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DIMENSION OF ELLIPTIC HARMONIC MEASURE OF SNOWSPHERES

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ABSTRACT. A metric space $\mathcal{S}$ is called a *quasisphere* if there is a quasisymmetric homeomorphism $f : S^2 \to \mathcal{S}$. We consider the *elliptic harmonic measure*, i.e., the push forward of 2-dimensional Lebesgue measure by f. It is shown that for certain self similar quasispheres $\mathcal{S}$ (*snowspheres*) the dimension of the elliptic harmonic measure is strictly less than the Hausdorff dimension of $\mathcal{S}$. This result is obtained by representing the self similarity of a snowsphere by a postcritically finite rational map, and showing a corresponding result for such maps. As a corollary a metric characterization of Lattès maps is obtained. Furthermore, a method to compute the dimension of elliptic harmonic measure numerically is presented, along with the (numerically computed) values for certain examples.

1. Introduction

A homeomorphism $f : X \to Y$ of metric spaces is called *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq \eta\left(\frac{|x - y|}{|x - z|}\right),$$

for all $x, y, z \in X, x \neq z$. Here and in the following, we use the *Polish notation* $|x - y|$ for the metric. See [Hei01] for general background on quasisymmetric maps. Every quasisymmetry is trivially quasiconformal, see [Väi71] for background on quasiconformal maps.

A metric space $\mathcal{S}$ is called a *quasisphere* if it is quasisymmetrically equivalent to the standard 2-sphere S^2, i.e., if there is a quasisymmetry $f : S^2 \to \mathcal{S}$.

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To give a geometric characterization of quasispheres is an open problem; the best results to date are due to Mario Bonk and Bruce Kleiner [BK02].

In [Mey02] and [Mey] *snowspheres* were constructed, which are topologically 2-dimensional analogs of snowflake curves. These were shown to be quasispheres. The construction is recalled briefly in the next section. Here, further properties of qs-parametrizations of such snowspheres are investigated.

THEOREM 1. *Let $\mathcal{S}$ be a snowsphere with uniformizing* (*quasisymmetric*) *map $f : S^2 \to \mathcal{S}$. There is $\alpha > 2/\dim_H(\mathcal{S})$ such that the following holds. For* (*Lebesgue*) *almost every $x \in S^2$,*

$$\lim_{y\to x} \frac{\log|f(x)-f(y)|}{\log|x-y|} = \alpha.$$

Note however, that the set where the above limit assumes a different value/does not exist is dense.

Denote by λ the 2-dimensional, normalized ($\lambda(S^2)=1$) Lebesgue measure on the sphere S^2. The *elliptic harmonic measure* μ of $\mathcal{S}$ induced by f is the pushforward of λ by f,

$$\mu = \mu_f = f_*\lambda. \tag{1.1}$$

Recall that $f_*\lambda(A) = \lambda(f^{-1}A)$ for any Borel measurable set $A \subset \mathcal{S}$.

The *dimension* of a probability measure μ is

$$\dim \mu := \inf\{\dim_H(E) : \mu(E) = 1\}, \tag{1.2}$$

where $\dim_H$ denotes the Hausdorff dimension.

THEOREM 2. *The elliptic harmonic measure μ of every snowsphere $\mathcal{S}$ satisfies*

$$\dim \mu = \frac{2}{\alpha} < \dim_H \mathcal{S}.$$

In concrete examples, the dimension $\dim \mu$ may be explicitly computed numerically. We do this for several snowspheres.

The uniformizing map $f : S^2 \to \mathcal{S}$ is not unique for a given quasisphere $\mathcal{S}$. The dimension of the elliptic harmonic measure however is independent of the particular map f.

LEMMA 1.1. *Let $f, g : S^2 \to \mathcal{S}$ be two quasisymmetric homeomorphisms. Then*

$$\dim \mu_f = \dim \mu_g.$$

Proof. The map

$$f^{-1} \circ g : S^2 \to S^2$$

is quasisymmetric, and therefore maps zero sets (of Lebesgue measure λ on the sphere S^2) to zero sets (see [Väi71], Section 33). So μ_f and μ_g are mutually absolutely continuous. $\square$

Hence, we can speak of *the* dimension of elliptic harmonic measure, which thus measures the size of the quasisphere $\mathcal{S}$ in a quasisymmetric fashion. A similar argument shows that Theorem 1 is independent of the specific quasisymmetric parametrization $f : S^2 \to \mathcal{S}$.

Note that the analog statement for quasicircles is false. This is due to the fact that quasisymmetric maps $h : S^1 \to S^1$ are not absolutely continuous in general [AB56]. Thus, given two quasisymmetric homeomorphisms $h_{1,2} : S^1 \to \mathcal{C}$, one has in general $\dim h_{1*}\lambda_1 \neq \dim h_{2*}\lambda_1$ (here λ_1 denotes 1-dimensional Lebesgue measure on the circle S^1). In fact, the infimum of $\dim h_*\lambda_1$ over all qs maps $h : S^1 \to \mathcal{C}$ is zero for all quasicircles $\mathcal{C}$ (see [Tuk89]). Thus, the notion of dimension of elliptic harmonic measure is unsuited to the 1-dimensional case. There are however interesting *quantitative* questions in this case, i.e., bounds on $\dim h_*\lambda_1$ depended on the distortion of h.

In [DT99], surfaces $\mathcal{S} \subset \mathbb{R}^3$ were constructed which admit a parametrization $h : \mathbb{R}^2 \to \mathcal{S}$ satisfying $1/C|x-y|^\alpha \leq |h(x)-h(y)| \leq C|x-y|^\alpha$, for a constant $C \geq 1$ and $0 < \alpha < 1$ (thus, h is a quasisymmetry). It follows that in these examples $\dim h_*\lambda = \dim_H \mathcal{S}$.

In [Mey02], it was shown how the self similarity of a snowsphere can be represented by a *postcritically finite* rational map R. The sphere S^2 can be equipped with a metric $|x-y|_{\mathcal{S}}$ with respect to which R acts as a *local similarity*. This means there is a $N > 1$ such that for all $x \in S^2$ there is a neighborhood $U(x)$ where

$$\frac{|R(x)-R(y)|_{\mathcal{S}}}{|x-y|_{\mathcal{S}}} = N, \tag{1.3}$$

for all $y \in U(x)$ $(x \neq y)$. In fact, N is the "scaling factor" of the self similar snowsphere $\mathcal{S}$. A piece of the snowsphere is isometric to a hemisphere in this metric.

Such metrics are constructed in [BM] for all postcritically finite rational maps that have no periodic critical points (equivalently, which have the whole sphere as their Julia set). In fact, such metrics were constructed for *expanding Thurston maps*, see below. Theorem 1 is a consequence of the following.

THEOREM 3. *Let R be a postcritically finite rational map without periodic critical points. Let $|x-y|_{\mathcal{S}}$ be a metric as above. Then there is $\alpha \geq 2/\dim_H(\mathcal{S})$ such that for (Lebesgue) almost every point $x \in S^2$*

$$\lim_{y\to x} \frac{\log|x-y|_{\mathcal{S}}}{\log|x-y|} = \alpha.$$

Here, $|x-y|$ denotes the spherical metric, $\dim_H(\mathcal{S})$ the Hausdorff dimension of $(S^2, |x-y|_{\mathcal{S}})$. Furthermore, $\alpha = 2/\dim_H(\mathcal{S})$ if and only if R is a Lattès map.

In fact, $\alpha = \frac{\log N}{\chi}$, where χ is the *Lyapunov exponent*.

The next theorem (and hence Theorem 2) is an easy consequence. Let $\dim(\lambda, \mathcal{S})$ be the dimension of Lebesgue measure with respect to the metric $|x-y|_{\mathcal{S}}$.

THEOREM 4. *With the setup as above*

$$\dim(\lambda, \mathcal{S}) = 2/\alpha < \dim_H(S^2, |x-y|_{\mathcal{S}}),$$

unless R is a Lattès map, in which case there is equality in the above.

Note that we can also express the dimension as $\dim(\lambda, \mathcal{S}) = \frac{2\chi}{\log N} = \frac{h}{\log N}$, where h is the entropy.

In [BM], *expanding Thurston maps* were considered. These are maps that behave *topologically* as a rational map, meaning they can be written locally as $z \mapsto z^n$ after suitable homeomorphic coordinate changes in domain and range. Furthermore, they are *postcritically finite* and satisfy an *expansion* property. Thurston has classified when such a map is *equivalent* to a rational map [DH93]. For such an expanding Thurston map, a metric $|x-y|_{\mathcal{S}}$ satisfying (1.3) was constructed in [BM], see also [HP09]. The metric is not unique, but two such metrics are snowflake equivalent.

DEFINITION 1.2. Two metric spaces $(X_1, |x-y|_1)$, $(X_2, |x-y|_2)$ are *snowflake equivalent* if there is a homeomorphism $\varphi\colon X_1 \to X_2$ constants $C \geq 1$, $\beta > 0$ such that

$$\frac{1}{C}|x-y|_1 \leq |\varphi(x) - \varphi(y)|_2^\beta \leq C|x-y|_1,$$

for all $x, y \in X_1$.

In [BM] (see also [HP09]), it is proved that $(S^2, |x-y|_{\mathcal{S}})$ is quasisymmetrically equivalent to S^2 (with the standard metric) if and only if R is topologically conjugate to a rational map. In the present paper, we obtain the following related theorem.

THEOREM 5. *Let R be an expanding Thurston map, and the metric $|x-y|_{\mathcal{S}}$ as in* (1.3). *Then*

$$(S^2, |x-y|_{\mathcal{S}}) \textit{ is snowflake equivalent to } (\textit{standard}) \; S^2$$

if and only if

$$R \textit{ is topologically conjugate to a Lattès map.}$$

We now give an outline of the paper. In the next section, the construction of snowspheres is recalled.

The construction of the rational map encoding the self similarity of a snowsphere is done in Section 3. This was only done for a specific example in [Mey02], so full proofs are given.

The proof of Theorem 3 becomes a relatively simple application of the thermodynamic formalism. The necessary facts are reviewed in Section 4.

The *invariant measure* μ of R that is absolutely continuous with respect to Lebesgue measure is constructed in Section 5.

The proofs of Theorem 3, Theorem 4, and Theorem 5 are done in Section 6.

In Section 7, the strict inequality $\alpha > 2/\dim_H(\mathcal{S})$ is shown (unless R is a Lattès map). This already follows from [Zdu90], but can be shown directly.

Section 8 shows how one can compute $\dim \mu$ using Birkhoff's ergodic theorem. The results for several examples are given.

In the last section, some open problems are presented.

1.1. Notation. Two nonnegative quantities A, B are *comparable* if there is a constant $C \geq 1$ such that

$$\frac{1}{C} A \leq B \leq CA.$$

We then write $A \asymp B$. The constant C is referred to as $C(\asymp)$. Similarly, we write $B \lesssim A$ if there is a constant $C > 0$ such that $B \leq CA$; we then refer to the constant C by $C(\lesssim)$.

The Riemann sphere is denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The extended real line is $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}$. The upper half plane is denoted by $\mathbb{H}^+$, the lower half plane by $\mathbb{H}^-$; their closures by $\overline{\mathbb{H}}^+$, $\overline{\mathbb{H}}^-$.

The jth *iterate* of a (rational) map R is denoted by R^j.

2. Snowspheres

Before presenting the general construction, let us give an example first. This *main* or *standard* example will serve throughout the whole paper to illustrate the construction. Start with the unit cube. Divide each side into 5×5 squares of side length $1/5$ ($1/5$-*squares*). Put a cube of side length $1/5$ (a $1/5$-*cube*) on the middle $1/5$-square of each side. This results in a polyhedron bounded by 6×29 $1/5$-squares. The construction is now iterated. Divide each $1/5$-square into 5×5 $1/25$-squares. Put a $1/25$-cube on the middle $1/25$-square of each $1/5$-square and so on. The limiting surface is called a *snowsphere* (which bounds a *snowball*).

In general, we divide each square into $N \times N$ $1/N$-squares (where N is an integer ≥ 2).

DEFINITION 2.1. An N-*generator* G is a polyhedral surface in $\mathbb{R}^3$ such that:

- Each face is a $1/N$-square.
- The boundary of G is the unit square, $\partial G = \partial[0,1]^2 \subset \mathbb{R}^3$. Here, we identify $\mathbb{R}^2 \supset [0,1]^2$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.
- G is homeomorphic to $[0,1]^2$.
- The generator is *symmetric* with respect to reflections on the planes $\{x = y\}, \{x+y=1\}, \{x=1/2\}, \{y=1/2\}$, hence it is also symmetric with respect to rotations around the axis $\{x=y=1/2\}$ by multiples of $\pi/2$.

- Only one $1/N$-square in G intersects each vertex of $[0,1]^2 \subset \mathbb{R}^3$.

This implies that all vertices of each $1/N$-square from which G is built are contained in the grid $\frac{1}{N}\mathbb{Z}^3$.

The *double pyramid* $\mathcal{P}$ is the union of the (solid) pyramid with base $[0,1]^2 \subset \mathbb{R}^3$ and tip $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and the (solid) pyramid with base $[0,1]^2$ and tip $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$.

A (self similar) snowsphere $\mathcal{S}$ is constructed by repeatedly replacing squares by scaled copies of the same generator. We exclude the trivial case where $G=[0,1]^2$. The limiting surface is a topological sphere embedded in $\mathbb{R}^3$ if the generator G used in the construction satisfies $G \cap \partial\mathcal{P} = \partial[0,1]^2$ (this implies that $G \subset \mathcal{P}$, by the last property of G). It was shown in [Mey] that it is quasisymmetrically equivalent to the standard sphere S^2, here $\mathcal{S}$ is equipped with the metric inherited from $\mathbb{R}^3$.

2.1. Abstract snowspheres. If the generator does not satisfy $G \cap \partial\mathcal{P} = \partial[0,1]^2$, we can still define the snowsphere generated by G abstractly as follows. Choose the homeomorphism $h\colon G \to [0,1]^2$ from Definition 2.1 such that it is the identity on $\partial[0,1]^2$. View $\mathcal{S}_0 := \partial[0,1]^3$ as a two-dimensional cell complex in the obvious way—faces, edges, vertices are the 2-, 1-, and 0-cells. In the same fashion, we view G as a two-dimensional cell complex. The cell complex $\mathcal{S}_1$ is constructed by replacing each face of $\mathcal{S}_0$ by G. Formally, map the 2-, 1-, and 0-cells of G by h to $[0,1]^2$ and then to each face of $\mathcal{S}_0$ by an isometry. Images of cells of G under this composition form the cell complex $\mathcal{S}_1$.

This procedure is now iterated. Namely, map cells of G by h to $[0,1]^2$ then to each $1/N$-square of G by a similarity. So each $1/N$-square of the generator G is a cell complex isomorphic to G. Map (the thus subdivided) G to $[0,1]^2$ by h and subsequently to each side of $\partial[0,1]^3$ by an isometry to obtain the cell complex S_2; and so on. Call the faces (2-cells) of $\mathcal{S}_j$ the *j-cylinders*. Note that any $(j+1)$-cylinder X_{j+1} is contained in exactly one j-cylinder X_j. The 1- and 0-cells of $\mathcal{S}_j$ are called *j-edges* and *j-vertices*.

A more general treatment of *subdivisions* can be found in [CFP01]. It should be emphasized that the above procedure is used to define $\mathcal{S}_j$ as a combinatorial object. The metric on the abstract snowsphere will be defined exclusively from the combinatorics.

Consider the set of sequences $x := \{X_j\}_{j\geq 0}$, where each X_{j+1} is a $(j+1)$-cylinder contained in (the j-cylinder) X_j. Another such sequence $y=\{Y_j\}$ is identified with x if and only if $X_j \cap Y_j \neq \emptyset$ for all j. The *abstract snowsphere* $\mathcal{S}$ generated by the generator G is the set of all equivalence classes of such sequences.

The metric on $\mathcal{S}$ is defined as follows. A *j-chain* is a sequence $Z_1,\dots,Z_n$ of j-cylinders such that $Z_k \cap Z_{k+1} \neq \emptyset$, $k=1,\dots,n-1$. The *length* of such a j-chain is n. We say that the j-chain *connects* the j-cylinders X_j and Y_j if $Z_1 = X_j$ and $Z_n = Y_j$. The chain is called *simple* if $Z_k \cap Z_m = \emptyset$ for $m > k+1$.

Given two sequences $x = \{X_j\}, y = \{Y_j\}$ of j-cylinders as above ($X_{j+1} \subset X_j, Y_{j+1} \subset Y_j$), define

(2.1) $$d_j(x,y) := N^{-j} \min\{\text{length of } j\text{-chain connecting } X_j, Y_j\}.$$

Recall that G was a N-generator, meaning it consists of $1/N$-squares.

THEOREM 6. *For all x, y as above, the limit*

$$d(x,y) = \lim_j d_j(x,y)$$

exists and is a metric on $\mathcal{S}$.

Proof. (1) We first show that d_j is *finite*; more precisely $d_j(x,y) \lesssim 1$, where $C(\lesssim)$ is independent of $x, y \in \mathcal{S}$ and j.

Consider the edges of length $1/N$ of the generator G; they are the edges of the $1/N$-squares from which G is built. Note that each of the four edges of $\partial G = \partial [0,1]^2$ consists of precisely N such edges.

This means that every j-edge can be covered by N $(j+1)$-edges. Hence, by N^k $(j+k)$-edges.

Let M be the number of edges (of length $1/N$) in the generator, this is the number of $(j+1)$-edges in each j-cylinder X_j. Consider now a point $x = \{X_j\} \in \mathcal{S}$. Then we can connect a given 0-vertex v_0 of X_0 to a 1-vertex v_1 of $X_1 \subset X_0$ by at most M 1-edges. Similarly, we can connect v_1 to X_2 by at most M 2-edges and so on. Thus, v_0 can be connected to X_j by at most

$$M(N^{j-1} + N^{j-2} + \dots + 1) \lesssim N^j$$

j-edges, where $C(\lesssim) = C(M,N)$. This shows the uniform boundedness of d_j.

For a given j-cylinder X, we define the *annulus* (or j-annulus)

$$A(X) := \bigcup \{j\text{-cylinder } Y : Y \neq X, Y \cap X \neq \emptyset\}.$$

Consider an m-chain $Z_1, \dots, Z_n$, where $m \geq j$. We say it *crosses* the annulus $A(X)$ if one m-cylinder Z_k intersects ∂X, and another intersects the other boundary component of $A(X)$.

(2) A $(j+1)$-chain crossing a j-annulus has length at least N.

Consider first a chain of $1/N$-squares in the generator G that connects two opposite (disjoint) sides of $\partial[0,1]^2$. This chain has length at least N.

The symmetry of the generator thus implies that a $(j+1)$-chain crossing a j-annulus has length at least N.

(3) We now show that d_j *converges*. Let $x = \{X_j\}, y = \{Y_j\} \in \mathcal{S}$ be arbitrary and distinct, $j_0 \geq 0$ be the smallest number such that $X_{j_0} \cap Y_{j_0} = \emptyset$. Thus,

(2.2) $$d_{j_0}(x,y) \geq 3N^{-j_0}.$$

We will show that $d_j(x,y) - 2N^{-j}$ is increasing for $j \geq j_0$. This shows convergence of d_j since it is bounded by (1). Let $j \geq j_0$. Consider a $(j+1)$-chain $Z_1, \dots, Z_n$ connecting X_{j+1} and Y_{j+1}.

We construct a j-chain $W_1, \dots, W_m$ connecting X_j and Y_j as follows. Let $W_1 := X_j$. Let Z_{i_1} be the last $(j+1)$-cylinder that is contained in a j-cylinder W_2 intersecting W_1 (equivalently, the last $(j+1)$-cylinder in $A(W_1)$). The $(j+1)$-chain $Z_2, \dots, Z_{i_1}$ crosses the j-annulus $A(W_1)$. In the same fashion, let Z_{i_2} be the last $(j+1)$-cylinder contained in a j-cylinder W_3 that intersects W_2. We continue to construct $\{W_i\}$ till W_{m-1} intersects $W_m := Y_j$. The chain $Z_2, \dots, Z_{n-1}$ crosses $m-2$ j-annuli $A(W_i)$. Thus, the length of the chain $\{Z_i\}$ is

$$n \geq (m-2)N + 2,$$

by (2). Therefore,

$$\begin{aligned} d_{j+1}(x,y) &\geq N^{-j-1}\big((m-2)N+2\big) \\ &\geq (m-2)N^{-j} + 2N^{-j-1} \geq d_j(x,y) - 2N^{-j} + 2N^{-j-1} \\ \Leftrightarrow \quad d_j(x,y) - 2N^{-j} &\leq d_{j+1}(x,y) - 2N^{-j-1}. \end{aligned}$$

Hence, $\lim_j (d_j(x,y) - 2N^{-j}) = \lim_j d_j(x,y) =: d(x,y)$ exists. It does not degenerate since

$$d_{j_0+k}(x,y) \geq d_{j_0}(x,y) - 2N^{-j_0} + 2N^{-j_0-k} \geq N^{-j_0},$$

for all $k \geq 0$ by (2.2). The symmetry of d is clear, the triangle inequality follows from the ones for d_j.

(4) Finally, we show that the definition of $d(x,y)$ is *independent of the representatives* x, y.

To verify this, let $x = \{X_j\} \sim \tilde{x} = \{\widetilde{X}_j\}$, $y = \{Y_j\} \sim \tilde{y} = \{\widetilde{Y}_j\}$ (meaning that $X_j \cap \widetilde{X}_j \neq \emptyset$, $Y_j \cap \widetilde{Y}_j \neq \emptyset$ for all j). Consider a j-chain connecting x and y. Adding $\widetilde{X}_j$ and $\widetilde{Y}_j$ to the beginning and end of this j-chain, yields a j-chain connecting $\tilde{x}$ to $\tilde{y}$. Thus,

$$|d_j(\tilde{x},\tilde{y}) - d_j(x,y)| \leq 2N^{-j}. \qquad \square$$

The proof above shows the following. If m is the smallest number such that $X_m \cap Y_m = \emptyset$ for given $x = \{X_j\}, y = \{Y_j\}$, then $d(x,y) \asymp N^{-m}$.

If $\mathcal{S}$ is a snowsphere embedded in $\mathbb{R}^3$, then $d(x,y)$ is comparable to the Euclidean distance of x, y (see [Mey], Section 2.4).

The proof in [Mey] applies here as well; each abstract snowsphere is quasisymmetrically equivalent to the standard sphere S^2.

3. Origami with rational maps

In [Mey02], it was shown that the self similarity of a snowsphere can be encoded by a rational map (this in turn was used to construct the quasisymmetry to S^2). We review the construction briefly.

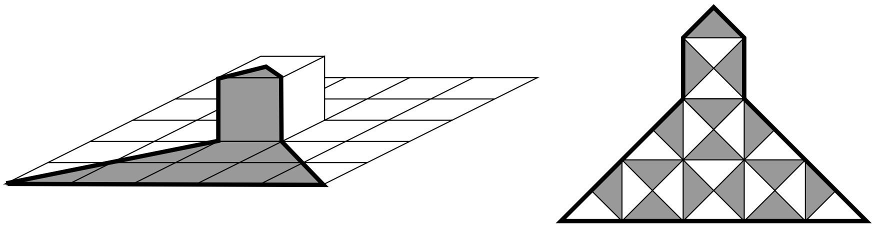

FIGURE 1. Triangular generator of main example.

3.1. The rational map representing the snowsphere. Cut the generator along the diagonals $\{x = y\}, \{x + y = 1\}$ into 4 pieces. We call one such piece the *triangular generator* G_T. Figure 1 shows the triangular generator of our main example.

LEMMA 3.1. *The triangular generator G_T satisfies the following.*

- *G_T is compact, connected, and simply connected.*
- *Divide each $1/N$-square of G into 4 $1/N$-triangles along the diagonals. Then G_T is built from such $1/N$-triangles.*
- *Thus, G_T may be viewed as a cell complex. The $1/N$-triangles, along with their edges/vertices are the 2-, 1-, and 0-cells.*

Proof. The first property follows from the symmetry of G. Recall that each $1/N$-square in G has vertices in the grid $\frac{1}{N}\mathbb{Z}^3$. This implies the second property. □

Note that each $1/N$-triangle has two vertices with angle $\pi/4$, and one with angle $\pi/2$.

There is a unique point of G_T which intersects the line $\{x = y = 1/2\}$, which we call the *tip* of the triangular generator G_T. The two vertices of $\partial[0,1]^2$ contained in G_T, together with the tip are the three *corners* of G_T. We think of G_T as a topological triangle.

LEMMA 3.2. *Consider the vertices (of the $1/N$-triangles) in the triangular generator G_T.*

- *Each of the two corners different from the tip is contained in exactly one $1/N$-triangle. The angle of the $1/N$-triangle at this corner is $\pi/4$.*
- *If N is odd the tip is contained in exactly one $1/N$-triangle, which forms an angle $\pi/2$ there. Otherwise, two $1/N$-triangles intersect at the tip, forming an angle $\pi/4$ there.*
- *The other vertices in ∂G_T are contained in at least two $1/N$-triangles.*
- *Vertices not on ∂G_T are contained in at least 4 $1/N$-triangles.*

Proof. The first property follows since only one $1/N$-square of the generator G intersects each vertex of $[0,1]^2$. The other properties follow from the symmetry of G, the fact that G “lives” in the grid $\frac{1}{N}\mathbb{Z}^3$, and is homeomorphic to $[0,1]^2$. □

Consider now two (identical) copies $G_T \times \{0\}, G_T \times \{1\}$ of the triangular generator. We now glue these two copies together along the boundary. More precisely, we identify $x \times \{0\}, x \times \{1\}$ for each $x \in \partial G_T$. This yields a cell complex which is topologically a 2-sphere.

We put a conformal structure on this complex in the following way. Let $2\pi\alpha$ be the total angle at a vertex v (here each 2-cell is viewed as a $1/N$-triangle). A chart around v is given by mapping all 2-cells containing v by $z \mapsto z^{1/\alpha}$. The uniformization theorem gives a conformal equivalence to the standard sphere S^2. We normalize the uniformizing map by mapping the three corners of G_T to $-1, 1, \infty$; where the tip is mapped to ∞. This implies that one copy of G_T is mapped to the upper half plane, while the other is mapped to the lower half plane. Indeed, otherwise the normalized uniformizing map would fail to be unique. So the common boundary of the two triangular generators contained in our cell complex maps to the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

For our standard example, the uniformizing map can be constructed explicitly as follows. Map the polygon on the right in Figure 1 to the upper half plane by the inverse of a Schwarz–Christoffel map.

The images of the 2-, 1-, 0-cells are called 1-*tiles*, 1-*edges*, and 1-*vertices* (these are all compact).

LEMMA 3.3. *The* 1*-tiles have the following properties.*

- *The two* 1*-tiles sharing a* 1*-edge are conformal reflections of each other along this* 1*-edge.*
- *At each* 1*-vertex an even number of* 1*-tiles intersect.*
- *Thus, we can color each* 1*-tile either black or white, such that any two which share a* 1*-edge are of different color.*

Proof. The first property follows from the choice of charts when defining the conformal structure.

Consider a 1-vertex $v \in \widehat{\mathbb{R}}$. Symmetry of the 1-tiles with respect to $\widehat{\mathbb{R}}$ implies that an even number of 1-tiles intersects in v. If $v \notin \widehat{\mathbb{R}}$, it is the image of a vertex $w \in G_T \setminus \partial G_T$. It is clear that at w an even number of $1/N$-triangles intersect. This implies the second property.

The third property follows from the second. □

Color the 1-tiles black and white as in the last lemma. Recall that each $1/N$-triangle in the triangular generator G_T had exactly one vertex with angle $\pi/2$. Label all 1-vertices which are images of such vertices (of $1/N$-triangles) under the uniformization by b. Label the two remaining 1-vertices of each white (black) 1-tile X' by a, c, such that a, b, c are mathematically positively (negatively) ordered in $\partial X'$. Note that the labeling is compatible, meaning that each 1-vertex gets exactly one of the labels a, b, c assigned.

Consider on a white 1-tile X' the Riemann map to the closed upper half plane $\overline{\mathbb{H}}^+$, normalized by mapping $a \mapsto 1, b \mapsto \infty, c \mapsto -1$. Let Y' be a black 1-tile sharing a 1-edge with X'. By the reflection principle, the Riemann map $X' \to \overline{\mathbb{H}}^+$ extends conformally to Y', which it maps to the closed lower half plane, again normalized by $a \mapsto 1, b \mapsto \infty, c \mapsto -1$.

Thus, we can define the *rational map R representing the snowsphere* as follows. On each white (black) 1-tile X' the map R is the Riemann map to the closed upper half plane $\overline{\mathbb{H}}^+$ (lower half plane $\overline{\mathbb{H}}^-$), normalized by mapping $a \mapsto 1, b \mapsto \infty, c \mapsto -1$. The map R is well defined and holomorphic on $S^2 = \widehat{\mathbb{C}}$ by the above (and the Riemann removability theorem). Hence, R is a rational map.

There are two different (black/white)-colorings of 1-tiles as above—switch the colors of white 1-tiles to black and vice versa. The two rational maps $R, \widetilde{R}$ thus obtained satisfy $\widetilde{R} = -R$. Both choices serve our intended purposes equally well.

The critical points are the 1-vertices (where more than two 1-tiles intersect), they are mapped by R to either $1, \infty$, or -1 (if their label is a, b, c). This allows to compute the map R explicitly in many concrete examples. Namely, one obtains two expressions for R, in terms of a's and b's, as well as in terms of b's and c's. Comparing coefficients yields a system of equations which can be solved numerically. For example, the rational map for our standard example from Figure 1 is

$$\begin{aligned}\widehat{R}(z) = 1 &+ \lambda(z-1)(z-a_1)^4(z-a_2)^5(z+a_3)^3 \\ &\times (z+a_4)^4(z-a_5)^4(z^2-a_6 z+a_7)^4 \\ &\Big/\big((z^2+t_1)^2(z^2+t_2)^2(z^2+t_3)^2 \\ &\times (z^2-t_4)^2(z^2-t_5)^2(z^4-t_6 z^2+t_7)^2\big).\end{aligned}$$

Here,

$$\begin{array}{lll} \lambda = -0.001870\ldots, & t_1 = 2712.82\ldots, & t_2 = 11.9805\ldots, \\ t_3 = 0.31506\ldots, & t_4 = 1.01913\ldots, & t_5 = 3.97391\ldots, \\ t_6 = 1.59735\ldots, & t_7 = 0.79867\ldots, & a_1 = 1.15921\ldots, \\ a_2 = 9.00678\ldots, & a_3 = 191.820\ldots, & a_4 = 0.96246\ldots, \\ a_5 = 0.48971\ldots, & a_6 = -1.98760\ldots, & a_7 = 1.99159\ldots. \end{array}$$

In [Mey02], several other explicit examples of rational maps encoding certain snowspheres can be found.

Denote the set of critical points of R by $\operatorname{crit}(R)$. The *postcritical set* is

$$\operatorname{post}(R) := \bigcup_{j\geq 1} R^j(\operatorname{crit}(R)).$$

LEMMA 3.4. *The map R is* postcritically finite, *meaning that* $\operatorname{post}(R)$ *is a finite set. In fact* $\operatorname{post}(R) = \{1, \infty, -1\}$. *Furthermore, R has* no critical periodic orbit, *meaning that*

$$R^k(z_0) = z_0 \quad \Rightarrow \quad z_0 \notin \operatorname{crit}(R),$$

for $k \geq 1$.

Proof. Recall that the number of j-tiles intersecting in a j-vertex is even (Lemma 3.3).

CLAIM. A point c is a critical point if and only if c is a 1-vertex at which at least four 1-tiles intersect.

Let c be a 1-vertex as above, then R maps tiles containing c alternatingly to the upper and lower half plane. Thus, c is a critical point. If only two 1-tiles intersect in the 1-vertex v it is not a critical point by the same argument.

To see the other implication, let $x \in \widehat{\mathbb{C}}$ be not a 1-vertex. Then x is contained in a set of the form $\operatorname{int} X' \cup \operatorname{int} Y' \cup \operatorname{int} E$. Here, X', Y' are 1-tiles sharing the 1-edge E. The map R maps this set conformally (to either $\mathbb{H}^+ \cup \mathbb{H}^- \cup (-1,1), \mathbb{H}^+ \cup \mathbb{H}^- \cup (1,\infty)$, or $\mathbb{H}^+ \cup \mathbb{H}^- \cup (-\infty,-1)$). So x is not a critical point.

The map R was constructed to map all 1-vertices (hence, all critical points) to $1, \infty$, or -1. Since these are in turn 1-vertices the map is postcritically finite.

It remains to show that R has no critical periodic orbits. Recall that $1, -1$ are the images of the two corners of the triangular generator G_T different from the tip. By Lemma 3.2, two 1-tiles intersect at both $1, -1$. Thus, they are not critical points. Again, by Lemma 3.2, the label at $1, -1$ is a or c. Thus, R maps $\{1, -1\}$ to $\{1, -1\}$.

Consider now ∞, which is the image of the tip of the triangular generator G_T under the uniformization. If N is odd, two 1-tiles contain ∞, and ∞ is labeled b (Lemma 3.2). Thus, ∞ is not a critical point and $R(\infty) = \infty$.

If N is even, four 1-tiles contain ∞, and ∞ is labeled a, or c. Thus, ∞ is a critical point and mapped to 1, or -1 by R. □

3.2. Embedding the snowsphere via the rational map. We briefly describe in what sense the rational map R "encodes" the self similarity of the snowsphere. We equip the sphere S^2 with a metric such that the upper and lower half planes $\mathbb{H}^+, \mathbb{H}^-$ are each isometric to a "triangular piece" of the snowball.

Consider one face (0-cylinder) X of the snowball $\mathcal{S}$. Let $E \subset X$ be a 0-edge, $E' = \partial X \setminus E$ be the union of the other 0-edges of X. A triangular piece of the snowball is defined as

$$\mathcal{S}_T := \{x \in X \mid \operatorname{dist}(x, E) \leq \operatorname{dist}(x, E')\}.$$

A *j-tile* is one component of the preimage of a 1-tile by R^{j-1} (equivalently, one component of the preimage of $\overline{\mathbb{H}}^+$, or $\overline{\mathbb{H}}^-$ by R^j). Preimages of $\{1,\infty,-1\}$ by R^j are called j-vertices. Note that each $(j+1)$-tile is contained in exactly one j-tile, since each 1-tile is contained in either $\overline{\mathbb{H}}^+$ or $\overline{\mathbb{H}}^-$.

LEMMA 3.5. *The size of j-tiles goes to zero*,

$$\max \operatorname{diam} X'_j \to 0 \quad \textit{as } j \to \infty.$$

Here, *the maximum is taken over all j-tiles X'_j.*

Proof. This is a straightforward application of Schwarz' lemma. We refer the reader to [Mey] Lemma 3.3 for details. □

We could use the j-tiles to define a metric on S^2 similarly as in Theorem 6.

We choose a slightly different approach here. Label all j-vertices that are preimages of ∞ by b. Form the union of j-tiles that intersect in a j-vertex labeled b to form a *j-quadrilateral.*

If N is odd there is a single j-quadrilateral containing only two j-tiles, namely the one containing ∞ (see Lemma 3.2 and the proof of Lemma 3.4). With this exception, all j-quadrilaterals contain exactly 4 j-tiles.

Use the j-quadrilaterals to define a metric as in Section 2.1,

$$d'_j(x,y) := N^{-j} \min\{\text{length of chain of } j\text{-quadrilaterals connecting } x,y\}.$$

THEOREM 7. *The limit $d'_j \to d'$ exists and is a metric on S^2. Furthermore*:

(1) *$(\mathcal{S}_T, d)$ is isometric to $(\overline{\mathbb{H}}^+, d')$ (as well as $(\overline{\mathbb{H}}^-, d')$).*
(2) *The map $(S^2, |x-y|) \to (S^2, d')$ is* quasisymmetric. *Here*, *$|x-y|$ is the spherical metric on S^2.*
(3) *The map $R\colon (S^2, d') \to (S^2, d')$ is a* local similarity. *More precisely*, *for any $x \in S^2$ there is a neighborhood U of x such that*

$$\frac{d'(R(x),R(y))}{d'(x,y)} = N,$$

for all $y \in U$.
(4) *The* Hausdorff dimension *of (S^2, d') is*

$$\dim_H(S^2, d') = \frac{\log \deg R}{\log N}.$$

Proof. (1) Observe first that for $z,w \in \mathbb{H}^+$ a shortest connecting chain of j-quadrilaterals is always in $\mathbb{H}^+$ (for j sufficiently large). The same is true for $x,y \in \operatorname{int} \mathcal{S}_T$. Note that the combinatorics of j-quadrilaterals in $\mathbb{H}^+$ is the same as the one for the j-cylinders in $\operatorname{int} \mathcal{S}_T$. Map points $x,y \in \operatorname{int} \mathcal{S}_T$ to points $z,w \in \mathbb{H}^+$ with the same combinatorics, then $d_j(x,y) = d'_j(z,w)$. The case when one or both points lie on the boundary is left to the reader.

From (1), Lemma 3.5, and the proof of Theorem 6 it follows that d' is a metric on S^2.

The proof for (2) is identical to the one in [Mey02], and will not be repeated here. It follows however essentially from the facts recorded in the next lemma.

(3) Let $x \in S^2$. Assume first that $x \neq \infty$, and that x is not a pole of R. The union of j-quadrilaterals containing x forms a neighborhood U of x. Assume j is large enough, such that $\infty \notin U$ and U does not contain a pole of R. Then every j-quadrilateral $\widetilde{X}_j$ (and every $(j+k)$-chain in $\widetilde{X}_j$) is mapped by R bijectively to a $(j-1)$-quadrilateral $\widetilde{X}_{j-1} = R(\widetilde{X}_j)$ (to a $(j+k-1)$-chain). As before, any shortest $(j+k)$-chain connecting $x, y \in \widetilde{X}_j$ can be chosen to lie in $\widetilde{X}_j$; similarly any shortest $(j+k-1)$-chain connecting $R(x), R(y) \in \widetilde{X}_{j-1}$ can be chosen to lie in $\widetilde{X}_{j-1}$. The statement follows.

The case where x is either ∞ or a pole of R are left to the reader.

(4) Since (S^2, d') satisfies the *open set condition*, its Hausdorff dimension is given by the pressure formula (see [Fal90], p. 118, Theorem 9.3). $\square$

Property (3) shows how R encodes the self similarity of the snowsphere (each j-cylinder $X_j \subset \mathcal{S}$ is similar to a $(j+k)$-cylinder, scaled by the factor N^{-k}).

It is easy to construct the quasisymmetric uniformizing map $f : S^2 \to \mathcal{S}$ from the above. Namely uniformize the (surface of the) cube as in Section 3.1. Map a 1-quadrilateral conformally into each of the uniformized faces. The combinatorics of the $(j+1)$-quadrilaterals is identical to the combinatorics of the j-cylinders, so each point on S^2 can be mapped to a point on $\mathcal{S}$. The last theorem shows that this map is a quasisymmetry.

Corollary 3.6. *Theorem 3 implies Theorem 1, and Theorem 4 implies Theorem 2.*

From now on, we will denote the metric d' from the last theorem by $|x-y|_{\mathcal{S}}$. The spherical metric is denoted by $|x-y|$.

Using the j-tiles, it is possible to measure distances in purely combinatorial terms. Let $x, y \in S^2$ be arbitrary. Then

$$j(x,y) := \min\{j \mid \text{ there exist disjoint } j\text{-tiles } X_j \ni x, Y_j \ni y\}. \tag{3.1}$$

This measures to what level of tiles one needs to descend to be able to distinguish x and y. The following are from [Mey] (Lemma 3.10, Lemma 2.4, Lemma 3.7, and Corollary 3.8).

Lemma 3.7. *For $x, y \in S^2$, $j = j(x,y)$ as above*

$$|x-y| \asymp \operatorname{diam} X'_j, \tag{3.2}$$

$$|x-y|_{\mathcal{S}} \asymp N^{-j}, \tag{3.3}$$

where $C(\asymp) = C(N)$, *and* X'_j *is a* j*-tile containing* x. *Furthermore,* (*for* j*-tiles* X'_j, Y'_j)

$$\operatorname{diam} X'_j \asymp \operatorname{diam} Y'_j, \quad \text{if } X'_j \cap Y'_j \neq \emptyset, \tag{3.4}$$

$$\operatorname{diam} X'_{j+1} \asymp \operatorname{diam} X'_j, \quad \text{for any } (j+1)\text{-tile } X'_{j+1} \cap X'_j \neq \emptyset, \tag{3.5}$$

$$\operatorname{area}(X'_j) \asymp (\operatorname{diam} X'_j)^2, \tag{3.6}$$

with a constant $C(\asymp)$.

Here, diameter and area are measured with respect to the spherical metric.

3.3. Expanding postcritically finite maps. Let R be a postcritically finite rational map that has no critical periodic orbit. In [BM], a metric $|x-y|_{\mathcal{S}}$ with respect to which R is a local similarity (as in Theorem 7(3)) was constructed. See also [HP09] and [CFKP03]. The map $\operatorname{id}(S^2|x-y|) \to (S^2, |x-y|_{\mathcal{S}})$ is again quasisymmetric. The Hausdorff dimension of $(S^2, |x-y|_{\mathcal{S}})$ is again given by the formula in Theorem 7(4). The metric $|x-y|_{\mathcal{S}}$ (as well as the expansion factor N) is not unique, two such metrics are snowflake equivalent.

Distances may again be measured in purely combinatorial terms. Pick a Jordan curve $\mathcal{C}$ through $\operatorname{post}(R)$, we require that $\mathcal{C}$ is a zero set with respect to 2-dimensional Lebesgue measure. The closure of one component of $R^{-j}(S^2 \setminus \mathcal{C})$ is called a j-tile. Lemma 3.7 continues to hold. By [BM] we can choose the curve $\mathcal{C} \supset \operatorname{post}(R)$ to be invariant for some iterate R^n, meaning that $R^n(\mathcal{C}) \subset \mathcal{C}$. This means that every j-tile defined in terms of R^n and $\mathcal{C}$ is contained in exactly one $(j-1)$-tile.

3.4. The orbifold associated to R. We remind the reader of the notion of the *orbifold* associated to a postcritically finite rational map R. See [Mil06a] (Appendix E) and [McM94] (Appendix A) for more background. The *ramification function* $\nu : \widehat{\mathbb{C}} \to \mathbb{N} \cup \{\infty\}$ is defined by the following. It is the smallest function such that

$$\nu(z) = 1, \quad \text{if } z \in \widehat{\mathbb{C}} \setminus \operatorname{post}(R),$$

and for all $p \in \operatorname{post}(R), R(q) = p$

$$\nu(p) \text{ is a multiple of } \deg_R(q)\nu(q).$$

Clearly, ν is finite on S^2 if and only if R has no critical periodic orbit (see Lemma 3.4). We call $\mathcal{O} = (S^2, \nu)$ the *orbifold* associated to R. The *signature* of $\mathcal{O}$ is the list of values of ν at the postcritical points. The *Euler characteristic* of $\mathcal{O}$ is given by

$$\chi(\mathcal{O}) = 2 + \sum_{p \in \operatorname{post}(R)} \left(\frac{1}{\nu(p)} - 1 \right).$$

It is nonpositive. The orbifold $\mathcal{O}$ is called *parabolic* if $\chi(\mathcal{O})=0$, hyperbolic otherwise. Parabolic orbifolds are completely classified ([DH93], Section 9). Indeed one checks directly that the only possible signatures are (∞,∞), $(2,2,\infty)$, $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$, $(3,3,3)$. The last four of those belong to rational maps R having no critical periodic orbit. Such a map is a *Lattès map*, i.e., obtained as a quotient of a map from a torus $\mathbb{T}^2=\mathbb{C}/\Lambda$ to itself, $\varphi:\mathbb{T}^2\to\mathbb{T}^2$ (see [Mil06b], [Mil06a],[1] and [Lat18]).

Consider the "trivial" snowsphere, i.e., the one with generator $G=[0,1]^2$. One checks directly that in this case the corresponding rational map R is a Lattès example with signature $(2,4,4)$.

In all other cases, there are vertices in the generator where not 4 $1/N$-squares intersect. The signature of the corresponding rational map will contain at least one number ≥ 12, thus R is not a Lattès map.

4. Ergodic theory and dynamics

General background for ergodic theory can be found in [Wal82] and [Pet83]. A survey of the methods used is given in [Urb03], the forthcoming book [PU] will contain an exhaustive treatment. The booklet by Michel Zinsmeister [Zin00] is very readable, though (or possibly because) it only deals with the hyperbolic case, not the subhyperbolic one at hand.

Let $R:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a postcritically finite rational map with no critical periodic orbit. The reader should think of R as the one constructed in Section 3, which represents the self similarity of a snowsphere and embeds it. In this case, the postcritical set is $\operatorname{post}(R)=\{-1,1,\infty\}$.

We consider two metrics on the sphere S^2. The "self similar" metric $|x-y|_{\mathcal{S}}$ is the one from Theorem 7 or from Section 3.3. The standard spherical metric is denoted by $|x-y|$. When writing $\widehat{\mathbb{C}}$, we always mean the sphere equipped with the spherical metric. Theorem 1 will follow from Theorem 3 by Theorem 7(1).

We denote the set of all j-tiles by $\mathbf{X}'_j$. Recall that if R is a rational map representing a snowsphere (Section 3.1) they are given as the set of preimages of $\overline{\mathbb{H}}^+$ and $\overline{\mathbb{H}}^-$ under the iterated map R^j. Let μ be an *R-invariant* probability measure, meaning that

$$R_*\mu(A)=\mu(R^{-1}A)=\mu(A) \tag{4.1}$$

for all Borel sets $A\subset\widehat{\mathbb{C}}$. Equivalently, for all $g\in L^1(\mu)$

$$\int g\circ R\,d\mu=\int g\,d\mu. \tag{4.2}$$

This is immediate for characteristic functions and follows for $L^1(\mu)$ functions by the usual approximation process. We will always assume that $\mu(X')=$

[1] Note that in earlier editions the definition of Lattès maps differs.

$\mu(\operatorname{int} X')$ for any j-tile X'. The (measure theoretic) *entropy* of μ is then given by

$$h = h_\mu = \lim_{j\to\infty} -\frac{1}{j} \sum_{X'_j \in \mathbf{X}'_j} \mu(X'_j) \log \mu(X'_j). \tag{4.3}$$

It is not very hard to show that the limit exists and is nonnegative. If μ is ergodic, the Shannon–McMillan–Breiman theorem (see, for example [Pet83]) says that

$$-\frac{1}{j} \log \mu(X'_j) \to h, \quad \text{as } j \to \infty, \tag{4.4}$$

for μ-almost every $\{x\} = \bigcap X'_j$, where $X'_j \in \mathbf{X}'_j$.

The *spherical derivative* of R is given by

$$R^{\#}(z) := |R'(z)| \frac{1 + |z|^2}{1 + |R(z)|^2},$$

here (and only here) the right-hand side is expressed in terms of the Euclidean metric. It satisfies the usual rules (chain rule, derivative of inverse), as well as

$$\left(\frac{1}{R}\right)^{\#} = R^{\#} \quad \text{and} \quad R\left(\frac{1}{z}\right)^{\#} = R^{\#}\left(\frac{1}{z}\right).$$

The area element on the sphere satisfies

$$d\lambda(R(w)) = (R^{\#})^2 \, d\lambda(w). \tag{4.5}$$

The *Lyapunov exponent* measures metrical expansion, in $(S^2, |x-y|)$ it is

$$\chi = \chi_\mu := \int \log R^{\#} \, d\mu, \tag{4.6}$$

in $(S^2, |x-y|_{\mathcal{S}})$ it is

$$\chi_{\mathcal{S}} = \chi_{\mu,\mathcal{S}} = \log N, \quad \text{by Theorem 7(3).} \tag{4.7}$$

The Lyapunov exponent χ is nonnegative (see [Prz93]). If R is ergodic, we obtain by Birkhoff's ergodic theorem and the chain rule

$$\frac{1}{k} \log (R^k)^{\#}(z) = \frac{1}{k} \sum_{j=0}^{k-1} \log R^{\#}(R^j z) \to \chi, \quad \text{as } k \to \infty \tag{4.8}$$

for μ-almost every $z \in S^2$. Essentially by combining equations (4.4) and (4.8), one obtains Manning's formula

$$\dim \mu | (S^2, |x-y|_{\mathcal{S}}) = \frac{h}{\chi_{\mathcal{S}}}, \tag{4.9}$$

when μ is ergodic and $\chi_\mu > 0$. This was originally proved (under stronger assumptions) in [Man84], the general case was done in [Mañ88]. A complete proof can also be found in [PU], Chapter 10.4. We will not directly use this formula.

Since the critical values of R are repelling fixed points, the Julia set of R is the whole sphere $\widehat{\mathbb{C}}$ (by the classification of Fatou components, see [CG93]). The map R then is *ergodic* (see Theorem 3.9 of [McM94]).

5. The invariant measure on the sphere

Lebesgue measure on the sphere is not invariant under the rational map R. There is however an invariant measure μ that is absolutely continuous with respect to (2-dimensional) Lebesgue measure λ on the sphere. We now proceed to define this measure. The technique to construct μ goes back to Sullivan and Patterson, the author learned it from [GS].

Remark. In the following, $R^{-1}(z)$ will denote two different things: the set of preimages, and a branch of the inverse function. We write $\{R^{-1}(z)\}$ to denote the former, $R^{-1}(z)$ for the latter.

For every $j \geq 0$, let $d\mu_j := \kappa_j \, d\lambda$, where

$$\kappa_j(z) := \sum_{w \in \{R^{-j}(z)\}} [(R^j)^{\#}(w)]^{-2},$$

and λ is *normalized* Lebesgue measure on the sphere S^2. Note that by the chain rule (for $j \geq 1$)

$$\kappa_j(z) = \sum_{y \in \{R^{-1}(z)\}} R^{\#}(y)^{-2} \kappa_{j-1}(y). \tag{5.1}$$

Lemma 5.1. *The measures μ_j are probability measures and satisfy*

$$R_* \mu_{j-1} = \mu_j,$$

for every $j \geq 1$.

Proof. By equation (5.1),

$$d\mu_j(z) = \kappa_j(z) \, d\lambda(z) = \sum_{y \in \{R^{-1}(z)\}} R^{\#}(y)^{-2} \kappa_{j-1}(y) \, d\lambda(z),$$

since $z = R(y)$, we have $d\lambda(z) = R^{\#}(y)^2 \, d\lambda(y)$,

$$= \sum_{y \in \{R^{-1}(z)\}} \kappa_{j-1}(y) \, d\lambda(y).$$

Thus, for any Borel set A

$$\mu_j(A) = \int_A d\mu_j = \int_{\{R^{-1}A\}} \kappa_{j-1} \, d\lambda = \mu_{j-1}(R^{-1}A) = R_* \mu_{j-1}(A).$$

A pushforward of a probability measure is a probability measure, thus all μ_j are probability measures. $\square$

So by iteration

$$\mu_j = R_*^j \lambda,$$

which we could have used as the definition. Since the set of probability measures is compact in the weak-$*$ topology, we can define μ as a weak-$*$ subsequential limit of

$$\overline{\mu}_j = \frac{1}{j}(\mu_1 + \cdots + \mu_j). \tag{5.2}$$

Since

$$\begin{aligned} &\left| \frac{1}{j}(\mu_1 + \cdots + \mu_j)(R^{-1}A) - \frac{1}{j}(\mu_1 + \cdots + \mu_j)(A) \right| \\ &\quad \le \frac{1}{j}\mu_{j+1}(A) + \frac{1}{j}\mu_1(A) \le \frac{2}{j}, \end{aligned}$$

for any Borel set $A \subset S^2$, the measure μ is invariant. It is a probability measure as the weak-$*$ limit of probability measures on a compact space. We need to show that μ is absolutely continuous with respect to Lebesgue measure. In the next lemma, it is shown that there is $G \in L^1(\lambda)$ such that $\kappa_j \le G$. This shows absolute continuity of μ.

To estimate the κ_j, we need to take preimages, more precisely estimate derivatives along inverse orbits. Since the rational map R is postcritically finite, this is not too hard. A branch of the inverse R^{-1} is defined in a (simply connected) neighborhood U of a point z if and only if U does not contain a critical value. Every preimage of such a U will again not contain any postcritical point by the postcritical finiteness. Thus, every branch of R^{-j} is then (univalently) defined on U. We can then use Koebe's theorem to control the derivative.

By $\deg(c) = \deg_R(c)$, we denote the *degree* of a critical point c, i.e.,

$$R(w) = p + a(w-c)^{\deg(c)} + \cdots,$$

in a neighborhood of c. It will simplify our discussion to allow $\deg(c) = 1$. For a postcritical point $p \in \operatorname{post}(R)$, let

$$\max\deg(p) := \max\bigl\{\deg_{R^j}(c) : c \in \{R^{-j}(p)\}, j \ge 1\bigr\}.$$

This is finite, since R has no critical periodic orbits (see Lemma 3.4).

LEMMA 5.2. *The densities $\kappa_j : \widehat{\mathbb{C}} \to \mathbb{R}$ satisfy the following.*

(1) *They are* equicontinuous *on $\widehat{\mathbb{C}} \setminus \operatorname{post}(R)$.*

(2) *Let $z \in \widehat{\mathbb{C}} \setminus \operatorname{post}(R)$, $\operatorname{dist}(z, \operatorname{post}(R)) \ge \varepsilon$. Then*

$$\kappa_j(z) \asymp 1,$$

where $C(\asymp) = C(\varepsilon)$ is independent of j.

(3) *In a neighborhood of a postcritical point* p,

$$\kappa_j(z) \asymp |z-p|^{-2(1-\frac{1}{\max \deg(p)})},$$

for $j \geq k_0$, *with* $C(\asymp)$ *independent of* j.

Proof. (1) Consider an arbitrary $z \in \widehat{\mathbb{C}} \setminus \operatorname{post}(R)$. Let $\delta := \operatorname{dist}(z, \operatorname{post}(R))$. Every branch of R^{-j} is then defined on the neighborhood $U_\delta(z)$. Let $\varepsilon < \delta$, then for $z' \in U_\varepsilon(z) \subset U_\delta(z)$ we have by Koebe distortion,

$$((R^j)^\#(w))^{-1} = (R^{-j})^\#(z) \asymp (R^{-j})^\#(z') = ((R^j)^\#(w'))^{-1},$$

where $w = R^{-j}(z)$, $w' = R^{-j}(z')$. Here $C(\asymp) = C(\varepsilon/\delta) \to 1$ as $\varepsilon/\delta \to 0$ (independent of j). This yields

$$\kappa_j(z) = \sum_{w \in \{R^{-j}(z)\}} ((R^j)^\#(w))^{-2} \asymp \sum_{w' \in \{R^{-j}(z')\}} ((R^j)^\#(w'))^{-2} = \kappa_j(z'),$$

(with $C(\asymp) = C(\varepsilon/\delta) \to 1$ as $\varepsilon/\delta \to 0$ as before). The statement follows from (2) which we prove next.

(2) Let $z \in \widehat{\mathbb{C}} \setminus \operatorname{post}(R)$. Let Y_1', Y_2' be the two 0-tiles, they are $\overline{\mathbb{H}}^+, \overline{\mathbb{H}}^-$ if R is as in Section 3.1. Without loss of generality, $z \in Y_1'$.

Assume first that $z \in Z_1' \in \mathbf{X}_1'$, where the 1-tile Z_1' does not contain a postcritical point. Let U be a neighborhood of Z_1' containing no postcritical point. Fix a branch of R^{-j} on U. Let $w := R^{-j}(z)$ and $R^{-j}(Z_1') =: Z_{j+1}' \in \mathbf{X}_{j+1}'$. There is (exactly) one j-tile $X_j' \ni w$ such that $R^j(X_j') = Y_1'$; conversely for each j-tile X_j' with $R^j(X_j') = Y_1'$ there is (exactly) one such $w \in X_j'$. By Koebe distortion

$$\operatorname{diam} Z_{j+1}' \asymp (R^{-j})^\#(z) \operatorname{diam} Z_1' \asymp ((R^j)^\#(w))^{-1}$$

and by Lemma 3.7

$$\operatorname{area} X_j' \asymp \operatorname{area} Z_{j+1}' \asymp ((R^j)^\#(w))^{-2}. \tag{5.3}$$

Here, $C(\asymp)$ is independent of j or the branch of R^{-j}.

By the above, we can estimate the area of all j-tiles X_j' that are preimages of Y_1'. To estimate the area of j-tiles $\widetilde{X}_j'$ that are preimages of Y_2', we *match* each such j-tile $\widetilde{X}_j$ to one X_j' (this is in fact a *perfect matching*). Fix a 0-edge E (which is $[-\infty,-1]$, $[-1,1]$, or $[1,\infty]$ for R as in Section 3.1). Consider now a j-tile $\widetilde{X}_j'$ that is mapped to Y_2', $R^j(\widetilde{X}_j') = Y_2'$. Then there is exactly one j-tile X_j' (which is mapped to Y_1' by R^j) that shares a preimage of E

with $\widetilde{X}'_j$. By Lemma 3.7, $\operatorname{area}\widetilde{X}'_j \asymp \operatorname{area} X'_j$. Thus,

$$\begin{aligned}
1 = \sum_{X'_j \in \mathbf{X}'_j} \operatorname{area} X'_j &\asymp \sum_{\substack{X'_j \in \mathbf{X}'_j \\ R^j(X'_j) = Y'_1}} \operatorname{area} X'_j, \quad \text{thus by (5.3)} \\
&\asymp \sum_{w \in \{R^{-j}(z)\}} ((R^j)^{\#}(w))^{-2} = \kappa_j(z).
\end{aligned}$$

To complete the statement, we consider $z \in Z'_k \in \mathbf{X}'_k$, where the k-tile Z'_k does not contain a postcritical point, and repeat the argument. Every point $z \in \widehat{\mathbb{C}} \setminus \operatorname{post}(R)$ with $\operatorname{dist}(z, \operatorname{post}(R)) \geq \varepsilon$ will be contained in such a Z'_k for $k = k(\varepsilon)$ sufficiently large by Lemma 3.5.

(3) Each postcritical point has a finite orbit, since R is postcritically finite. Let $\operatorname{post}_p(R)$ be the set of preperiodic postcritical points of R, and $\operatorname{post}_f(R)$ be the set of periodic postcritical points of R,

$$\operatorname{post}(R) = \operatorname{post}_p(R) \cup \operatorname{post}_f(R).$$

Each point $p \in \operatorname{post}_f(R)$ is a (repelling) fixed point for a suitable iterate R^{n_0}. The proof for (3) is broken up into three parts. In (3b), we assume that each $p \in \operatorname{post}_f(R)$ is a fixed point.

(3a) Consider first a (preperiodic) point $p \in \operatorname{post}_p(R)$. Then there is a $k_0 \geq 1$ such that no $c \in \{R^{-k_0}(p)\}$ is a postcritical point. Let $z \in U_\varepsilon(p)$, where ε is smaller than the distance of p to another postcritical point. Then for any $y \in \{R^{-k_0}(z)\}$

$$\begin{aligned}
|z - p| = |R^{k_0}(y) - p| &\asymp |y - c|^d, \quad \text{and} \\
(R^{k_0})^{\#}(y) \asymp |y - c|^{d-1} &\asymp |z - p|^{\frac{d-1}{d}},
\end{aligned}$$

where $c \in \{R^{-k_0}(p)\}$, $d = d(y) = \deg_{R^{k_0}}(c)$, and $C(\asymp)$ does only depend on ε. Thus, (see (5.1))

$$\begin{aligned}
\kappa_{j+k_0}(z) &= \sum_{y \in \{R^{-k_0}(z)\}} (R^{k_0})^{\#}(y)^{-2} \underbrace{\kappa_j(y)}_{\asymp 1 \text{ by (2)}} \\
&\asymp \sum_{y \in \{R^{k_0}(z)\}} |z - p|^{-2(1 - \frac{1}{d(y)})} \asymp |z - p|^{-2(1 - \frac{1}{\max \deg(p)})}.
\end{aligned}$$

For $j < k_0$, the same argument as above yields $\kappa_j(z) \lesssim |z - p|^{-2(1 - \frac{1}{\max \deg(p)})}$.

(3b) Consider now a point $p \in \operatorname{post}_f(R)$. We assume here that p is a (repelling) fixed point of R. The *multiplier* (of R at p) is defined as

$$\Lambda = \Lambda(p) := |R'(p)| = R^{\#}(p) > 1.$$

By Koenigs' linearization theorem (see [CG93], II.2 and II.3), there is a neighborhood $U = U(p)$ of p in which R is conformally conjugate to $z \mapsto R'(p)z$. Let R_p^{-k} be the branch of R^{-k} that keeps p fixed (defined on U).

Let $z \in U$, and set $z_k := R_p^{-k}(z)$. Then (by Koebe)

$$|z_k - p| \asymp \Lambda^{-k}|z-p| \quad \text{and} \quad (R^k)^{\#}(z_k) \asymp \Lambda^k,$$

where $C(\asymp)$ is independent of k. Consider now a point $w \in \{R^{-1}(z_k)\}$, $w \neq z_{k+1}$. Then w is in the neighborhood of a point $p' \in \{R^{-1}(p)\}$, where $p' \notin \operatorname{post}_f(R)$, and with $d = d(w) = \deg_R(p')$

$$\begin{aligned} |w-p'|^d &\asymp |z_k - p| \asymp \Lambda^{-k}|z-p|, \\ R^{\#}(w) &\asymp |w-p'|^{d-1} \asymp \Lambda^{-k(\frac{d-1}{d})}|z-p|^{1-\frac{1}{d}}, \\ (R^{k+1})^{\#}(w) = R^{\#}(w)(R^k)^{\#}(z_k) &\asymp \Lambda^{\frac{k}{d}}|z-p|^{1-\frac{1}{d}}. \end{aligned}$$

Thus, by (3a) for $j \geq k_0$

$$\begin{aligned} \kappa_j(w) &\asymp |w-p'|^{-2(1-\frac{1}{\max\deg(p')})} \\ &\asymp \Lambda^{\frac{2k}{d}(1-\frac{1}{\max\deg(p')})}|z-p|^{\frac{-2}{d}(1-\frac{1}{\max\deg(p')})}, \end{aligned}$$

$$[(R^{k+1})^{\#}(w)]^{-2}\kappa_j(w) \asymp \Lambda^{\frac{-2k}{d\max\deg(p')}}|z-p|^{-2(1-\frac{1}{d\max\deg(p')})}. \tag{5.4}$$

Here, $C(\asymp)$ does only depend on U. If $j < k_0$, we obtain in the previous two expressions "$\lesssim$" instead of "$\asymp$." Thus, (using (5.1))

$$\begin{aligned} \kappa_j(z) = &\sum_{\substack{w\in\{R^{-1}(z)\}\\ w\neq z_1}} R^{\#}(w)^{-2}\kappa_{j-1}(w) \\ &+ \sum_{\substack{w\in\{R^{-1}(z_1)\}\\ w\neq z_2}} [(R^2)^{\#}(w)]^{-2}\kappa_{j-2}(w) \\ &\vdots \\ &+ \sum_{\substack{w\in\{R^{-1}(z_{j-1})\}\\ w\neq z_j}} [(R^j)^{\#}(w)]^{-2}\underbrace{\kappa_{j-j}(w)}_{=1} \\ &+ [(R^j)^{\#}(z_j)]^{-2}. \end{aligned}$$

Hence, by (5.4), for $j > k_0$

$$\asymp |z-p|^{-2(1-\frac{1}{\max\deg(p)})}\underbrace{(1+\Lambda^{\frac{-2}{\max\deg(p)}}+\cdots+\Lambda^{\frac{-2(j-1)}{\max\deg(p)}})}_{\asymp 1}+\Lambda^{-2j}.$$

This proves property (3) in this case.

(3c) Consider now a $p \in \operatorname{post}_f(R)$, where we now drop the assumption that p is a fixed point R. Let n_0 be the period of p. Then we know the behavior of

κ_{jn_0} near p by (3b). It is therefore enough to show that if κ_j has the desired behavior near p, then κ_{j+1} has this behavior as well.

Let $z \in U(p)$. Then $\{R^{-1}(z)\}$ contains one point z' in a neighborhood of $p' \in \operatorname{post}_f(R)$. We have $R^{\#}(z') \asymp 1$, and $\max\deg(p) = \max\deg(p')$. Thus,

$$|z' - p'| \asymp |z - p|.$$

We assume now that for $j \geq k_0$

$$\begin{aligned} \kappa_j(z') &\asymp |z' - p'|^{-2(1-\frac{1}{\max\deg p'})}, \quad \text{this yields} \\ &\asymp |z-p|^{-2(1-\frac{1}{\max\deg p})}. \end{aligned}$$

Consider now $w \in \{R^{-1}(z)\}, w \neq z'$. Note that w lies in a neighborhood of $q \in \{R^{-1}(p)\}$, where $q \notin \operatorname{post}_f(R)$. Then we obtain as before with $d = \deg(q)$ and a constant $C(\asymp)$ (only depending on $U(p)$)

$$\begin{aligned} w - q^d &\asymp |z-p|, \\ R^{\#}(w) &\asymp |w-q|^{d-1} \asymp z - p^{1-\frac{1}{d}}, \\ \kappa_j(w) &\asymp |w-q|^{-2(1-\frac{1}{\max\deg(q)})}, \quad \text{from (3a) for } j \geq k_0, \\ &\asymp |z-p|^{-\frac{2}{d}(1-\frac{1}{\max\deg(q)})}, \\ R^{\#}(w)^{-2}\kappa_j(w) &\asymp |z-p|^{-2(1-\frac{1}{d\max\deg(q)})}. \end{aligned}$$

Thus, (using (5.1)) for $j \geq k_0$

$$\begin{aligned} \kappa_{j+1}(z) &= R^{\#}(z')^{-2}\kappa_j(z') + \sum_{\substack{w \in \{R^{-1}(z)\} \\ w \neq z'}} R^{\#}(w)^{-2}\kappa_j(w) \\ &\asymp |z-p|^{-2(1-\frac{1}{\max\deg p})}. \end{aligned}$$ □

Lemma 5.3. *The averages $\overline{\mu}_j := \frac{1}{j}(\mu_1 + \cdots + \mu_j)$ converge weak-$*$ to an R-invariant ergodic probability measure μ. Furthermore*:

(1) *$d\mu = \kappa\, d\lambda$ is absolutely continuous with respect to Lebesgue measure.*
(2) *The density κ is continuous on $\widehat{\mathbb{C}} \setminus \operatorname{post}(R)$.*
(3) *Let $U = \bigcup_{p \in \operatorname{post}(R)} U_\varepsilon(p)$ be an ε-neighborhood of the postcritical set. Then*

$$\kappa(z) \asymp 1,$$

for $z \in \widehat{\mathbb{C}} \setminus U$, where $C(\asymp) = C(\varepsilon)$.

(4) *In a neighborhood of a postcritical point p,*

$$\kappa(z) \asymp |z-p|^{-2(1-\frac{1}{\max\deg(p)})}.$$

Proof. Every weak-$*$ subsequential limit μ of $(\overline{\mu}_j)$ satisfies (3) and (4) by the last lemma. Note that all κ_j are dominated by a $G \in L^1(\lambda)$. Thus, μ is absolutely continuous with respect to Lebesgue measure. It is well known (see [McM94], Theorem 3.9) that μ is ergodic, since the Julia set of R is $\widehat{\mathbb{C}}$.

Recall that two ergodic invariant probability measures are either singular or identical. Thus, $(\overline{\mu}_j)$ converges. The measure μ is a probability measure as a weak-$*$ limit of probability measures on a compact space.

The averages $\frac{1}{j}(\kappa_1 + \cdots + \kappa_j)$ are equicontinuous on $\widehat{\mathbb{C}} \setminus \operatorname{post}(R)$ by the last lemma. Taking another subsequence from the above, we obtain by Arzelà-Ascoli that κ is continuous. Uniqueness of ergodic, nonsingular measures yields (2). $\square$

6. Proof of the theorems

In this section, we prove Theorem 3 and Theorem 4 (and thus, Theorem 1 and Theorem 2) for

$$\alpha = \frac{\log N}{\chi}.$$

Recall that $\mathcal{S}$ was constructed from an N-generator. Let us first express the entropy and the Lyapunov exponent in terms of the invariant measure μ constructed in the previous section.

The *Jacobian* describes the expansion of R with respect to μ,

$$J_\mu(x) := \frac{d\mu \circ R}{d\mu} = \lim_j \frac{\mu(R(X'_j))}{\mu(X'_j)} = R^{\#}(z)^2 \frac{\kappa(Rz)}{\kappa(z)}. \tag{6.1}$$

Here, the first term denotes the Radon–Nikodym derivative, in the second term $x \in X'_j \in \mathbf{X}'_j$. Note that

$$\int J_\mu \, d\mu = \int d\mu \circ R = \sum_{X' \in \mathbf{X}'_1} \int_{X'} d\mu \circ R \overset{Rz=w}{=} \deg R. \tag{6.2}$$

The entropy may be expressed via the Jacobian

$$h = h_\mu = \int \log J_\mu \, d\mu. \tag{6.3}$$

This is known as *Rohlin's formula* (see for example [PU], Chapter 1). Thus, (using (4.2))

$$h = \int \log(R^{\#})^2 \, d\mu + \int \log \kappa(Rz) \, d\mu - \int \log \kappa(z) \, d\mu = 2\chi. \tag{6.4}$$

This is a trivial instance of Manning's formula (4.9), it just says that Lebesgue measure (on S^2) has dimension 2.

By (6.4), (6.3), (6.2), and *Jensen's inequality*

$$2\chi = h = \int \log J_\mu \, d\mu \le \log \int J_\mu \, d\mu = \log \deg R, \tag{6.5}$$

where equality occurs if and only if $J_\mu = \text{const} = \deg R$ (μ almost everywhere). The measure satisfying this is called the measure of *maximal entropy*, it is actually the Hausdorff measure on the snowsphere—in the Hausdorff dimension of it. It was shown in [Zdu90] that Lebesgue measure is not absolutely continuous to the measure of maximal entropy unless R is a *Lattès map* (has *parabolic orbifold*). Thus, $2\chi < \log\deg R$, unless R is Lattès in which case $2\chi = \log\deg R$. However, we can derive this relatively easily given the setup of the last section. This is done for the convenience of the reader in the next section.

Recall that $\dim_H(\mathcal{S}) = \frac{\log\deg R}{\log N}$ (Theorem 7(4)). Hence,

$$\begin{aligned} \alpha &= \frac{\log N}{\chi} = \frac{2}{\dim_H(\mathcal{S})}, \quad &&\text{if } R \text{ is Lattès,} \\ \alpha &> \frac{2}{\dim_H(\mathcal{S})}, &&\text{otherwise} \end{aligned}$$

as desired.

Let $\operatorname{dist}(x, \operatorname{post}(R)) > \varepsilon > 0$ and $x \in X'_j \in \mathbf{X}'_j$. By Lemma 3.7 and Lemma 5.3(3), we obtain for sufficiently large j

$$\mu(X'_j) \asymp (\operatorname{diam} X'_j)^2, \tag{6.6}$$

where $C(\asymp) = C(\varepsilon)$. Thus, using Shannon–McMillan–Breiman (4.4)

$$h = \lim_j -\frac{1}{j}\log\mu(X'_j) = 2\lim_j -\frac{1}{j}\log\operatorname{diam} X'_j, \tag{6.7}$$

for Lebesgue almost every $\{x\} = \bigcap X'_j$. We are now ready to prove Theorem 3.

Proof of Theorem 3. In the following, we write $a = b \pm C$ if $b - C \leq a \leq b + C$.

Consider a $\{x\} = \bigcap X'_j$ satisfying (6.7). Let $j = j(x,y)$ (see (3.1)), then $y \to x \Leftrightarrow j \to \infty$. Thus,

$$\lim_{y\to x} \frac{\log|x-y|_{\mathcal{S}}}{\log|x-y|} = \lim_j \frac{-j(\log N \pm C/j)}{\operatorname{diam} X'_j \pm C} \quad \text{by (3.2) and (3.3).}$$

Thus, using (6.7) and (6.4)

$$= \lim_j \frac{\log N \pm C/j}{-\frac{1}{j}\operatorname{diam} X'_j \pm C/j} = \frac{\log N}{h/2} = \frac{\log N}{\chi} = \alpha. \qquad \square$$

Proof of Theorem 4. This is an easy consequence of Theorem 3. Indeed consider a homeomorphism $f : \mathbf{X} \to \mathbf{Y}$ between metric spaces, such that

$$\begin{aligned} \frac{\log|f(x) - f(y)|}{\log|x-y|} &\leq \alpha + \varepsilon, \\ \frac{\log|f(x) - f(y)|}{\log|x-y|} &\geq \alpha - \varepsilon, \end{aligned}$$

for all $0 < |x-y| < 1/n$. Then we obtain for the Hausdorff dimension of the spaces $\dim\mathbf{Y} \geq \dim\mathbf{X}/(\alpha+\varepsilon)$ and $\dim\mathbf{Y} \leq \dim\mathbf{X}/(\alpha-\varepsilon)$. This follows directly from the definition of Hausdorff dimension. Thus, Theorem 4 is proved, since for every $\varepsilon > 0$ we can exhaust a set of full measure where the above holds (by Theorem 3). $\square$

Proof of Theorem 5. Let $(S^2, |x-y|_{\mathcal{S}})$ be snowflake equivalent to S^2. In particular they are quasisymmetric. Thus, by [BM], the expanding Thurston map R is topologically conjugate to a rational map. So we can assume that R is in fact a rational map (which is postcritically finite and has no periodic critical points).

Let $\varphi\colon (S^2, |x-y|_{\mathcal{S}}) \to S^2$ be the snowflake equivalence. This means that there is a $\beta > 0$, such that

$$|\varphi(x) - \varphi(y)|^\beta \asymp |x-y|_{\mathcal{S}}$$

for all $x, y \in S^2$. Clearly, φ changes the (Hausdorff) dimension by the factor β. Thus, we have $\beta = 2/d$, where $d = \dim_H(S^2, |x-y|_{\mathcal{S}})$. The map φ maps d-dimensional Hausdorff measure to (2-dimensional) Lebesgue measure (up to multiplicative constants).

Assume now that R is not a Lattès map. Then by Theorem 4 there is a set $A \subset S^2$ that has full (2-dimensional) Lebesgue measure λ, yet is a (Hausdorff) d-dimensional zero set in $(S^2, |x-y|_{\mathcal{S}})$.

The map $\mathrm{id}\colon S^2 \to (S^2, |x-y|_{\mathcal{S}})$ is a quasisymmetry (see [BM] and [Mey]). We now get a contradiction since the composition

$$S^2 \xrightarrow{\mathrm{id}} (S^2, |x-y|_{\mathcal{S}}) \xrightarrow{\varphi} S^2,$$

is a quasiconformal map that maps A to a zero set (with respect to Lebesgue measure λ); hence is not absolutely continuous.

Assume now that R is topologically conjugate to a Lattès map. So assume that R is a Lattès map. Then R is obtained as a quotient of a linear map by a *wallpaper group*. We can choose $|x-y|_{\mathcal{S}}$ to be the projection of the Euclidean metric. Then $(S^2, |x-y|_{\mathcal{S}})$ is easily seen to be bi-Lipschitz to S^2.

To be more explicit, we remind the reader of Theorem 3.1 in [Mil06b]. The map R may be described as follows.

- There is a flat torus $\mathbb{T}^2 \cong \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice of rank 2.
- Furthermore, there is an affine map $L = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$; satisfying $L\Lambda \subset \Lambda$ (it holds $|a|^2 = \deg R$).
- Then R is conformally conjugate to
$$L/G_n\colon \mathbb{T}^2/G_n \to \mathbb{T}^2/G_n,$$
for some n. Here, G_n is the group of nth roots of unity, acting on $\mathbb{T}^2$ by rotation around a base point.

The space $\mathbb{T}^2/G_n$ is topologically a sphere. In fact, in the flat metric inherited from the torus $\mathbb{T}^2$ the sphere $\mathbb{T}^2/G_n$ is isometric to (see Section 4 in [Mil06b]) the path metric on:

- a tetrahedron if the signature of R is $(2, 2, 2, 2)$,
- two triangles glued together along their boundaries, in the other cases.

Clearly, these spheres are bi-Lipschitz to the standard sphere. Note that $\mathrm{id} : (S^2, |x-y|_{\mathcal{S}}) \to S^2$ is *not* bi-Lipschitz. □

7. The Jacobian of the invariant measure

In this section, we show that $\alpha = \frac{\log N}{\chi} > \frac{2}{\dim_H(\mathcal{S})}$, unless R is a Lattès map, in which case equality holds. By (6.5), it is enough to show that for our given measure μ the Jacobian is not constant.

LEMMA 7.1. *The Jacobian J_μ of the invariant measure μ satisfies the following.*

(1) *It is continuous on* $\widehat{\mathbb{C}} \setminus (\mathrm{crit}(R) \cup \mathrm{post}(R))$. *For* $\mathrm{dist}(x, \mathrm{crit}(R) \cup \mathrm{post}(R)) \geq \varepsilon$

$$J_\mu(x) \asymp 1,$$

where $C(\asymp) = C(\varepsilon)$.

(2) *In the neighborhood of a point* $q \in \mathrm{crit}(R) \cup \mathrm{post}(R)$, *with* $p = R(q)$

$$J_\mu(z) \asymp |z-q|^{-2\left(\frac{1}{\max \deg(q)} - \frac{\deg(q)}{\max \deg(p)}\right)}.$$

Note that $\max \deg(p) \geq \deg(q) \max \deg(q)$, so the exponent in the last expression is nonpositive.

Proof. (1) is clear from (6.1) and Lemma 5.3(1), (3).

(2) Let z be contained in a suitably small neighborhood of $q \in \mathrm{crit}(R) \cup \mathrm{post}(R)$. Let $p = R(q) \in \mathrm{post}(R)$, $d = \deg_R(q)$. Note that d or $\max \deg(q)$ may be 1, though not both at the same time. We obtain (using Lemma 5.3)

$$\begin{aligned} R^{\#}(z) &\asymp |z-q|^{d-1}, \\ \kappa(R(z)) &\asymp R(z) - p^{-2\left(1 - \frac{1}{\max \deg(p)}\right)} \\ &\asymp |z-q|^{-2d\left(1 - \frac{1}{\max \deg(p)}\right)}, \\ \kappa(z) &\asymp |z-q|^{-2\left(1 - \frac{1}{\max \deg(q)}\right)}. \end{aligned}$$

Thus,

$$J_\mu(z) = R^{\#}(z)^2 \frac{\kappa(R(z))}{\kappa(z)} \asymp |z-q|^{-2\left(\frac{1}{\max \deg(q)} - \frac{\deg(q)}{\max \deg(p)}\right)},$$

as desired. □

LEMMA 7.2. *It holds $\alpha > \frac{2}{\dim_H(\mathcal{S})}$ unless R is a Lattès map, in which case* $\alpha = \frac{2}{\dim_H(\mathcal{S})}$.

Proof. The reader should review the terminology from Section 3.4. From (6.5) and Theorem 7 (3.1) it follows that $\alpha = \frac{\log N}{\chi} \geq \frac{2 \log N}{\log \deg R} = \frac{2}{\dim_H(\mathcal{S})}$. Assume now that $\alpha = \frac{2}{\dim_H(\mathcal{S})}$, which is equivalent to $J_\mu = \deg R$ almost everywhere. This means in particular that the exponent in Lemma 7.1(2) is 0, or $\max \deg(p) = \deg(q) \max \deg(q)$ (for $R(q) = p$). Thus, for all $x, y \notin \mathrm{post}(R)$

with $R^n(x) = R^n(y)$ we have $\deg_{R^n}(x) = \deg_{R^n}(y)$. This implies that the ramification function equals the maximal degree, $\nu(p) = \max\deg(p)$ for all $p \in \operatorname{post}(R)$. Let $k = \#\operatorname{post}(R)$, and $(\nu(p_1), \dots, \nu(p_k))$ be the signature of the orbifold associated to R.

The argument will be a simple counting argument using the Euler characteristic. Fix a $j \geq 0$. Consider the j-tiles, j-edges, and j-vertices; they are called faces, edges, vertices in the following for simplicity. Let F, E, V be the number of faces, edges, vertices. Since every face is incident to k edges, and every edge is incident to two faces we have

$$kF = 2E.$$

Consider now a vertex q that is mapped by R^j to (the first postcritical point) p_1. Then q is contained in $2\deg_{R^j}(q)$ faces. Note that each face contains exactly one such vertex q. Let n_1 be the number of such vertices q. By the above $\deg_{R^j}(q) = \nu(p_1)$ if $q \notin \operatorname{post}(R)$, otherwise $\deg_{R^j}(q) \leq \nu(p_1)$. Therefore,

$$2(n_1 - k)\nu(p_1) \leq F \leq 2n_1\nu(p_1).$$

Analog expressions of course hold for all $p_l \in \operatorname{post}(R)$. Thus,

$$\frac{F}{2}\sum_l \frac{1}{\nu(p_l)} \leq \sum_l n_l = V \leq \frac{F}{2}\sum_l \frac{1}{\nu(p_l)} + k^2.$$

This yields

$$\begin{aligned} 2 = F - E + V &= F - \frac{k}{2}F + \frac{1}{2}F\sum_l \frac{1}{\nu(p_l)} + O(1) \\ &= \frac{F}{2}\left(2 - k + \sum_l \frac{1}{\nu(p_l)}\right) + O(1) \\ &= \frac{F}{2}\underbrace{\left(2 + \sum_l \left(\frac{1}{\nu(p_l)} - 1\right)\right)}_{\chi(\mathcal{O})} + O(1). \end{aligned}$$

Since $F = 2(\deg R)^j$, this can only be satisfied if $\chi(\mathcal{O}) = 0$. This means that the orbifold is parabolic, thus R is a Lattès map. $\square$

8. Numerical experiments

In principle, one could use the explicit description of the invariant measure μ to calculate the entropy or the Lyapunov exponent by equation (6.3) or (4.6) and thus the dimension of elliptic harmonic measure. However, we do not know the rate of convergence of the densities $\kappa_j \to \kappa$. We use Birkhoff's

TABLE 1. Experimental values of the dimensions by equation (8.1)

Rational map	Dimension of snowsphere	$\dim \mu$ average	$\dim \mu$ maximal	$\dim \mu$ minimal	Standard deviation
$\widehat{R}$	$\frac{\log 29}{\log 5} = 2.092\dots$	2.030	2.034	2.026	0.0017
R	$\frac{\log 13}{\log 3} = 2.335\dots$	2.115	2.125	2.105	0.0044
R_6	$\frac{\log 6}{\log 2} = 2.585\dots$	2.359	2.376	2.342	0.0068
R_7	$\frac{\log 10}{\log 2} = 3.322\dots$	2.594	2.625	2.563	0.012
R_1	2	1.999	2.001	1.996	0.00079
R_2	2	1.9998	2.0003	1.9987	0.00025
R_3	2	1.9999	2.0003	1.9996	0.00012
R_4	2	1.9997	2.0023	1.9943	0.0012

ergodic theorem instead to calculate the Lyapunov exponent

$$\frac{1}{n}\sum_{j=0}^{n-1} \log R^{\#}(R^j z_0) \to \chi, \quad \text{as } n \to \infty, \tag{8.1}$$

for (μ or Lebesgue) almost every z_0. The dimension of μ with respect to the metric $|x-y|_{\mathcal{S}}$ is then $\dim \mu = \frac{2\chi}{\log N}$. The ergodic sum is very easy to calculate. There is no way to determine however that we picked a *generic point* z_0. Even if we did, we do not know how fast the sum converges. So the results in this section are to be understood as *numerical experiments* only. Still the values should give a good indication of what to expect.

The dimensions thus found are recorded in Table 1. We picked 100 random starting values $z_0 \in [0,1]^2$, uniformly distributed. The number of iterations was in each case $n = 10{,}000$. The computations are done for our standard example (see Section 3) $\widehat{R}$. The other examples are from [Mey02]. The map R embeds a snowsphere where the generator is a square divided into 3×3 squares, with a 1/3-cube put on the middle 1/3-square. The maps R_6, R_7 embed "triangular" snowspheres. More precisely, the generator is a unit equilateral triangle divided into 4 equilateral triangles of side-length 1/2. The snowsphere represented by R_6 has a generator where the middle triangle is replaced by a tetrahedron. The snowsphere represented by R_7 has a generator where the middle triangle is replaced by a octahedron. The snowspheres represented by R, R_6, R_7 do have self intersections when embedded in $\mathbb{R}^3$.

The maps R_1, R_2, R_3, R_4 are Lattès maps, and serve as a "control group" for our computations. Their signatures are $(2,4,4)$, $(3,3,3)$, $(2,2,2,2)$, and $(2,3,6)$. The computations were done with Maple 9, using machine precision.

9. Open problems

We conclude with some questions.

Open Problem 1. Is it true that

$$\dim \mu > 2?$$

Here, μ is an elliptic harmonic measure on a self similar snowsphere.

The answer is generally expected to be yes. For corresponding results on the dimension of harmonic measure, see [Wol95]. There is some hope that the description through the rational map R might help answering this question.

Open Problem 2. Is there a nontrivial upper bound for the dimension of an elliptic harmonic measure? More precisely, is there an $\varepsilon > 0$, such that

$$\dim \mu \leq 3 - \varepsilon,$$

for an elliptic harmonic measure μ of any snowsphere $\mathcal{S} \subset \mathbb{R}^3$?

The corresponding statement for harmonic measure is true by [Bou87], see also [Mak85] for the two-dimensional case.

Open Problem 3. Is there a *geometric way* to "see" the exponent α/the dimension of the elliptic harmonic measure? This would characterize the "quasisymmetric size" of a quasisphere. In general, one expects to have different upper and lower dimensions ($\dim^* \mu = \dim \mu$ as before, $\dim_* \mu = \inf\{\dim_H K \mid \mu(K) > 0\}$).

Open Problem 4. Given the elliptic harmonic measure on a snowsphere $\mathcal{S} \subset \mathbb{R}^3$, is there a *relation to other natural measures* on $\mathcal{S}$? In particular, is there a relation to the harmonic measure or the 3-harmonic measure on $\mathcal{S}$?

References

[AB56] L. V. Ahlfors and A. Beurling, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142. MR 0086869

[BK02] M. Bonk and B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres*, Invent. Math. **150** (2002), 127–183. MR 1930885

[BM] M. Bonk and D. Meyer, *Expanding Thurston maps*, in preparation.

[Bou87] J. Bourgain, *On the Hausdorff dimension of harmonic measure in higher dimension*, Invent. Math. **87** (1987), 477–483. MR 0874032

[CFKP03] J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry, *Constructing rational maps from subdivision rules*, Conform. Geom. Dyn. **7** (2003), 76–102. MR 1992038

[CFP01] J. W. Cannon, W. J. Floyd and W. R. Parry, *Finite subdivision rules*, Conform. Geom. Dyn. **5** (2001), 153–196 (electronic). MR 1875951

[CG93] L. Carleson and T. W. Gamelin, *Complex dynamics*, Springer, New York, 1993. MR 1230383

[DH93] A. Douady and J. H. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), 263–297. MR 1251582

[DT99] G. David and T. Toro, *Reifenberg flat metric spaces, snowballs, and embeddings*, Math. Ann. **315** (1999), 641–710. MR 1731465

[Fal90] K. Falconer, *Fractal geometry. Mathematical foundations and applications*, Wiley, Chichester, 1990. MR 1102677

[GS] J. Graczyk and S. Smirnov, *Non-uniform hyperbolicity in complex dynamics*, Invent. Math. **175** (2009), 335–415. MR 2470110

[Hei01] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer, New York, 2001. MR 1800917

[HP09] P. Haïssinsky and K. M. Pilgrim, *Coarse expanding conformal dynamics*, Astérisque, vol. 325, Société Mathématique de France, Paris, 2009.

[Lat18] S. Lattès, *Sur l'itération des substitutions rationnelles et les fonctions de Poincaré*, C. R. Acad. Sci. Paris **166** (1918), 26–28 (French).

[Mak85] N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. (3) **51** (1985), 369–384. MR 0794117

[Man84] A. Manning, *The dimension of the maximal measure for a polynomial map*, Ann. of Math. (2) **119** (1984), 425–430. MR 0740898

[Mañ88] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Dynamical systems, Valparaiso 1986, Lecture Notes in Math., vol. 1331, Springer, Berlin, 1988, pp. 86–117. MR 0961095

[McM94] C. T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994. MR 1312365

[Mey] D. Meyer, *Snowballs are quasiballs*, Trans. Amer. Math. Soc. **362** (2010), 1247–1300.

[Mey02] ———, *Quasisymmetric embedding of self similar surfaces and origami with rational maps*, Ann. Acad. Sci. Fenn. Math **27** (2002), 461–484. MR 1922201

[Mil06a] J. Milnor, *Dynamics in one complex variable*, 3rd ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006. MR 2193309

[Mil06b] ———, *On Lattès maps*, Dynamics on the Riemann sphere, Eur. Math. Soc., Zürich, 2006, pp. 9–43.

[Pet83] K. Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, 1983. MR 0833286

[Prz93] F. Przytycki, *Lyapunov characteristic exponents are nonnegative*, Proc. Amer. Math. Soc. **119** (1993), 309–317. MR 1186141

[PU] F. Przytycki and M. Urbański, *Conformal fractals-Ergodic Theory Methods*, London Math. Soc. Lect. Note Series, vol. 371. Cambridge University Press, Cambridge, 2010.

[Tuk89] P. Tukia, *Hausdorff dimension and quasisymmetric mappings*, Math. Scand. **65** (1989), 152–160. MR 1051832

[Urb03] M. Urbański, *Measures and dimensions in conformal dynamics*, Bull. Amer. Math. Soc. **40** (2003), 281–312. MR 1978566

[Väi71] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, vol. 229, Springer, Berlin, 1971. MR 0454009

[Wal82] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer, New York, 1982. MR 0648108

[Wol95] T. H. Wolff, *Counterexamples with harmonic gradients in* $\mathbf{R}^3$, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), Princeton Math. Ser., vol. 42, Princeton Univ. Press, Princeton, NJ, 1995, pp. 321–384. MR 1315554

[Zdu90] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Math. **99** (1990), 627–649. MR 1032883

[Zin00] M. Zinsmeister, *Thermodynamic formalism and holomorphic dynamical systems*, SMF/AMS Texts and Monographs, vol. 2, Amer. Math. Soc., Providence, RI, 2000. MR 1724307

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