4) is the unique Loewy filtration of ${}_AA$. The form $(\cdot,\cdot)$ pairs it with another Loewy filtration, and hence with itself. This yields that $(\cdot,\cdot)$ induces a nondegenerate pairing between A/I_j and I_{N-j} for every j and the claim follows from Theorem 5. □

Some other examples to which Theorem 5 can be applied come from the category $\mathcal{O}$ and will be discussed later on (see Example 25). If A is not symmetric (or if it is symmetric but does not satisfy the assumptions of Theorem 5), we cannot realize A as a centralizer subalgebra of some symmetric quasi-hereditary algebra, but instead as an idempotent subquotient. This goes as follows.

Assume that (4) is the radical filtration of A. We form a new algebra $\tilde{A}$ by attaching, for every vertex k, a vertex $\tilde{k}$ and an arrow $k \to \tilde{k}$, keeping the original relations R, defining the algebra A. Then A is a centralizer subalgebra of $\tilde{A}$ (corresponding to nontilded vertices) in the natural way, and $\operatorname{Rad}\tilde{A}$ has nilpotency degree $N+1$. Moreover, the algebra A is also an idempotent quotient of $\tilde{A}$, obtained by factoring out the two-sided ideal, generated by idempotents, associated with the new (tilded) vertices. Set $\mathbf{N} = \{1,\dots,n\}$, $\tilde{\mathbf{N}} = \{\tilde{1},\dots,\tilde{n}\}$, and $\overline{\mathbf{N}} = \mathbf{N} \cup \tilde{\mathbf{N}}$.

Now $\operatorname{soc}_{\tilde{A}}\tilde{A}$ consists of simple modules with indices $\tilde{k}$. The right projective $e_k\tilde{A}$ for $\tilde{A}$, corresponding to a vertex $k \in \mathbf{N}$, is the same as the right projective for A at the same vertex. The right projective $e_{\tilde{k}}\tilde{A}$ at vertex $\tilde{k} \in \tilde{\mathbf{N}}$ is an extension of the simple at $\tilde{k}$ with the right projective at k (the simple extending the top of e_kA), hence has a longer Loewy length. Therefore, $e_k \operatorname{Rad}^N \tilde{A} = 0$ or, equivalently, the Loewy length N_k^r of $e_k\tilde{A}$ is strictly less than the nilpotency degree of $\operatorname{Rad}\tilde{A}$ (which is $N+1$). Let $\tilde{A}e_k$ be the left projective at vertex k, N_k^l its Loewy length.

We now take $\mathfrak{C} = \mathfrak{C}(\tilde{A})$ (with respect to the radical filtration) and form the trivial extension $\mathfrak{D} = \mathfrak{D}(\tilde{A})$ of $\mathfrak{C}$ with its "restricted dual" $\mathfrak{C}$-bimodule

$$\mathfrak{C}^* := \bigoplus_{i,j\in\mathbb{Z};x,y\in\overline{\mathbf{N}}} \operatorname{Hom}_{\Bbbk}(e_{y,j}\mathfrak{C}e_{x,i},\Bbbk)$$

(see [Ha, Section 3.1]). Being a trivial extension of $\mathfrak{C}$, the algebra $\mathfrak{D}$ is automatically symmetric. To make the notation consistent with the previous section, from now on we assume that the nilpotency degree of $\operatorname{Rad}\tilde{A}$ is N.

We now extend our first order in the following way: for $(k,i),(l,j) \in \overline{\mathbf{N}} \times \mathbb{Z}$ we set $(k,i) > (l,j)$ if $i > j$ or if $i = j$, $k \in \mathbf{N}$ and $l \in \tilde{\mathbf{N}}$. We will again call this order the first order.

PROPOSITION 7. *The algebra $\mathfrak{D}(\tilde{A})$ is quasi-hereditary with respect to the first order and for left standard $\mathfrak{D}$-modules we have $\Delta^{1,l}_{\mathfrak{D}}(k,i) = \Delta^{1,l}_{\mathfrak{C}}(k,i)$, $k \in \overline{\mathbf{N}}$, $i \in \mathbb{Z}$.*

Proof. We first consider $\mathfrak{C}$. Let $e_{k,i}$ denote the idempotent in $\tilde{A}$ at the vertex $k \in \overline{\mathbf{N}}$, in matrix position i,i. With respect to our first order, left standard modules $\Delta^{1,l}_{\mathfrak{C}}(k,i)$ are uniserial with a filtration with composition factors

$$L^l(k,i), \qquad L^l(k,i-1), \qquad \dots, \qquad L^l(k,i-N+1)$$

read from top to bottom (see Proposition 2(i)). Then, by (2), the left projective $\mathfrak{C}e_{k,i}$ for $\mathfrak{C}$ has a filtration with subquotients

$$\Delta^{1,l}_{\mathfrak{C}}(k,i), \qquad \bigoplus_{j\in J_1} \Delta^{1,l}_{\mathfrak{C}}(j,i+1), \qquad \dots, \qquad \bigoplus_{j\in J_{N^l_k}} \Delta^{1,l}_{\mathfrak{C}}(j,i+N^l_k),$$

where $\operatorname{Rad}^m \tilde{A}e_k / \operatorname{Rad}^{m+1} \tilde{A}e_k \cong \bigoplus_{j\in J_m} L^l(j)$.

Similarly, the right projective $e_{k,i}\mathfrak{C}$ has a filtration with subquotients

$$\Delta^{2,r}_{\mathfrak{C}}(k,i), \qquad \bigoplus_{j\in \hat{J}_1} \Delta^{2,r}_{\mathfrak{C}}(j,i-1), \qquad \dots, \qquad \bigoplus_{j\in \hat{J}_{N^r_k}} \Delta^{2,r}_{\mathfrak{C}}(j,i-N^r_k),$$

where $e_k \operatorname{Rad}^m \tilde{A}/e_k \operatorname{Rad}^{m+1} \tilde{A} \cong \bigoplus_{j\in \hat{J}_m} L^r(j)$. Hence, the left injective $(e_{k,i}\mathfrak{C})^*$ has a filtration with subquotients

$$\bigoplus_{j\in \hat{J}_{N^r_k}} \nabla^{2,l}_{\mathfrak{C}}(j,i-N^r_k), \qquad \dots, \qquad \bigoplus_{j\in \hat{J}_1} \nabla^{2,l}_{\mathfrak{C}}(j,i-1), \qquad \nabla^{2,l}_{\mathfrak{C}}(k,i)$$

and thus, by the isomorphism $\nabla^{2,l}_{\mathfrak{C}}(k,i) \cong \Delta^{1,l}_{\mathfrak{C}}(k,i+N)$ (Corollary 4), a filtration with subquotients

$$\bigoplus_{j\in \hat{J}_{N^r_k}} \Delta^{1,l}_{\mathfrak{C}}(j,i+N-N^r_k), \qquad \dots,$$

$$\bigoplus_{j\in \hat{J}_1} \Delta^{1,l}_{\mathfrak{C}}(j,i+N-1), \qquad \Delta^{1,l}_{\mathfrak{C}}(k,i+N).$$

We now claim that $\mathfrak{D} = \mathfrak{D}(\tilde{A})$ is quasi-hereditary with $\Delta^{1,l}_{\mathfrak{D}}(k,i) = \Delta^{1,l}_{\mathfrak{C}}(k,i)$. As the projective module $\mathfrak{D}e_{k,i}$ has a filtration with subquotients $\mathfrak{C}e_{k,i}$ and $(e_{k,i}\mathfrak{C})^*$, which both have $\Delta^{1,l}_{\mathfrak{D}}$-filtrations by above, $\mathfrak{D}e_{k,i}$ also has a $\Delta^{1,l}_{\mathfrak{D}}$-filtration. So it suffices to check that all standard modules appearing in $(e_{k,i}\mathfrak{C})^*$ have larger index than (k,i). To see this, we need to distinguish two cases.

The first case is when $k \in \mathbf{N}$. In this case, the smallest second index of the standard modules appearing in $(e_{k,i}\mathfrak{C})^*$ is $i + N - N^r_k$. But, as seen above, for $k \in \{1,\dots,n\}$, $N^r_k < N$, so $i + N - N^r_k > i$, which is what we need.

The second case is when $k \in \tilde{\mathbf{N}}$. In this case, the smallest second index of the standard modules appearing in $(e_{k,i}\mathfrak{C})^*$ can well be i, however, in this case $P^r(k)$ has simple top $L^r(k)$ and all other composition factors are of the form $L^r(j)$, with $j \in \{1, \dots, n\}$. Therefore, the standard modules appearing in $(e_{k,i}\mathfrak{C})^*$ with smallest second index, namely $\Delta^{1,l}_{\mathfrak{C}}(j,i)$, have first index j where $L^r(j)$ occurs in $e_k \operatorname{Rad}^{N^r_k} \tilde{A}$, so $j \in \mathbf{N}$, and $(k,i) < (j,i)$. This completes the proof that $\mathfrak{D}$ is quasi-hereditary. □

From Proposition 7 and [MT1, Corollary 5] it follows that, with respect to the first order, right $\mathfrak{D}$-projectives also have standard filtrations. The corresponding standard modules are described as follows.

LEMMA 8. *The right standard module $\Delta^{1,r}_{\mathfrak{D}}(i,k)$ for $\mathfrak{D}$ is an extension of the $\mathfrak{C}$-modules $\Delta^{1,r}_{\mathfrak{C}}(i,k)$ and $\nabla^{2,r}_{\mathfrak{C}}(i-N+1,k)$.*

Proof. The right projective module $e_{k,i}\mathfrak{D}$ has a filtration with subquotients $e_{k,i}\mathfrak{C}$ and $(\mathfrak{C}e_{k,i})^*$. The module $e_{k,i}\mathfrak{C}$ is filtered by

$$\Delta^{1,r}_{\mathfrak{C}}(k,i), \qquad \Delta^{1,r}_{\mathfrak{C}}(k,i+1), \qquad \dots, \qquad \Delta^{1,r}_{\mathfrak{C}}(k,i+N-1)$$

and the module $\mathfrak{C}e_{k,i}$ is filtered by

$$\Delta^{2,l}_{\mathfrak{C}}(k,i), \qquad \Delta^{2,l}_{\mathfrak{C}}(k,i-1), \qquad \dots, \qquad \Delta^{2,l}_{\mathfrak{C}}(k,i-N+1).$$

Therefore, the module $(\mathfrak{C}e_{k,i})^*$ is filtered by

$$\nabla^{2,r}_{\mathfrak{C}}(k,i-N+1), \qquad \nabla^{2,r}_{\mathfrak{C}}(k,i-N+2), \qquad \dots, \qquad \nabla^{2,r}_{\mathfrak{C}}(k,i).$$

Let X denote the quotient of $e_{k,i}\mathfrak{D}$ modulo the trace of all $e_{k,j}\mathfrak{D}$, $j > i$. Obviously $\Delta^{1,r}_{\mathfrak{C}}(k,i)$ is a quotient of X. Since none of modules $e_{k,j}\mathfrak{D}$, $j > i$, contains $L^r(k,i-N+1)$, $\nabla^{2,r}_{\mathfrak{C}}(k,i-N+1)$ is a subquotient of X as well. By definition, none of other $\Delta^{1,r}_{\mathfrak{C}}(k,j)$ contributes to X, which yields that X has a quotient $\tilde{X}$, which is an extension of $\Delta^{1,r}_{\mathfrak{C}}(k,i)$ by $\nabla^{2,r}_{\mathfrak{C}}(k,i-N+1)$.

As $\mathfrak{C}$ is quasi-hereditary with respect to the second order, we also have a quotient $\Delta^{2,r}_{\mathfrak{C}}(k,i)$ which is uniserial with a filtration $L^r(k,i), L^r(k,i+1), \dots, L^r(k,i+N-1)$. Since $L^r(k,i+1)$ is in the top of the kernel of $e_{k,i}\mathfrak{D} \twoheadrightarrow X$, we know that $\Delta^{1,r}_{\mathfrak{D}}(k,i+1)$ appears as a subquotient of a standard filtration of $e_{k,i}\mathfrak{D}$. Inductively, we obtain that the modules

$$\Delta^{1,r}_{\mathfrak{D}}(k,i), \qquad \Delta^{1,r}_{\mathfrak{D}}(k,i+1), \qquad \dots, \qquad \Delta^{1,r}_{\mathfrak{D}}(k,i+N-1)$$

appear as subquotients of a standard filtrations of $e_{k,i}\mathfrak{D}$. Each of those $\Delta^{1,r}_{\mathfrak{D}}(k,j)$ has a quotient which is an extension of $\Delta^{1,r}_{\mathfrak{C}}(k,j)$ by $\nabla^{2,r}_{\mathfrak{C}}(k,j-$

$N+1$) and we see that this exhausts the whole module. Hence, the surjection of X onto $\tilde{X}$ must be an isomorphism and right standard modules in the first order for $\mathfrak{D}$ are of the desired form. □

Corollary 9. *The algebra $\mathfrak{D}$ is quasi-hereditary with respect to the second order as well.*

Proof. Since $\mathfrak{D}$ is symmetric, projective and injective $\mathfrak{D}$-modules coincide. By Proposition 7, left standard $\mathfrak{D}$-modules with respect to the first order are uniserial and coincide with the corresponding $\mathfrak{C}$-modules. Take a standard filtration of a left projective $\mathfrak{D}$-module. Applying duality, we get a costandard filtration of a right injective $\mathfrak{D}$-module, which is also a right projective.

Taking into account that these right costandard modules coincide, up to shift, with right standard modules with respect to the second order, we obtain that right projective $\mathfrak{D}$-modules have a filtration by right standard modules with respect to the second order. Since all shifts are the same (by N), it follows that this filtration satisfies the necessary ordering condition. The claim follows. □

Remark 10. Assume that the right projective $\tilde{A}$-module at vertex k has radical filtration with subquotients $P^1, \dots, P^s$. Then the indecomposable right projective $\mathfrak{C}$-module at (k, i) looks as follows:

$$\begin{array}{ccccccccc}
 & & & & & P_i^1 & & & \\
 & & & & P_{i+1}^1 & & P_{i-1}^2 & & \\
 & & & P_{i+2}^1 & & P_i^2 & & \ddots & \\
 & & & & \ddots & & \ddots & & P_{i-s+1}^s \\
 & & & & & & & P_{i-s+2}^s & \\
P_{i+N-1}^1 & & & & & & & & \\
 & P_{i+N-2}^2 & & & & & & & \\
 & & & & & & & & \\
 & & & P_{i+N-s}^s & & & & &
\end{array} \tag{5}$$

If the left projective $\tilde{A}$-module at vertex k has a radical filtration with subquotients $Q^1, \dots, Q^t$, then the indecomposable right injective $\mathfrak{C}$-module at (k, i)

looks as follows:

$$
\begin{array}{ccccccccc}
 & & & & & Q^t_{i-N+t} & & & \\
 & & & & Q^t_{i-N+t+1} & & Q^{t-1}_{i-N+t-1} & & \\
 & & & Q^t_{i-N+t+2} & & Q^{t-1}_{i-N+t} & & \ddots & \\
 & & & & \ddots & & \ddots & & Q^1_{i-N+1} \\
 & & & & & & & Q^1_{i-N+2} & \\
 & Q^t_{i+t-1} & & & & & & & \\
 & & Q^{t-1}_{i+t-2} & & & & & & \\
 & & & & & & & & \\
 & & & & Q^1_i & & & &
\end{array} \tag{6}
$$

For $k \in \tilde{\mathbf{N}}$, s can reach N, but $t = 1$. For $k \in \mathbf{N}$, t can reach N, but s is always less than N and Q^t only has composition factors indexed by $k \in \mathbf{N}$.

The corresponding indecomposable injective $\mathfrak{D}$-module is obtained by gluing (5) and (6). The standard filtrations of this module with respect to the first and the second order can be organized using the left and the right diagrams from (3), respectively.

Proposition 11. *A is an idempotent subquotient of $\mathfrak{D}$ as follows*:

$$A \cong \frac{1_{\tilde{A}_i}\mathfrak{D}1_{\tilde{A}_i}}{1_{\tilde{A}_i}\mathfrak{D}\tilde{e}_i\mathfrak{D}1_{\tilde{A}_i}}, \quad \textit{where } \tilde{e}_i = \sum_{k=1}^{n} e_{\tilde{k},i} \in \mathfrak{D}.$$

Proof. It is obvious that $1_{\tilde{A}_i}\mathfrak{D}1_{\tilde{A}_i}$ is isomorphic to the trivial extension S of $\tilde{A}$ by $\tilde{A}^*$. Now we claim that the ideal $\tilde{A}^*$ is contained in the ideal, generated by $\tilde{e} := \sum_{k=1}^{n} e_{\tilde{k}}$.

Consider the right projective S-module $e_k S$ at vertex $k \in \mathbf{N}$. This module has a filtration by the right projective $\tilde{A}$-module $e_k\tilde{A}$ (which is the dual of the corresponding left injective A-module and only has composition factors $L^r(j)$ for $j \in \mathbf{N}$), and the right injective $\tilde{A}$-module at the vertex k, which has a semisimple quotient consisting of simples $L^r(r)$ for $r \in \tilde{\mathbf{N}}$ and sitting on a submodule isomorphic to the right injective A-module at the vertex k. Hence, right projectives for $S/S\tilde{e}S$ look like right projectives for A, so $S/S\tilde{e}S \cong A$. □

4. Triangular decomposition

Recall (see [Ko1], [Ko2]) that a directed subalgebra B of a basic quasi-hereditary algebra A is called a (*strong*) *exact Borel subalgebra* provided that A and B have the same simple modules, the tensor induction functor $A\otimes_B \_$ is exact and maps simple modules to standard modules. Dually, one defines (*strong*) Δ-*subalgebras* (again see [Ko1]). There is an obvious generalization of these notions to $\Bbbk$-linear categories (our algebras). We keep the setup of the previous section and identify the algebra A/I_1 with some maximal semisimple subalgebra of A, say S. Then S is a maximal semisimple subalgebra (in particular, a subspace) of all algebras A/I_i for all $i > 0$.

PROPOSITION 12. *The algebra*

$$\tilde{\mathcal{B}} := \begin{pmatrix} \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & S & S & \ddots & S & 0 & 0 & \cdots \\ \cdots & 0 & S & S & \ddots & S & 0 & \cdots \\ \cdots & 0 & 0 & S & S & \ddots & S & \cdots \\ \cdots & 0 & 0 & 0 & S & S & \ddots & \cdots \\ \cdots & 0 & 0 & 0 & 0 & S & S & \ddots \\ \cdots & 0 & 0 & 0 & 0 & 0 & S & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(*here, each row contains exactly N nonzero entries*) *is a strong exact Δ-subalgebra of both $\mathfrak{C}$ and $\mathfrak{D}$ with respect to the first order.*

Proof. The algebra $\tilde{\mathcal{B}}$ is obviously a subalgebra of both $\mathfrak{C}$ and $\mathfrak{D}$. It is directed by definition and thus quasi-hereditary with respect to the first order. Corresponding right standard modules are just simple modules, corresponding left standard modules are projectives and look as follows:

$$\begin{pmatrix} \vdots \\ A/I_1 \\ A/I_1 \\ A/I_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

These coincide with left standard modules for both $\mathfrak{C}$ and $\mathfrak{D}$ (by Proposition 2(i) and Proposition 7). Therefore, using [Ko1, Theorem A], we deduce that $\tilde{\mathcal{B}}^{\mathrm{op}}$ is an exact Borel subalgebra for $\mathfrak{C}^{\mathrm{op}}$ and $\mathfrak{D}^{\mathrm{op}}$. Thus, by [Ko1, Theorem B], we have that $\tilde{\mathcal{B}}$ is a Δ-subalgebra for $\mathfrak{C}$ and $\mathfrak{D}$. That $\tilde{\mathcal{B}}$ is strong follows from the definitions. This completes the proof. □

Assume now that the algebra A is positively graded, $A = \bigoplus_{i=0}^{\infty} A_i$ and that the filtration (4) coincides with the grading filtration, that is $I_j = \bigoplus_{i=j}^{\infty} A_i$. In this case, we have $I_j/I_{j+1} \cong A_j$ for all i, in particular, I_j/I_{j+1} can be realized as a canonical subspace of A.

PROPOSITION 13. *Under the above assumptions, the algebra*

$$\mathcal{B} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & A_0 & 0 & 0 & 0 & \cdots \\ \cdots & A_1 & A_0 & 0 & 0 & \cdots \\ \cdots & A_2 & A_1 & A_0 & 0 & \cdots \\ \cdots & A_3 & A_2 & A_1 & A_0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a strong exact Borel subalgebra of $\mathfrak{C}$ *with respect to the first order.*

Proof. That $\mathcal{B}$ is a subalgebra follows from the definitions and the fact that A is graded (i.e., $A_i A_j \subset A_{i+j}$). Note that A_0 is a maximal semi-simple subalgebra of A and hence simple A-modules can be identified with simple A_0-modules. Therefore, simple $\mathfrak{C}$-modules (shifted simple A-modules) and $\mathcal{B}$-modules (shifted simple A_0-modules) can be identified as well.

The algebra $\mathcal{B}$ is directed by definition hence quasi-hereditary with respect to the first order. Left standard $\mathcal{B}$-modules are simple. Right standard $\mathcal{B}$-modules are projective. Left costandard $\mathcal{B}$-modules are dual to right standard $\mathcal{B}$-modules and hence have the following form:

$$\begin{pmatrix} \vdots \\ A_2^* \\ A_i^* \\ A_0^* \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

As $A_j \cong I_j/I_{j+1}$ for all j, from Corollary 3(i) we obtain that these costandard modules are restrictions of costandard $\mathfrak{C}$-modules. Hence, $\mathcal{B}$ is an exact Borel subalgebra by [Ko1, Theorem A]. That $\mathcal{B}$ is strong follows from the definitions. This completes the proof. □

REMARK 14. If we assume the existence of a Borel subalgebra, the condition of left costandard modules for this algebra being isomorphic to left costandard modules for $\mathfrak{C}$ forces the Borel subalgebra to have the following form:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & X_{0,i-1} & 0 & 0 & 0 & \cdots \\ \cdots & X_{1,i-1} & X_{0,i} & 0 & 0 & \cdots \\ \cdots & X_{2,i-1} & X_{1,i} & X_{0,i+1} & 0 & \cdots \\ \cdots & X_{3,i-1} & X_{2,i} & X_{1,i+1} & X_{0,i+2} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $X_{j,i}$ are subspaces of I_j providing a splitting of $I_j \twoheadrightarrow I_j/I_{j+1}$. Furthermore, we must have $X_{j,i}X_{i-k,k} \subseteq X_{j+i-k,k}$ for this to be a subalgebra. If we assume that the Borel subalgebra is stable under the shift, i.e., that $X_{j,i} = X_{j,i+1}$ for all i, j, then the above is simply the condition that A is graded. Hence, the existence of a Borel subalgebra which is invariant under the shift is equivalent to A being graded with respect to the filtration (4).

We further assume that A is positively graded. Then the trivial extension $\overline{A} = A \oplus A^*$ of A inherits a natural $\mathbb{Z}$-grading by assigning degree $-i$ to the space A_i^*, $i \geq 0$. We would need to redefine this natural grading as follows: set $\deg A_i^* = N - 1 - i$. For $i \in \mathbb{Z}$, set $\overline{A}_i = A_i \oplus A^*_{N-1-i}$ and, because of $\mathrm{Rad}^N(A) = 0$, we have $\overline{A}_i = 0$ for all $i < 0$.

Proposition 15. *Under the assumptions of Proposition 13, the algebra*

$$\overline{\mathcal{B}} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \overline{A}_0 & 0 & 0 & 0 & \cdots \\ \cdots & \overline{A}_1 & \overline{A}_0 & 0 & 0 & \cdots \\ \cdots & \overline{A}_2 & \overline{A}_1 & \overline{A}_0 & 0 & \cdots \\ \cdots & \overline{A}_3 & \overline{A}_2 & \overline{A}_1 & \overline{A}_0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a strong exact Borel subalgebra of $\mathfrak{D}$ with respect to the first order.

Proof. That $\overline{\mathcal{B}}$ is a directed subalgebra of $\mathfrak{D}$ and that simple $\overline{\mathcal{B}}$ and $\mathfrak{D}$ modules can be identified follows from the construction. Using Lemma 8, the rest is proved just as in the proof of Proposition 13. □

Denote by $\mathtt{S}_{\mathbb{Z}}$ the subalgebra

$$\tilde{\mathcal{B}} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & A_0 & 0 & 0 & \cdots \\ \cdots & 0 & A_0 & 0 & \cdots \\ \cdots & 0 & 0 & A_0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of $\mathfrak{C}$. Note that $\mathtt{S}_{\mathbb{Z}}$ is a semi-simple subalgebra of $\mathfrak{D}$, $\tilde{\mathcal{B}}$, $\mathcal{B}$ and $\overline{\mathcal{B}}$. Propositions 13 and 15 allow us to deduce the following *triangular decompositions* for the algebras $\mathfrak{C}$ and $\mathfrak{D}$:

THEOREM 16. *Under the assumptions of Proposition 13, we have:*

(1) *Multiplication in $\mathfrak{C}$ induce the following isomorphism of left $\tilde{\mathcal{B}}$- and right $\mathcal{B}$-modules: $\mathfrak{C} \cong \tilde{\mathcal{B}} \otimes_{\mathtt{S}_{\mathbb{Z}}} \mathcal{B}$.*

(2) *Multiplication in $\mathfrak{D}$ induce the following isomorphism of left $\tilde{\mathcal{B}}$- and right $\overline{\mathcal{B}}$-modules: $\mathfrak{D} \cong \tilde{\mathcal{B}} \otimes_{\mathtt{S}_{\mathbb{Z}}} \overline{\mathcal{B}}$.*

Proof. This follows from Propositions 13 and 15 and [Ko2]. □

Similarly, one obtains the following.

THEOREM 17. *With respect to the second order, we have the following:*

(1) *The algebra $\tilde{\mathcal{B}}$ is a strong exact Borel subalgebra of both $\mathfrak{C}$ and $\mathfrak{D}$.*

(2) *Under the assumptions of Proposition 13, the algebra $\mathcal{B}$ is a strong exact Δ-subalgebra of $\mathfrak{C}$.*

(3) *Under the assumptions of Proposition 13, the algebra $\overline{\mathcal{B}}$ is a strong exact Δ-subalgebra of $\mathfrak{D}$.*

Proof. Left to the reader. □

COROLLARY 18. *Under the assumptions of Proposition 13, we have that* $A-$mod *embeds into $\mathcal{F}(\Delta^{1,l}_{\mathfrak{C}})$.*

Proof. As $\mathcal{B}$ is a Borel subalgebra of $\mathfrak{C}$, we have that $\mathcal{B}-$mod embeds into $\mathcal{F}(\Delta^{1,l}_{\mathfrak{C}})$ via exact tensor induction. As A is an idempotent subquotient of $\mathcal{B}$ by construction, the claim follows. □

Similarly we have the following.

COROLLARY 19. *Under the assumptions of Proposition 13, we have that* mod$-A$ *embeds into $\mathcal{F}(\Delta^{2,r}_{\mathfrak{C}})$.*

Let B be the path algebra of the quiver

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

modulo the relations that any composition of N arrows is zero.

COROLLARY 20. *The category* $B-$mod *embeds into $\mathcal{F}(\Delta^{2,l}_{\mathfrak{C}})$.*

Proof. The algebra $\tilde{B}$ consists of direct summands, each of which is isomorphic to B. As $\tilde{B}$ is a Δ-subalgebra of $\mathfrak{C}$, we have that $\tilde{B}-$mod, and hence $B-$mod, embeds into $\mathcal{F}(\nabla^{1,l}_{\mathfrak{C}})$. However, up to a shift, costandard modules in the first order are the same as standard modules in the second order by Corollary 4, so $\mathcal{F}(\nabla^{1,l}_{\mathfrak{C}}) = \mathcal{F}(\Delta^{2,l}_{\mathfrak{C}})$. This completes the proof. □

Similarly, we have the following.

COROLLARY 21. *The category* mod$-B$ *embeds into $\mathcal{F}(\Delta^{1,r}_{\mathfrak{C}})$.*

Corollary 22. (1) *The category* mod$-B$ *embeds into* $\mathcal{F}(\Delta_{\mathfrak{D}}^{1,r})$.

(2) *The category* $B-$mod *embeds into* $\mathcal{F}(\Delta_{\mathfrak{D}}^{2,l})$.

(3) *Under the assumptions of Proposition* 13, *we have that* $\overline{A}-$mod *embeds into* $\mathcal{F}(\Delta_{\mathfrak{D}}^{1,l})$.

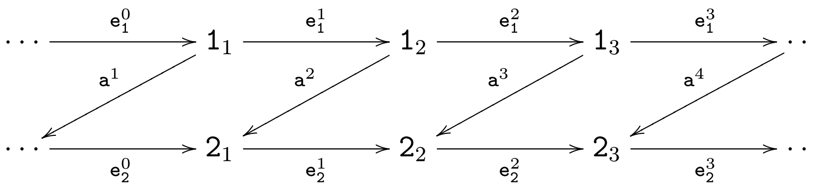
(4) *Under the assumptions of Proposition* 13, *we have that* mod$-\overline{A}$ *embeds into* $\mathcal{F}(\Delta_{\mathfrak{D}}^{2,r})$.

5. Examples

Example 23 (*An easy quiver algebra*). Let A be the path algebra of the following quiver:

$$\mathtt{1} \xrightarrow{\mathtt{a}} \mathtt{2}.$$

Assume that (4) is the radical filtration of A. Let $\mathtt{e}_1$ and $\mathtt{e}_2$ be the idempotents of A, corresponding to the vertices $\mathtt{1}$ and $\mathtt{2}$, respectively. In this case, the algebra $\mathfrak{C}(A)$ is the path algebra of the following quiver:

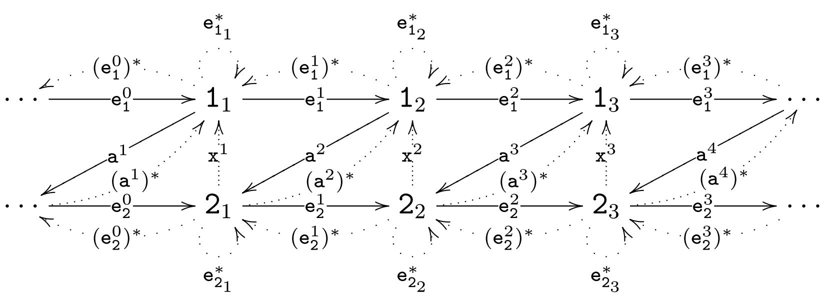


modulo the ideal, generated by the following relations:

$$\mathtt{e}_1^{i+1}\mathtt{e}_1^{i} = \mathtt{e}_2^{i+1}\mathtt{e}_2^{i} = 0, \qquad \mathtt{e}_2^{i-1}\mathtt{a}^{i} = \mathtt{a}^{i+1}\mathtt{e}_1^{i}, \tag{7}$$

where $i \in \mathbb{Z}$.

We also have $A \cong \tilde{\Bbbk}$ (where $\mathtt{2} = \tilde{\mathtt{1}}$). In this case, the algebra $\mathfrak{D}(\tilde{\Bbbk})$ is the path algebra of the following quiver (the dual part $\mathfrak{C}^*$ is depicted using the dotted arrows):



(here, $x^i = (\mathtt{e}_2^{i-1}\mathtt{a}^i)^*$) modulo the ideal, generated by the relations (7), the relations saying that the product of any two dotted arrows is zero, and the relations defining the natural $\mathfrak{C}$-bimodule structure on $\mathfrak{C}^*$.

EXAMPLE 24 (*Schur algebras for* GL_2). Let A be a block of a Schur algebras for GL_2, say with $ap^k + r$ simple modules ($1 \leq a \leq p-1, k \geq 0, 1 \leq r \leq p^k$). These have been extensively studied in [MT1] and [MT2] and in particular have been shown to be hereditary idempotent subquotients of certain infinite-dimensional symmetric quasi-hereditary algebras. Instead of taking an idempotent subquotient, one might also take a centralizer subalgebra B which is again symmetric, such that it corresponds to the endomorphism ring of the first ap^k projectives for the Schur algebra. From the explicit description in terms of quivers and relations in [MT2], it is easily seen that this has a $\mathbb{Z}$-grading, which coincides with the radical filtration, hence has semisimple subquotients. By [MV, Theorem 3.3], any connected finite-dimensional self-injective positively graded algebra is rigid. Therefore, we can apply Corollary 6 to obtain a symmetric quasi-hereditary algebra. This will however give an algebra that is significantly larger than the symmetric quasi-hereditary algebra given in [MT1], [MT2].

EXAMPLE 25 (*Category* $\mathcal{O}$). Let $\mathfrak{g}$ be a semi-simple finite-dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathcal{O}_0^{\mathfrak{p}}$ denote the principal block of the $\mathfrak{p}$-parabolic category $\mathcal{O}$ for $\mathfrak{g}$, and $A^{\mathfrak{p}}$ denote the endomorphism algebra of the multiplicity-free direct sum of all indecomposable projective-injective modules in $\mathcal{O}_0^{\mathfrak{p}}$.

The algebra $A^{\mathfrak{p}}$ is positively graded and symmetric (see [MS]) and simple $A^{\mathfrak{p}}$-modules are naturally indexed by the elements of some right cell for the Weyl group W of $\mathfrak{g}$. In the special case $\mathfrak{g} = \mathfrak{sl}_n$, the parabolic subalgebra $\mathfrak{p}$ is given by some composition of n and the algebra $A^{\mathfrak{p}}$ can be used to model the corresponding Specht module (for the symmetric group or Hecke algebra) via the action of some exact functors on $A^{\mathfrak{p}}-\mathrm{mod}$, see [KMS]. The algebra $A^{\mathfrak{p}}$ has a simple preserving duality, which yields that all indecomposable projective $A^{\mathfrak{p}}$-modules are self-dual. Since the trace form on $A^{\mathfrak{p}}$ respects grading, it follows that this form induces a nondegenerate pairing between the components of the grading filtration of $A^{\mathfrak{p}}$ as required in the formulation of Theorem 5. Thus, from Theorem 5 it follows that the quasi-hereditary algebra $\mathfrak{C}(A^{\mathfrak{p}})$ of $A^{\mathfrak{p}}$ is symmetric and thus $A^{\mathfrak{p}}$ is a centralizer subalgebra of a symmetric quasi-hereditary algebra. It would be interesting to understand the algebra $\mathfrak{C}(A^{\mathfrak{p}})$. Note that the natural grading filtration on $A^{\mathfrak{p}}$ does not have to coincide with the radical filtration.

In the special case $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+$, the algebra $\mathfrak{C}(A^{\mathfrak{p}})$ is closely related to the algebras from [MT1].

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