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JENSEN MEASURES AND ANNIHILATORS OF HOLOMORPHIC FUNCTIONS

NIHAT GOGUS AND EVGENY A. POLETSKY

ABSTRACT. For a relatively compact domain M in a complex manifold, we completely characterize in terms of Jensen measures the annihilating measures of the algebra $A(\overline{M})$ of holomorphic functions and the space $h(\overline{M})$ of pluriharmonic functions continuous on $\overline{M}$. We also establish the equivalence of Mergelyan type approximation properties of a domain for different function spaces.

1. Introduction

Jensen measures, introduced by Bishop in [B], found many applications to uniform algebras. The most comprehensive discussion of the subject can be found in the book [Ga] of Th. Gamelin, where Jensen measures were used as representing measures.

Of course, not all representing measures are Jensen measure. A representing measure is a point evaluation plus an annihilator of the algebra. Annihilators are usually complex measures and their nature is quite mysterious.

The main goal of this paper is to relate Jensen measures and annihilators of the uniform algebra $A(\overline{M})$ of functions continuous on the closure $\overline{M}$ of a relatively compact domain M in a complex manifold N and holomorphic on M.

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and let $M' = \mathbb{D} \times M \subset \mathbb{C} \times N$. In Section 3, we introduce a continuous operator $G^* : C^*(\overline{M'}) \to C^*(\overline{M})$ with the following important property: for any $z \in M$ it transforms the set $J((z,0), M')$ of all Jensen measures on M' with barycenters at $(z,0)$ into measures in $A^\perp(\overline{M})$. In the same section, we introduce an operator $P^* : C^*(\overline{M}) \to C^*(\overline{M'})$ which is the right inverse of G^*.

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In Section 4, we obtain the description of the space $A^{\perp}(\overline{M})$ in terms of Jensen measures: for any $z \in M$ the set $A^{\perp}(\overline{M})$ is the weak-$*$ closure of the real positive cone over $G^*(J((z,0),M'))$. Moreover, Corollary 4.4 produces a holomorphicity test similar to the F. and M. Riesz Theorem.

Weak-$*$ closures are rather difficult to handle. To alleviate this problem, we consider two subspaces of $C(\overline{M'})$: the space $S(\overline{M'})$ of functions holomorphic in average (see Section 4 for the definition) and the space $h(\overline{M'})$ of functions pluriharmonic on M'. In Theorems 4.5 and 6.1, we show that $P^*(A^{\perp}(\overline{M})) = S^{\perp}(\overline{M'})$ while $G^*(S^{\perp}(\overline{M'})) = A^{\perp}(\overline{M})$ and $P^*(A^{\perp}(\overline{M})) \subset h^{\perp}(\overline{M'})$ while $G^*(h^{\perp}(\overline{M'})) = A^{\perp}(\overline{M})$. The new spaces are more flexible and should be easier to study.

The real annihilators of $h(\overline{M})$ are annihilators of the space $h_{\mathbb{R}}(M)$ of real pluriharmonic functions on M. Theorem 5.2 provides a simple description of this space: for any $z \in M$ the space $h_{\mathbb{R}}^{\perp}(M)$ is the weak-$*$ closure of the set $L(z,M) = \{a(\mu - \nu) : a \in \mathbb{R}, \mu, \nu \in J(z,M)\}$.

In the final Section 6, we address the Mergelyan property for $A(\overline{M})$ which is known only for strongly pseudoconvex domains and some particular cases. We show in Theorem 6.1 that it is equivalent to the Mergelyan property for spaces $S(\overline{M'})$ and $h(\overline{M'})$.

2. Measures generated by analytic disks

Let $\mathbb{D}$ be the unit disk and $\mathbb{T} = \partial\mathbb{D}$. For a complex manifold M of dimension m, we denote by $\mathcal{H}(M)$ be the set of all holomorphic mappings defined on neighborhoods of $\overline{\mathbb{D}}$ which map $\overline{\mathbb{D}}$ into M. Let $C(M)$ be the set of all continuous functions on M.

Let $K(z,M)$ be the set of all measures $\nu_{f,\alpha}$ on M such that for a function $\phi \in C(M)$

$$\nu_{f,\alpha}(\phi) = \frac{1}{2\pi i} \int_{|\zeta|=1} \phi(f(\zeta))\alpha(\zeta)\, d\zeta,$$

where $f \in \mathcal{H}(M)$, $f(0) = z$, and $\alpha \in \mathcal{H}(\mathbb{D})$.

The norm

$$\|\nu_{f,\alpha}\| = \sup\{|\nu_{f,\alpha}(\phi)| : |\phi| \le 1 \text{ on } M\}$$

of $\nu_{f,\alpha}$ does not exceed

$$\|\alpha\|_{H^1} = \frac{1}{2\pi} \int_0^{2\pi} |\alpha(e^{i\theta})|\, d\theta \le 1.$$

The following example shows that $\|\nu_{f,\alpha}\|$ can be strictly less than $\|\alpha\|_{H^1}$.

Example 2.1. Let $M = \mathbb{C}$, $\alpha \equiv 1$ and

$$f(\zeta) = \zeta \frac{\zeta + \overline{a}}{1 + a\zeta},$$

where $|a| < 1$. If $G_1(\xi)$ and $G_2(\xi)$ are two branches of the solution to the equation

$$(1 + a\zeta)\xi = \zeta(\zeta + \overline{a}),$$

then

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \phi(f(\zeta))\,d\zeta = \frac{1}{2\pi i}\int_{|\xi|=1} \phi(\xi)\big(G_1'(\xi) + G_2'(\xi)\big)\,d\xi.$$

But $G_1(\xi) + G_2(\xi) = a\xi - \overline{a}$ and, therefore,

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \phi(f(\zeta))\,d\zeta = \frac{a}{2\pi i}\int_{|\xi|=1} \phi(\xi)\,d\xi.$$

Thus, $\|\nu_{f,\alpha}\| = |a| < 1$. In particular, if $a = 0$ then $\nu_{f,1} = 0 = \nu_{0,1}$.

Thus, this example shows that the mapping $(f,\alpha) \to \nu_{f,\alpha}$ is not an injection and the support of $\nu_{f,\alpha}$ need not to coincide with $f(\mathbb{T})$ although $\operatorname{supp}\nu_{f,\alpha}$ always belongs to $f(\mathbb{T})$.

However, if $f \in \mathcal{H}(M)$ and there is a set $E \subset \mathbb{T}$ of full measure such that f is an injection on E, then, evidently, $\operatorname{supp}\nu_{f,\alpha} = f(\mathbb{T})$ and $\|\nu_{f,\alpha}\| = \|\alpha\|_{H^1}$. Clearly, the set of such f is dense in $\mathcal{H}(M)$.

Another example of measures generated by analytic disks $f \in \mathcal{H}(M)$ are measures

$$\mu_f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))\,d\theta.$$

We denote by $J(z,M)$ the set of all such measures with $f(0) = z$.

The following result was proved in [P1] and [R]. There $\mathrm{PSH}(M)$ stands for the set of all plurisubharmonic functions on M.

Theorem 2.2. *If ϕ is an upper semicontinuous function on M, then its envelope*

$$\mathcal{E}_M\phi(z) = \inf\{\mu(\phi) : \mu \in J(z,M)\}$$

is plurisubharmonic on M and coincides with

$$\sup\{v(z) : v \in \mathrm{PSH}(M), v \le \phi\}$$

for every $z \in M$.

Now we assume that M is a relatively compact domain in a complex manifold N. We denote by $K(z,\overline{M})$ the weak-$*$ closure of $K(z,M)$ in $C^*(\overline{M})$ and by $J(z,\overline{M})$ the weak-$*$ closure of $J(z,M)$ in $C^*(\overline{M})$. Let us remind that a Jensen measure on M with barycenters at $z \in M$ is a probability measure μ with compact support on M such that $\phi(z) \le \mu(\phi)$ for all $\phi \in \mathrm{PSH}(M)$. The following result was obtained in [BS] in the case when M is a domain in $\mathbb{C}^n$.

Theorem 2.3. *Suppose that M is a relatively compact domain in a complex manifold N. Then the set $J(z,\overline{M})$ is convex and the set of all Jensen measures on M is a subset of $J(z,\overline{M})$.*

Proof. To show that the set $J(z,\overline{M})$ is convex it suffices to show that for any $\mu_f,\mu_g \in J(z,M)$ and any a, $0 \le a \le 1$, the measure $a\mu_f + (1-a)\mu_g$ belongs to $J(z,\overline{M})$.

If N is a Stein manifold, then we denote by F an imbedding of N into $\mathbb{C}^m$. Let $\tilde{f} = F \circ f$ and $\tilde{g} = F \circ g$. By [GR, Theorem 8.C.8] there are a neighborhood W of $\tilde{f}(\overline{\mathbb{D}}) \cup \tilde{g}(\overline{\mathbb{D}})$ in $\mathbb{C}^m$ and a holomorphic retraction P of W onto $F(N)$. By [P2, Lemma 2.2], there is a sequence $\{p_j\} \subset \mathcal{H}(W)$ such that the measures μ_{p_j} converge weak-$*$ to $a\mu_{\tilde{f}} + (1-a)\mu_{\tilde{g}}$. Moreover, the sets $p_j(\overline{\mathbb{D}})$ lie in any neighborhood of $\tilde{f}(\overline{\mathbb{D}}) \cup \tilde{g}(\overline{\mathbb{D}})$ when j is sufficiently large. Therefore, if $\tilde{h}_j = P \circ p_j$ then $a\mu_{\tilde{f}} + (1-a)\mu_{\tilde{g}}$ is the weak-$*$ limit of $\mu_{\tilde{h}_j}$. Consequently, the measure $a\mu_f + (1-a)\mu_g$ is the weak-$*$ limit of μ_{h_j}, where $h_j = F^{-1} \circ \tilde{f}$, and, therefore, belongs to $J(z,\overline{M})$.

In the general case, since $f,g \in \mathcal{H}(M)$, there is $r > 1$ such that f and g are defined on the closure of the disk $\mathbb{D}_r$ of radius r centered at the origin. Let $M'_r = M \times \mathbb{D}_r$ and let $\tilde{f}(\zeta) = (f(\zeta),\zeta)$ and $\tilde{g}(\zeta) = (g(\zeta),\zeta)$ be the mappings of $\mathbb{D}_r$ into M'_r. Then the set $\tilde{f}(\mathbb{D}_r) \cup \tilde{g}(\mathbb{D}_r)$ is a complex subvariety in M'_r and, by Siu's theorem in [S], has a Stein neighborhood W. By the previous paragraph, there is a sequence $\{\tilde{h}_j\} \subset \mathcal{H}(W)$ such that the measures $\mu_{\tilde{h}_j}$ converge weak-$*$ to $a\mu_{\tilde{f}} + (1-a)\mu_{\tilde{g}}$. If $\tilde{h}_j(\zeta) = (h_j(\zeta), p_j(\zeta))$, $h_j(\zeta) \in M$ and $p_j(\zeta) \in \mathbb{D}_r$, then $a\mu_f + (1-a)\mu_g$ is the weak-$*$ limit of μ_{h_j}.

Thus, the set $J(z,\overline{M})$ is convex. Let us show that the set of all Jensen measures on M is a subset of the set $J(z,\overline{M})$. If not, there is a Jensen measure ν with barycenter at $z \in M$ which does not belong to $J(z,\overline{M})$. Then we can find a continuous function $\phi \in C(\overline{M})$ such that

$$\nu(\phi) < \inf\{\mu(\phi) : \mu \in J(z,\overline{M})\} = \inf\{\mu(\phi) : \mu \in J(z,M)\}.$$

Note that the function $v(w) = \mathcal{E}_M\phi(w) \le \phi(w)$ and is plurisubharmonic on M. Hence,

$$\nu(v) \le \nu(\phi) < v(z).$$

So we got a contradiction and $\nu \in J(z,\overline{M})$. □

We will denote by $A(\overline{M}) \subset C(\overline{M})$ the uniform algebra of functions continuous on $\overline{M}$ and holomorphic on M. Clearly, $K(z,\overline{M}) \subset A^{\perp}(\overline{M})$ while $J(z,\overline{M})$ consists of positive representing measures for $A(\overline{M})$ at z.

3. G^*- and P^*-transforms

Suppose that M is a relatively compact domain in a complex manifold N. Let $M' = M \times \mathbb{D}$. Consider a continuous linear operator $G : C(\overline{M}) \to C(\overline{M'})$ defined as $G\phi(z,\zeta) = \zeta\phi(z)$. Note that the range $\operatorname{ran} G$ of G is closed in $C(\overline{M'})$, $\ker G = \{0\}$ and $\|G\| = 1$.

Additionally, we consider a continuous linear operator $P : C(\overline{M'}) \to C(\overline{M})$ defined as

$$P\psi(z) = \frac{1}{2\pi i}\int_0^{2\pi} \psi(z, e^{i\theta})e^{-i\theta}\, d\theta.$$

Clearly, $\operatorname{ran} P = C(\overline{M})$, $\|P\| = 1$, $PG\phi = \phi$ and $\psi - GP\psi \in \ker P$. Thus, $C(\overline{M'}) = \operatorname{ran} G \oplus \ker P$ with the decomposition of $\psi \in C(\overline{M'})$ given by the formula

$$\psi = GP\psi + (\psi - GP\psi).$$

The G^**-transform* is the adjoint of G mapping $C^*(\overline{M'})$ into $C^*(\overline{M})$ and is uniquely defined by the formula

$$G^*\mu(\phi) = \mu(G\phi) \tag{3.1}$$

for $\phi \in C(\overline{M})$ and $\mu \in C^*(\overline{M'})$. Clearly, $\ker G^* = (\operatorname{ran} G)^{\perp}$ and, since $\ker G = \{0\}$ and $\operatorname{ran} G^*$ is weak-$*$ closed, $\operatorname{ran} G^* = C^*(\overline{M})$.

The P^**-transform* is the adjoint of P mapping $C^*(\overline{M})$ into $C^*(\overline{M'})$ and is uniquely defined by the formula

$$P^*\nu(\psi) = \nu(P\psi) \tag{3.2}$$

for $\psi \in C(\overline{M'})$ and $\nu \in C^*(\overline{M})$. Clearly, $\ker P^* = \{0\}$ and, since $\operatorname{ran} P^*$ is weak-$*$ closed, $\operatorname{ran} P^* = (\ker P)^{\perp}$. Moreover, $G^*P^*\nu = \nu$.

It follows that

$$C^*(\overline{M'}) = (\operatorname{ran} G)^{\perp} \oplus (\ker P)^{\perp} = \ker G^* \oplus \operatorname{ran} P^*$$

with the decomposition of $\mu \in C^*(\overline{M'})$ given by the formula

$$\mu = (\mu - P^*G^*\mu) + P^*G^*\mu.$$

By the definition of P^* if $\nu \in C^*(M)$, $\|\nu\| = 1$ and $|\nu(\phi)| = 1$ for $\phi \in C(M)$ with $\|\phi\| = 1$, then $|P^*\nu(\psi)| = 1$ for $\psi(z,\zeta) = \zeta\phi(z)$. Thus, P^* imbeds $C^*(M)$ isometrically into $C^*(\overline{M'})$.

Proposition 3.1. *For $z \in M$, the G^*-transform maps $J((z,0), M')$ onto $K(z,M)$ and $J((z,0),\overline{M'})$ onto $K(z,\overline{M})$.*

Proof. Suppose that $\nu = \nu_{f,\alpha} \in K(z,M)$, where $f \in \mathcal{H}(M)$ and $\alpha \in \mathcal{H}(\mathbb{D})$. Then the mapping $g : \overline{\mathbb{D}} \to M'$, $g(\zeta) = (f(\zeta), \zeta\alpha(\zeta))$ is in $\mathcal{H}(M')$. If $\phi \in C(\overline{M})$ and $\tilde{\phi} = G\phi$, then

$$\mu_g(\tilde{\phi}) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\tilde{\phi}(g(\zeta))}{\zeta}\, d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=1} \phi(f(\zeta))\alpha(\zeta)\, d\zeta = \nu_{f,\alpha}(\phi).$$

Thus, $\nu_{f,\alpha} = G^*\mu_g$. Hence, G^* maps $J((z,0), M')$ onto $K(z,M)$.

Now suppose that $\nu \in K(z,\overline{M})$ and measures $\nu_j \in K(z,M)$ converge weak-$*$ to ν. Take measures $\mu_j \in J((z,0), M')$ such that $G^*\mu_j = \nu_j$. Since the unit ball in $C^*(\overline{M'})$ is compact, we can take a subsequence μ_{j_k} converging weak-$*$

to a measure $\mu \in J((z,0),\overline{M'})$. Then $G^*\mu = \nu$ and we see that G^* maps $J((z,0),\overline{M'})$ onto $K(z,\overline{M})$. □

THEOREM 3.2. *The weak-$*$ closure $K(z,\overline{M})$ of $K(z,M)$ in the dual space $C^*(M)$ is convex and balanced.*

Proof. The convexity follows from the convexity of the set $J((z,0),\overline{M'})$ and the last statement of the previous proposition.

If $a \in \overline{\mathbb{D}}$, then $\nu_{f,a\alpha} = a\nu_{f,\alpha}$. Thus, the set $K(z,M)$ and, consequently, $K(z,\overline{M})$ are balanced. □

4. The algebra $A(\overline{M})$ and the space $S(\overline{M})$

Suppose that M is a relatively compact domain in a complex manifold N. Let us define an operator $\mathcal{L}$ on the $C(\overline{M})$ by

$$\mathcal{L}\phi(z) = \inf\{\mathbf{Re}\,\nu(\phi) : \nu \in K(z,M)\}$$

for any point $z \in M$.

PROPOSITION 4.1. *The operator $\mathcal{L}$ is a continuous mapping of $C(\overline{M})$ into the cone of non-positive plurisubharmonic functions equipped with the uniform norm and it has the following properties*:

(1) $-\|\phi\| \le \mathcal{L}\phi \le 0$;
(2) $\mathcal{L}(\alpha\phi) = |\alpha|\mathcal{L}\phi$ *when* $\alpha \in \mathbb{C}$;
(3) $\mathcal{L}(\phi_1 + \phi_2) \ge \mathcal{L}(\phi_1) + \mathcal{L}(\phi_2)$;
(4) $\mathcal{L}\phi(z_0) = 0$ *for some* $z_0 \in M$ *if and only if* $\phi \in A(M)$.

Proof. By Proposition 3.1 for every $\nu \in K(z,M)$, there is a measure $\mu \in J((z,0),M')$ such that $G^*\mu = \nu$. In view of (3.1),

$$\mathbf{Re}\,\nu(\phi) = \mathbf{Re}\,G^*\mu(\phi) = \mathbf{Re}\,\mu(G\phi) = \mu(\mathbf{Re}\,G\phi).$$

Since $G^*(J((z,0),M')) = K(z,M)$ we see that

$$\mathcal{L}\phi(z) = \mathcal{E}_{M'}\,\mathbf{Re}\,G\phi(z,0)$$

for every $z \in M$. By Theorem 2.2, $\mathcal{L}\phi$ is plurisubharmonic on M. The continuity of $\mathcal{L}$ is evident.

Since the set $K(z,M)$ is balanced and lies in the unit ball of $C^*(\overline{M})$, (1) follows. Properties (2) and (3) are evident.

In (4), the part "if" is evident. To show "only if," we note that since $\mathcal{L}\phi$ is plurisubharmonic and nonpositive it follows from the maximum principle that $\mathcal{L}\phi(z) \equiv 0$ on M. By Morera's Theorem, ϕ is holomorphic on every complex line and, consequently, holomorphic on M. □

We denote by $C(z,M)$ the real positive cone over $K(z,M)$ and let $\overline{C}(z,M)$ be its weak-$*$ closure in $C^*(\overline{M})$. Since $K(z,\overline{M})$ is convex and balanced, the set $\overline{C}(z,M)$ is a weak-$*$ closed complex linear subspace of $C^*(\overline{M})$. The following theorem and its corollaries describes the space $A^\perp(\overline{M})$ in terms of analytic disks.

THEOREM 4.2. *Suppose that M is a relatively compact domain in a complex manifold N and $z_0 \in M$. Then*

$$A^\perp(\overline{M}) = \overline{C}(z_0, M).$$

Proof. If a function $\phi \in A(\overline{M})$, then $\mu(\phi) = 0$ for every $\mu \in K(z_0, M)$. Therefore, $\mu(\phi) = 0$ for every $\mu \in \overline{C}(z_0, M)$. Thus, $\overline{C}(z_0, M) \subset A^\perp(\overline{M})$.

If $\mu \in A^\perp(\overline{M})$ and $\mu \notin \overline{C}(z_0, M)$, then we take a function $\phi \in C(\overline{M})$ such that $\mu(\phi) = 1$ while $\nu(\phi) = 0$ for all $\nu \in \overline{C}(z_0, M)$. By property (4) of Proposition 4.1, $\phi \in A(\overline{M})$. Thus, $\mu(\phi) = 0$ and this contradiction proves the theorem. □

As an immediate corollary of this theorem, we see that to check whether is holomorphic or not it suffices to test it on the set $K(z_0, M)$ for only one point $z_0 \in M$.

COROLLARY 4.3. *Suppose that M is a relatively compact domain in a complex manifold N and $z_0 \in M$. Then a continuous function $\phi \in A(\overline{M})$ if and only if $\mu(\phi) = 0$ for every $\mu \in K(z_0, M)$.*

In its turn, the result above produces a version of F. and M. Riesz Theorem.

COROLLARY 4.4. *A continuous function $\phi \in A(\overline{M})$ if and only if there exists a point $z_0 \in M$ so that*

$$\int_{|\zeta|=1} \phi(f(\zeta))\zeta^k \, d\zeta = 0$$

for every integer $k \geq 0$ and every mapping $f \in \mathcal{H}(M)$ with $f(0) = z_0$.

Proof. If $\alpha \in \mathcal{H}(\mathbb{D})$, then $\alpha(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$. The result is then a direct consequence of Corollary 4.3. □

To understand how operators G^* and P^* work on the space $A^\perp(\overline{M})$, we introduce the space $S(\overline{M'})$ which consists of all continuous complex-valued functions $u(z, \zeta)$ on $\overline{M'}$ such that the function

$$Pu(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z, e^{i\theta}) e^{-i\theta} \, d\theta$$

is holomorphic on M. Clearly, $Pu \in A(\overline{M})$ and $S(\overline{M'})$ is closed in $C(\overline{M'})$.

The following theorem shows the relationship between spaces $A^\perp(\overline{M})$ and $S^\perp(\overline{M'})$.

THEOREM 4.5. *The space $P^*(A^\perp(\overline{M})) = S^\perp(\overline{M'})$ while $G^*(S^\perp(\overline{M'})) = A^\perp(\overline{M})$. Moreover, $P^* G^* \mu = \mu$ when $\mu \in S^\perp(\overline{M'})$.*

Proof. If $u\in S(\overline{M'})$ then the function $Pu\in A(\overline{M})$. Hence, if $\nu\in A^{\perp}(\overline{M})$ then $P^*\nu(u)=\nu(Pu)=0$ and $P^*(A^{\perp}(\overline{M}))\subset S^{\perp}(\overline{M'})$.

If $\nu\in S^{\perp}(\overline{M'})$ and not in $P^*(A^{\perp}(\overline{M}))$, then there is $u\in C(\overline{M'})$ such that

$$\nu(u)<0\le\mu(u)\quad\text{for all } \mu=P^*\mu_1,\mu_1\in A^{\perp}(\overline{M}).$$

This tells us that $P^*\mu_1(u)=0$ for all $\mu_1\in A^{\perp}(\overline{M})$. But $P^*\mu_1(u)=\mu_1(Pu)$ and, therefore, $Pu\in A(\overline{M})$ and $u\in S(\overline{M'})$. Hence, $\nu(u)=0$ and we get a contradiction. Thus, $P^*(A^{\perp}(\overline{M}))=S^{\perp}(\overline{M'})$.

Now if $\mu\in S^{\perp}(\overline{M'})$ and $\phi\in A(\overline{M})$, then $G\phi\in S(\overline{M'})$ and, therefore, $G^*\mu(\phi)=0$. Thus $G^*(S^{\perp}(\overline{M'}))\subset A^{\perp}(\overline{M})$. But if $\nu\in A^{\perp}(\overline{M})$, then $P^*\nu\in S^{\perp}(\overline{M'})$. Since $G^*P^*\nu=\nu$ we see that $G^*(S^{\perp}(\overline{M'}))=A^{\perp}(\overline{M})$.

If $\mu\in S^{\perp}(\overline{M'})$, then there is $\nu\in A^{\perp}(\overline{M})$ such that $P^*\nu=\mu$. Hence,

$$P^*G^*\mu=P^*G^*P^*\nu=P^*\nu=\mu. \qquad\square$$

5. The space of pluriharmonic functions

Another space closely related to complex analysis is the space $h(\overline{M})$ of all continuous complex-valued functions u on $\overline{M}$ such that $\mathbf{Re}\,u$ and $\mathbf{Im}\,u$ are pluriharmonic on M. As we show below, the annihilators to this space also can described in terms of Jensen measures.

We will need a lemma.

LEMMA 5.1. *A real valued function $\phi\in C(\overline{M})$ is pluriharmonic if and only if $\mathcal{E}_M\phi(z_0)+\mathcal{E}_M(-\phi)(z_0)=0$ for some point $z_0\in M$.*

Proof. If ϕ is pluriharmonic, then $\mathcal{E}_M\phi\equiv\phi$ and

$$\mathcal{E}_M\phi(z_0)+\mathcal{E}_M(-\phi)(z_0)\equiv 0.$$

By the properties of envelopes, $\mathcal{E}_M\phi\le\phi$ on M. Hence, $\mathcal{E}_M\phi+\mathcal{E}_M(-\phi)\le\phi-\phi=0$ and for any $\phi\in C(\overline{M})$ the function $E=\mathcal{E}_M\phi+\mathcal{E}_M(-\phi)$ is plurisubharmonic and nonpositive on M. So if $E(z_0)=0$ for some point $z_0\in M$, then $E\equiv 0$. Consequently, $\mathcal{E}_M\phi=\phi$ and $\mathcal{E}_M(-\phi)=-\phi$. Thus, both ϕ and $-\phi$ are plurisubharmonic and this implies that ϕ is pluriharmonic. $\square$

Now we can describe the space $h^{\perp}(\overline{M})$ in terms of Jensen measures. If z is any point in M, let

$$L(z,M)=\{a(\mu-\nu):a\in\mathbb{R},\mu,\nu\in J(z,M)\}$$

and let $\overline{L(z,M)}$ be its weak-$*$ closure.

THEOREM 5.2. *The space $h^{\perp}(\overline{M})=\overline{L(z,M)}\oplus i\overline{L(z,M)}$ for every $z\in M$.*

Proof. Clearly, $\overline{L(z,M)}\oplus i\overline{L(z,M)}\subset h^{\perp}(\overline{M})$. To show the reverse inclusion, we note that if $\lambda\in h^{\perp}(\overline{M})$ then both its real and imaginary parts are in

$h^\perp(\overline{M})$. So we may assume that λ is real. If $\lambda \notin \overline{L(z,M)}$, then there exists a real function $\phi \in C(\overline{M})$ such that

$$\lambda(\phi) < \inf\{a(\mu(\phi) - \nu(\phi)), a \in \mathbb{R}, \mu, \nu \in J(z,M)\}.$$

It follows that $\mu(\phi) - \nu(\phi) = 0$ for every $\mu, \nu \in J(z,M)$. Hence, $\mathcal{E}_M\phi(z) + \mathcal{E}_M(-\phi)(z) = 0$ and by Lemma 5.1 the function $\phi \in h(\overline{M})$. But then $\lambda(\phi) = 0$ and, by contradiction, $\lambda \in \overline{L(z,M)}$. □

The following theorem describes the actions of P^* and G^* on annihilators $A^\perp(\overline{M}))$ and $h^\perp(\overline{M'})$.

Theorem 5.3. *The space* $P^*(A^\perp(\overline{M})) \subset h^\perp(\overline{M'})$ *while* $G^*(h^\perp(\overline{M'})) = A^\perp(\overline{M})$.

Proof. If $u = u_1 + iu_2 \in h(\overline{M'})$, then u is harmonic in ζ. Therefore,

$$Pu(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z,e^{i\theta})e^{-i\theta}\,d\theta = \frac{1}{2\pi i}\int_0^{2\pi} u(z,e^{i\theta})\,d\overline{\zeta} = \frac{\partial u}{\partial \zeta}(z,0).$$

Since u is pluriharmonic, $Pu \in A(\overline{M})$ and we see that $h(\overline{M'}) \subset S(\overline{M'})$. By Theorem 4.5 $P^*(A^\perp(\overline{M})) \subset h^\perp(\overline{M'})$.

By the same theorem, $A^\perp(\overline{M}) \subset G^*(h^\perp(\overline{M'}))$. Now if $\mu \in h^\perp(\overline{M'})$ and $\phi \in A(\overline{M})$, then $G\phi \in h(\overline{M'})$ and, therefore, $G^*\mu(\phi) = 0$. Thus, $G^*(h^\perp(\overline{M'})) \subset A^\perp(\overline{M})$. □

6. Mergelyan property

Suppose that we assign to any relatively compact domain M in a complex manifold N a uniformly closed subspace $X(M) \subset C(M)$ such that the restrictions of functions from $X(M_1)$ to M_2 belong to $X(M_2)$ when $M_2 \subset M_1$. Let us define $X_o(\overline{M})$ as the uniform closure of the union of the restrictions to $\overline{M}$ of the spaces $X(V)$, where V runs over all domains containing $\overline{M}$ in its interior. We also let $X(\overline{M})$ be the space of all functions in $C(\overline{M})$ whose restrictions to M belong to $X(M)$. We say that a relatively compact domain M has the *Mergelyan property* with respect to the spaces $X(M)$ if $X(\overline{M}) = X_o(\overline{M})$ or, equivalently, $X^\perp(\overline{M}) = X_o^\perp(\overline{M})$. Clearly, $X_o(\overline{M}) \subset X(\overline{M})$ and $X^\perp(\overline{M}) \subset X_o^\perp(\overline{M})$.

Let us fix a relatively compact domain W in N. If the closure of a domain $V \subset N$ belongs to W, then there is a natural isometric imbedding of $C^*(\overline{V})$ into $C^*(\overline{W})$. In the sequel, we will identify $C^*(\overline{V})$ with its imbedding into $C^*(\overline{W})$. It is easy to see that operators G^* and P^* defined for W map, respectively, $C^*(\overline{V'})$ into $C^*(\overline{V})$ and $C^*(\overline{V})$ into $C^*(\overline{V'})$. Therefore, in the future we need not to specify for which domain operators G^* and P^* are defined.

Let $h_o^0(M')$ be the space of all pluriharmonic functions $u(z,\zeta)$ on M' such that the function $u(z,0)=0$ on M. The following theorem relates the Mergelyan property for different function spaces.

THEOREM 6.1. *Let M be a relatively compact domain in a complex manifold N. Consider the following Mergelyan properties of function spaces*:

(1) $A_o(\overline{M})=A(\overline{M})$;
(2) $S_o(\overline{M'})=S(\overline{M'})$;
(3) $h_o(\overline{M'})=h(\overline{M'})$;
(4) $h_o^0(\overline{M'})=h^0(\overline{M'})$.

Then (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (1), *and* (1) $\Rightarrow$ (4).

Proof. Suppose that M is the intersection of a decreasing sequence of open sets $\{V_j\}$ lying in $1/j$-neighborhoods of $\overline{M}$. First, we show that (1) $\Rightarrow$ (2). By Theorem 4.5, $P^*(A^\perp(\overline{V}_j))=S^\perp(\overline{V}_j)$. Hence,

$$S^\perp(\overline{M})\subset S_o^\perp(\overline{M})=\bigcap_{j=1}^\infty S^\perp(\overline{V}_j)=\bigcap_{j=1}^\infty P^*(A^\perp(\overline{V}_j)).$$

If $\mu\in\bigcap_{j=1}^\infty P^*(A^\perp(\overline{V}_j))$, then for all j we can find $\nu_j\in A^\perp(\overline{V}_j)$ such that $\mu=P^*\nu_j$. Since P^* is an isometry, $\|\nu_j\|=\|\mu\|$. Hence, there is a subsequence ν_{j_k} weak-$*$ converging to ν. Since the spaces $A^\perp(\overline{V}_j)$ are closed and $A^\perp(\overline{V}_j)\subset A^\perp(\overline{V}_k)$ when $j>k$, we see that $\nu\in\bigcap_{j=1}^\infty A^\perp(\overline{V}_j)=A_o^\perp(\overline{M})$ and $P^*\nu=\mu$. Hence, $\bigcap_{j=1}^\infty P^*(A^\perp(\overline{V}_j))\subset P^*(A_o^\perp(\overline{M}))=S^\perp(\overline{M})$ and, consequently, $S_o(\overline{M'})=S(\overline{M'})$.

The proof that (2) $\Rightarrow$ (1) follows the same lines. We use the second part of Theorem 4.5 stating that $G^*(S^\perp(\overline{M'}))=A^\perp(\overline{M})$ and the fact that G^* is an isometry on $S^\perp(\overline{M'})$.

Let us show that (3) $\Rightarrow$ (1). Clearly, $A_o(\overline{M})\subset A(\overline{M})$. Suppose that there is $\phi\in A(\overline{M})$ which does not belong to $A_o(\overline{M})$. Since $A_o(\overline{M})$ is a complex linear space, by the complex version of the Hahn–Banach theorem there is $\nu\in C^*(\overline{M})$ such that $\nu\in A_o^\perp(\overline{M})$ and $\nu(\phi)\neq 0$. Consequently, $\nu\in A^\perp(\overline{V})$ for every relatively compact set $V\subset N$ containing $\overline{M}$.

By Lemma 5.3, $P^*\nu\in h^\perp(\overline{V'})$ and, consequently, $P^*\nu\in h_o^\perp(\overline{M'})$. Since the last space is equal to $h^\perp(\overline{M'})$, we see that $P^*\nu\in h^\perp(\overline{M'})$ and, again by Lemma 5.3, $\nu=G^*P^*\nu\in A^\perp(\overline{M})$. Hence, $\nu(\phi)=0$ and we got a contradiction.

Now we show that (1) $\Rightarrow$ (4). Let $u(z,\zeta)$, $z\in\overline{M}$ and $\zeta\in\mathbb{D}$, be a function in $h^0(\overline{M'})$. Fix $\varepsilon>0$ and find numbers $r<s<1$ and the function $u_1(z,\zeta)=u(z,r\zeta)$ on the closure $\overline{M'_s}$ of $M'_s=\overline{M}\times\mathbb{D}_{1/s}$ such that $\|u-u_1\|_{\overline{M'}}<\varepsilon$.

Letting $\zeta=\alpha+i\beta$, $\alpha,\beta\in\mathbb{R}$, we define the function v on $\overline{M'_r}$ as

$$v_1(z,\zeta)=\int_0^\zeta\left(-\frac{\partial u_1}{\partial\beta}\,d\alpha+\frac{\partial u_1}{\partial\alpha}\,d\beta\right).$$

As the path of integration, we can take the segment $[0,\zeta]$. Clearly, v_1 is continuous on $\overline{M'_s}$, real analytic on M'_s, $v_1(z,0)=0$ and for every $z\in M'$ the function $v_1(z,\zeta)$ is the harmonic conjugate of $u_1(z,\zeta)$. Thus, the function $f(z,\zeta)=u_1(z,\zeta)+iv_1(z,\zeta)$ is continuous on $\overline{M'_s}$ and holomorphic in ζ, so we can write

$$f(z,\zeta)=\sum_{j=0}^{\infty}a_j(z)\zeta^j \quad \text{and} \quad u_1(z,\zeta)=\frac{1}{2}\sum_{j=0}^{\infty}\bigl(a_j(z)\zeta^j+\overline{a}_j(z)\overline{\zeta}^j\bigr).$$

The coefficients

$$a_j(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(z,\zeta)}{\zeta^{j+1}}\,d\zeta=\frac{1}{2j!}\frac{\partial^j u_1}{\partial\zeta^j}(z,0)$$

are continuous on $\overline{M}$ and, by pluriharmonicity of u_1, holomorphic on M when $j\geq 1$. Note that $a_0(z)=f(z,0)=u(z,0)=0$ on M.

If $A=\|u\|_{\overline{M}'}$, then $\|a_j(z)\|_{\overline{M}}\leq Ar^{-j}$. This implies that the series

$$\frac{1}{2}\sum_{j=0}^{\infty}\bigl(a_j(z)\zeta^j+\overline{a}_j(z)\overline{\zeta}^j\bigr)$$

converges to u_1 uniformly on $\overline{M}'_s$. Therefore, we can find k such that

$$\left|u_1(z,\zeta)-\frac{1}{2}\sum_{j=0}^{k}\bigl(a_j(z)\zeta^j+\overline{a}_j(z)\overline{\zeta}^j\bigr)\right|<\varepsilon$$

on $\overline{M}'_s$. We let $b_0=a_0\equiv 0$ and for every $1\leq j\leq k$ we can find a neighborhood V_j of $\overline{M}$ and a function $b_j\in A(\overline{V}_j)$ such that

$$\left|\frac{1}{2}\sum_{j=0}^{k}\bigl(b_j(z)\zeta^j+\overline{b}_j(z)\overline{\zeta}^j\bigr)-\frac{1}{2}\sum_{j=0}^{k}\bigl(a_j(z)\zeta^j+\overline{a}_j(z)\overline{\zeta}^j\bigr)\right|<\varepsilon$$

on $\overline{M}'_s$. If

$$u_2(z,\zeta)=\frac{1}{2}\sum_{j=0}^{k}\bigl(b_j(z)\zeta^j+\overline{b}_j(z)\overline{\zeta}^j\bigr)$$

and $V=\bigcap_{j=1}^{k}V_j$, then $u_2\in h^0(\overline{V'})$ and $\|u_2-u\|_{\overline{M'}}<3\varepsilon$. Hence, $h^0_o(\overline{M'})=h^0(\overline{M'})$. ☐

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Nihat Gogus, Faculty of Engineering and Natural Sciences, Sabanci University, Orhanli, Tuzla, 34956 Istanbul, Turkey

E-mail address: nggogus@sabanciuniv.edu

Evgeny A. Poletsky, Department of Mathematics, Syracuse University, 215 Carnegie Hall, Syracuse, NY 13244-1150, USA

E-mail address: eapolets@syr.edu