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REAL MEROMORPHIC FUNCTIONS AND A RESULT OF HINKKANEN AND ROSSI

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ABSTRACT. Let f be a transcendental meromorphic function such that all but finitely many of the poles of f and zeroes of f' are real. Generalising a result of Hinkkanen and Rossi (*Proc. Amer. Math. Soc.* **92** (1984) 72–74), we characterize those f such that f' takes some nonzero value only finitely often, and show that all but finitely many of the zeroes of f'' are real in this case. We also prove a related asymptotic result about real meromorphic functions with a nonzero deficient value α and only finitely many nonreal zeroes, poles and α-points.

1. Introduction

The starting point for this paper is the following theorem of Hinkkanen and Rossi [12]. Here and henceforth, meromorphic should be taken to mean meromorphic in the plane unless stated otherwise.

THEOREM A ([12]). *Suppose that f is a nonentire real transcendental meromorphic function with only real poles, and that the zeroes of f and f' are real. If f' omits a nonzero value α, then the omitted value is real and*

$$f(z) = \alpha z - \lambda \tan(cz + d) + A, \tag{1.1}$$

where λ, c, d and A are real and $\lambda, c \neq 0$. Furthermore, the zeroes of f'' are real.

A meromorphic function f is said to be real if $f(z)$ is real or infinite whenever z is real. Theorem A arose from an endeavour to determine all meromorphic functions f with only real poles for which f, f' and f'' each have only real zeroes. Hellerstein, Shen and Williamson ([7], [8], [9]) settled this question for all entire functions and for those meromorphic functions that are

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not a constant multiple of a real function. Further, they proved in [10] that Theorem A holds in the $\alpha = 0$ case, under the additional assumptions that f has at least one zero and f'' has only real zeroes.

We aim to generalize Theorem A by considering functions with arbitrary zeroes and finitely many nonreal poles and critical points. In addition, the derivative must either take some nonzero value only finitely often, or at least have a nonzero finite deficient value. The notation and techniques of the value distribution theory of meromorphic functions [5] will be used throughout this paper.

The first result below characterizes all real functions that fail to satisfy Hinkkanen and Rossi's hypothesis at finitely many points. In this case, the restriction on the zeroes of f is shown to be a consequence rather than a prerequisite.

THEOREM 1. *Suppose that f is a real transcendental meromorphic function such that all but finitely many of the zeroes and poles of f' are real, and $f'(z) = \alpha$ only finitely often for some finite nonzero α. Then f can be written in the form*

$$f(z) = \alpha z + i\lambda \frac{P(z)e^{icz} - \overline{P(\overline{z})}e^{-icz}}{P(z)e^{icz} + \overline{P(\overline{z})}e^{-icz}} + A, \tag{1.2}$$

where α, λ and A are real constants, $\alpha\lambda \neq 0$, $c > 0$ and P is a polynomial with zeroes $a_1, \ldots, a_N$ (repeated to multiplicity) such that $a_j \neq \overline{a_k}$.

In the converse direction, if f is given by (1.2) then all but finitely many of the zeroes and poles of f and f'' are real and the equation $f'(z) = \alpha$ has at most $2N$ solutions, counting with multiplicities. Moreover, all but finitely many of the zeroes of f' are real if and only if either $0 < \lambda c/\alpha < 1$ or

$$\lambda c = \alpha \quad \textit{and} \quad \sum_{j=1}^{N} \frac{\operatorname{Im} a_j}{|x - a_j|^2} < 0 \quad \textit{as real } x \to \pm\infty. \tag{1.3}$$

Note that if $\lambda c = \alpha$ then the condition (1.3) is satisfied if $\sum \operatorname{Im} a_j < 0$ and is not satisfied if $\sum \operatorname{Im} a_j > 0$.

Before proceeding, we shall briefly consider some examples. If we take $P(z) \equiv e^{id}$, then we see that (1.2) simply reduces to (1.1). Choosing instead $P(z) = z + i$ and $c = 1$ gives

$$f(z) = \alpha z + \lambda \frac{z \sin z + \cos z}{\sin z - z \cos z} + A, \qquad f'(z) = \alpha - \lambda \frac{z^2}{(\sin z - z \cos z)^2}.$$

In this case, the derivative omits α, showing that the relevant part of Theorem 1 cannot be changed to "$f'(z) = \alpha$ has $2N$ solutions."

Kohs and Williamson proved in [14] that Hinkkanen and Rossi's Theorem A essentially continues to hold without the demand that f is real and transcendental. By an extension of the method of Kohs and Williamson, we show that in the statement of Theorem 1 we may replace the assumption that the function is real by the condition that it has infinitely many poles.

THEOREM 2. *Let g be a transcendental meromorphic function such that all but finitely many of the zeroes and poles of g' are real, and $g'(z)=\beta$ only finitely often for some finite nonzero β. Then all but finitely many of the zeroes of g'' are real and either*

(i) *there exists a constant c_0 such that $f=g/\beta+c_0$ is a real function satisfying the hypothesis of Theorem 1 with $\alpha=1$; or*
(ii) *we have $g(z)=R(z)e^{icz}+\beta z+d$, where R is rational, c and d are constants and c is real.*

The following example demonstrates that case (ii) can occur, and hence also that Theorem 1 may fail for strictly nonreal functions with finitely many poles. Let α be nonzero and take

$$g(z)=\alpha z+\frac{3-iz}{z-i}\alpha e^{iz}, \qquad g'(z)=\alpha+\left(\frac{z+i}{z-i}\right)^2\alpha e^{iz}.$$

Then g has only one pole and clearly cannot be written in the form (1.2). However, the derivative only takes the value α at one point and has finitely many nonreal zeroes. To establish this last claim, write

$$\frac{(z-i)^2g'(z)}{\alpha e^{iz/2}}=(z-i)^2e^{-iz/2}+(z+i)^2e^{iz/2}.$$

It will be shown in Lemma 6 that functions of this form have only finitely many nonreal zeroes.

We now weaken the hypotheses of Theorems A and 1 by allowing $f'(z)=\alpha$ infinitely often, and just requiring α to be a deficient value of f'. Under these conditions, f has the same asymptotic behavior away from the real axis as was found in the two earlier theorems.

THEOREM 3. *Let f be a real transcendental meromorphic function of positive lower order. Assume that f' has a nonzero finite deficient value α and that all but finitely many of the poles of f, and the zeroes and α-points of f', are real. Then α is real and, for $\varepsilon>0$,*

$$f(z)\sim\alpha z \quad \textit{as } z\to\infty \textit{ with } \varepsilon<|\arg z|<\pi-\varepsilon.$$

An example of such a function where the derivative does have infinitely many α-points is given by

$$h(z)=\frac{1}{3}\tan^3 z-3\tan z+4z, \qquad h'(z)=(\tan^2 z-1)^2.$$

Observe that h has only real poles and the derivative has only real zeroes. Since $\tan^2 z$ omits -1, we see that $h'(z)=4$ if and only if $\tan z=\pm\sqrt{3}$. As all the zeroes of $\tan z\pm\sqrt{3}$ are real and simple it follows that $h'(z)=4$ only for real z, and that 4 is a deficient value of h' with $\delta(4,h')=\frac{1}{2}$.

Theorem 3 follows from our final result.

THEOREM 4. *Let g be a real transcendental meromorphic function of positive lower order. Assume that g has a nonzero real finite deficient value α and that all but finitely many of the zeroes, poles and α-points of g are real. Then, for $\varepsilon > 0$,*

$$g(z) \sim \alpha \quad \text{as } z \to \infty \text{ with } \varepsilon < |\arg z| < \pi - \varepsilon.$$

REMARKS. We mention the following points for completeness.

(1) If instead α is nonreal, then the function g can only take the values α and $\overline{\alpha}$ finitely often. In this case, $(g-\alpha)/(g-\overline{\alpha})$ has finitely many zeroes and poles and it can be shown that

$$g(z) = \frac{\alpha P(z)e^{icz} + \overline{\alpha P(\overline{z})}e^{-icz}}{P(z)e^{icz} + \overline{P(\overline{z})}e^{-icz}},$$

where c is real and P is a polynomial.

(2) For any g with zero lower order and a deficient value α, there exist by [3] a positive constant d and a set of radii r with upper logarithmic density one such that $\log|g(re^{i\theta}) - \alpha| < -dT(r,g)$. That is, $g(z) \sim \alpha$ on whole circles of suitable radius.

To deduce Theorem 3, first note that if α were nonreal then the real function f' could have no real α-points. Hence, the transcendental derivative f' would take the values α and $\overline{\alpha}$ only finitely often, but this is not possible [5, p. 59]. Using the fact that f and f' have equal lower order [6], Theorem 3 is now established by applying Theorem 4 to f' and integrating.

2. Preliminaries

The following lemma is contained in a more general result due to Edrei [1].

LEMMA 1 ([1]). *Let f be meromorphic with only finitely many nonreal zeroes and poles and only finitely many nonreal roots of $f^{(n)}(z) = \alpha$, for some $n \geq 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$. If*

$$\delta(0,f) + \delta(\infty,f) + \delta\big(\alpha, f^{(n)}\big) > 0$$

then the order of f does not exceed one.

Hille's method. The proof of Theorem 1 involves studying the solutions of differential equations of the form

$$w'' + b(z)w = 0 \tag{2.1}$$

where $b(z)$ is a rational function. Hille's method [11, Section 5.6] can be used to give an asymptotic description of these solutions if $b(z) \sim dz^n$ as $z \to \infty$ where $n \geq -1$. We shall only consider the $n = 0$ case, so that as $z \to \infty$ we have $b(z) = d + O(|z|^{-1})$ for a nonzero constant d.

The *critical rays* are defined to be those rays $\arg z = \theta$ for which either $\theta = -(\arg d)/2$ or $\theta = \pi - (\arg d)/2$. Now assume that $\arg z = \theta_0$ is a critical ray, let $\delta > 0$ and let R_1 be large and positive. Define the region

$$S_1 = \{z : |z| > R_1,\ |\arg z - \theta_0| < \pi - \delta\}$$

and the transformation

$$Z = \int_{R_1 e^{i\theta_0}}^{z} b(t)^{1/2}\,dt = d^{1/2} z + O(\log|z|), \quad z \in S_1,\ z \to \infty.$$

By Hille's method, there then exist principal solutions $u_+(z)$ and $u_-(z)$ of (2.1) on S_1 given by

$$u_\pm(z) = b(z)^{-1/4} \exp\bigl(\pm iZ + o(1)\bigr).$$

These principal solutions are analytic in S_1 and have no zeroes there. However, any linear combination $\mu u_+ + \nu u_-$, where μ and ν are nonzero constants, has infinitely many zeroes near the critical ray $\arg z = \theta_0$.

3. Proof of Theorem 1—Part one

Let f be a real transcendental meromorphic function such that all but finitely many of the zeroes and poles of f' are real, and $f'(z) = \alpha$ only finitely often for some finite nonzero α. This section is devoted to proving that f can be written in the form (1.2) with α, λ and A real, $\lambda \neq 0$, $c > 0$ and P a polynomial without a pair of complex conjugate roots.

It is immediate that α is real, since otherwise the real transcendental derivative f' only takes the values α and $\overline{\alpha}$ finitely often. Let

$$H(z) = f(z) - \alpha z \tag{3.1}$$

and note that by Lemma 1 the order of H satisfies $\rho(H) = \rho(f') \leq 1$.

Our aim is to write f in the form (1.2) by expressing H as a quotient of solutions to the differential equation (2.1), in which the function $b(z)$ is equal to half the Schwarzian derivative of H.

Lemma 2. *The Schwarzian derivative*

$$S(H) = \frac{H'''}{H'} - \frac{3}{2}\left(\frac{H''}{H'}\right)^2$$

is rational.

Proof. Since H has finite order the lemma of the logarithmic derivative gives that $m(r, S(H)) = O(\log r)$. Recall that the Schwarzian derivative $S(H)$ has poles only at the multiple points of H. Therefore, to show that $S(H)$ is rational we shall show that H has only finitely many multiple points. As $H' = f' - \alpha$ has finitely many zeroes, our task is reduced to showing that H has only finitely many multiple poles.

Define the real function $g(z)$ by

$$f' = \alpha + 1/g. \tag{3.2}$$

Denote by $a_1, \dots, a_N$ the poles of g and by $b_1, \dots, b_M$ and $c_1, c_2, \dots,$ respectively the nonreal and real zeroes of $g + 1/\alpha$, all repeated according to multiplicity. The sequence c_n must be infinite as f' cannot take the values 0 and α both only finitely often. Lemma 1 gives $\rho(g) = \rho(f') \leq 1$, so that we have the Weierstrass product representation [5, p. 21]

$$g(z) + \frac{1}{\alpha} = z^p e^{az+b} \frac{\prod_{n=1}^{M}(z - b_n)}{\prod_{n=1}^{N}(z - a_n)} \prod_{\substack{n=1 \\ c_n \neq 0}}^{\infty} \left(1 - \frac{z}{c_n}\right) e^{z/c_n}$$

for some real constants a and b, and $p = \#\{n : c_n = 0\}$. We calculate

$$\left(\frac{g'}{g + 1/\alpha}\right)' = \sum_{n=1}^{N} \frac{1}{(z - a_n)^2} - \sum_{n=1}^{M} \frac{1}{(z - b_n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z - c_n)^2}. \tag{3.3}$$

If z is restricted to real values with $|z|$ large, then the expression in (3.3) is negative, as can be seen by truncating the infinite sum to a large number of terms.

By (3.1) and (3.2), the multiple poles of H correspond to zeroes of g of order greater than 2. At these zeroes, the left-hand side of (3.3) vanishes, and hence there can only be finitely many of them on the real axis. Since H has only finitely many nonreal poles, this completes the proof. □

Let

$$b(z) = \frac{1}{2} S(H)(z). \tag{3.4}$$

Theorem 6.1 of [15] states that if $D \subseteq \mathbb{C}$ is a simply-connected domain on which b is analytic, then (2.1) has two linearly independent analytic solutions w_1, w_2 on D such that $H = w_1/w_2$ there. We may assume that these solutions are normalized by $w_1 w_2' - w_1' w_2 = 1$. It follows that, on D,

$$H' = \frac{-1}{w_2^2}, \qquad \frac{H'}{H} = \frac{-1}{w_1 w_2}, \qquad \frac{H'}{H^2} = \frac{-1}{w_1^2},$$

and therefore w_1^2, $w_1 w_2$ and w_2^2 all have meromorphic extensions to the complex plane. Hence, if v_1, v_2 are any solutions of (2.1) on D then v_1^2, $v_1 v_2$ and v_1/v_2 each extend meromorphically to the whole complex plane. Furthermore, these extensions have order at most one, and v_1^2 has poles only at the (finitely many) poles of b. The latter claim can be proved by noting that v_1^2 is a solution of $4b(z)w^2 + 2ww'' - (w')^2 = 0$.

It is through studying equation (2.1) and its solutions that we will be able to express $f = H + \alpha z$ in the form (1.2).

Lemma 3. *The rational function $b(z)$ has a nonzero real value at infinity.*

Proof. That $b(z)$ is rational and real follows from (3.4). Moreover, $b(z)$ does not vanish identically because H is not a Möbius map. Hence, we must show that $b(\infty) \neq 0, \infty$. As the order of H does not exceed one, Corollary 1 of [4] gives a ray on which

$$\left|\frac{H''}{H'}\right| \leq |z|^{\varepsilon}, \qquad \left|\frac{H'''}{H'}\right| \leq |z|^{2\varepsilon}.$$

Therefore, using (3.4) again, the rational function $b(z)$ must be finite at infinity.

Suppose now that $b(z) = O(|z|^{-2})$ as $z \to \infty$. Let $N \subseteq \overline{\mathbb{C}}$ be a simply-connected neighborhood of infinity such that b is analytic on $N^* = N \setminus \{\infty\}$ and $N_0 = N^* \setminus \mathbb{R}_{\geq 0}$ is also simply-connected. In this case, equation (2.1) has a regular singular point at infinity [11, Sections 5.2–5.3] and therefore has a pair of linearly independent solutions on N of the form either

(i) $z^{\sigma_1} g_1(z)$ and $z^{\sigma_2} g_2(z)$; or
(ii) $z^{\sigma_1} g_1(z)$ and $z^{\sigma_2} g_2(z) + c_1 z^{\sigma_1} g_1(z) \log z$,

where g_1, g_2 are analytic on N. By the discussion preceding this lemma, the quotient of any two solutions must be meromorphic on N^* and so we must have case (i). Then $z^{\sigma_1 - \sigma_2}(g_1/g_2)$ is meromorphic on N^* and so $\sigma_1 - \sigma_2$ is an integer. The discussion above also gives that on N_0 we can write H as the quotient of two linear combinations of $z^{\sigma_1 - \sigma_2} g_1$ and g_2. This means that H can be meromorphically extended to infinity, contradicting the fact that it is transcendental.

Now suppose instead that $b(z) = cz^{-1}(1 + o(1))$ as $z \to \infty$ where $c \neq 0$. For F a solution of (2.1), put

$$z = u^2, \qquad h(u) = u^{-1/2} F(u^2). \tag{3.5}$$

Then h solves

$$h'' + c_0(u) h = 0, \tag{3.6}$$

where

$$c_0(u) = 4u^2 b(u^2) - 3/(4u^2) = 4c\big(1 + o(1)\big), \quad u \to \infty.$$

By applying Hille's method, we can find a principal solution to equation (3.6) that tends to zero exponentially fast as u goes to infinity in a sector with angular opening $\pi - 2\delta$. Using (3.5), this leads to an exponentially decaying solution $v(z)$ to (2.1) on a sector D of angular opening $2\pi - 4\delta$. By the discussion preceding this lemma, v^2 extends to be meromorphic on the plane with finitely many poles and order not exceeding one. As v^2 is bounded on D the Phragmén–Lindelöf principle implies that v^2 is rational, contradicting the exponential decay of v on D. □

Let C be the nonzero real value taken by b at infinity, and choose c so that $c^2 = C$. We again use Hille's method to find solutions of (2.1) on

$$S_1 = \{z : |z| > R_1,\ |\arg z - \theta_0| < \pi - \delta\},$$

where $\arg z = \theta_0$ is a critical ray, R_1 is large and $0 < \delta < \pi/4$. We obtain principal solutions

$$u_\pm(z) = b(z)^{-1/4} \exp\bigl(\pm icz + O(\log|z|)\bigr), \quad z \to \infty, \tag{3.7}$$

which are analytic and non-zero on S_1.

The next lemma shows that we may take c to be real and positive.

LEMMA 4. *The value C is positive.*

Proof. Suppose that $C < 0$ and so c is purely imaginary. In this case, the critical ray $\arg z = \theta_0$ lies along the imaginary axis and if μ, ν are nonzero constants then $\mu u_+ + \nu u_-$ has infinitely many zeroes near this critical ray.

By the discussion of (3.4) above, $H = w_1/w_2$ and $H' = -1/{w_2}^2$ on S_1, where w_1 and w_2 are linear combinations of u_+ and u_-. Since H has only finitely many nonreal poles, w_2 must be a multiple of a principal solution, $w_2 = \kappa u_\pm$. Then using (3.7), we see that $H'(z) = -1/(\kappa u_\pm)^2$ tends to either zero or infinity as $|z| \to \infty$ with z real. Hence, $H'(z) + \alpha = 0$ has only finitely many real roots. On recalling that $f' = H' + \alpha$ has only finitely many nonreal zeroes, we uncover a contradiction: the transcendental derivative f' takes both of the values 0 and α only finitely often. $\square$

We now choose $c = \sqrt{C} > 0$.

LEMMA 5. *We can write*

$$H(z) = \frac{kP(z)e^{icz} + lQ(z)e^{-icz}}{P(z)e^{icz} + Q(z)e^{-icz}}, \tag{3.8}$$

where $k, l \in \mathbb{C}$ and P and Q are polynomials without common zeroes.

Proof. For $z \in S_1$, let

$$v_\pm(z) = u_\pm(z)e^{\mp icz}.$$

Referring again to the discussion preceding Lemma 3, we find that the functions $v_\pm^2$ extend to be meromorphic on the plane, with finitely many poles and orders not exceeding one. Also, (3.7) gives that if $z \in S_1$ then $v_\pm^2 = O(|z|^M)$ for some M, so that the Phragmén–Lindelöf principle shows these functions to be rational. Moreover, as $v_\pm$ is analytic on S_1, we can write

$$v_\pm^2 = \frac{r_\pm}{s_\pm}, \tag{3.9}$$

where $r_\pm$ and $s_\pm$ are polynomials and $s_\pm$ has no zeroes in S_1. In particular, we may define an analytic branch of $(s_+s_-)^{1/2}$ on S_1.

The discussion of (3.4) above gives that, on S_1, we can write H as a quotient of solutions of (2.1),

$$H = \frac{\mu_1 u_+ + \nu_1 u_-}{\mu_2 u_+ + \nu_2 u_-} = \frac{\mu_1 v_+ e^{icz} + \nu_1 v_- e^{-icz}}{\mu_2 v_+ e^{icz} + \nu_2 v_- e^{-icz}}. \tag{3.10}$$

Multiplying through by $(s_+ s_-)^{1/2}$ and then taking $P = \mu_2 v_+ (s_+ s_-)^{1/2}$ and $Q = \nu_2 v_- (s_+ s_-)^{1/2}$ we see that (3.10) becomes (3.8) on S_1. These functions P and Q are analytic on S_1 and by (3.9) both P^2 and Q^2 are polynomial. Neither P nor Q can vanish identically, since if $\mu_2 \nu_2 = 0$ then $H'(z) = r(z) e^{\pm 2icz}$ for some rational function $r(z)$, and this contradicts the reality of H.

We may assume that the polynomials P^2 and Q^2 have no common zeroes in the plane. To see this, suppose that $P^2(z_0) = Q^2(z_0) = 0$. If $z_0 \in S_1$, then P and Q are analytic at z_0 and we may divide both by $(z - z_0)$. Otherwise, $z_0 \notin S_1$ and we may divide both P and Q by a branch of $(z - z_0)^{1/2}$ that is analytic on S_1.

We complete the proof by showing that P and Q are themselves polynomial, so that (3.8) must hold on the whole plane by the Identity Theorem. We shall prove that P and Q may be analytically continued along any path, and then the Monodromy Theorem shows P and Q to be analytic, and hence polynomial, on the plane.

Let

$$\gamma : [0, \infty) \to \mathbb{C}, \qquad \gamma(0) \in S_1$$

be a path starting in S_1. Suppose that $0 < t_0 < \infty$ is maximal such that both P and Q can be analytically continued along the path $\gamma(t)$ for $0 \leq t < t_0$. As P^2 and Q^2 are polynomial the point $\gamma(t_0)$ must be a zero of either P^2 or Q^2. Suppose that $P(\gamma(t_0))^2 = 0$ (the proof is identical if instead $Q(\gamma(t_0))^2 = 0$). Then $\gamma(t_0)$ is not a zero of Q^2, and so Q admits analytic continuation along $\gamma(t)$ for $t < t_0 + \varepsilon$. Since H is meromorphic on the plane, (3.8) defines a meromorphic continuation of P along $\gamma(t)$ for $t < t_0 + \varepsilon$; namely,

$$P(\gamma(t)) = \frac{l - H(\gamma(t))}{H(\gamma(t)) - k} Q(\gamma(t)) e^{-2ic\gamma(t)}.$$

As P^2 is a polynomial this continuation must be analytic, contradicting the maximality of t_0. □

The function H is real and satisfies (3.8) so we must have that, for real x,

$$\operatorname{Im}\big(k|P(x)|^2 + l|Q(x)|^2 + kP(x)\overline{Q(x)}e^{2icx} + l\overline{P(x)}Q(x)e^{-2icx}\big) = 0. \tag{3.11}$$

Let $R(x) = \operatorname{Re}(P(x)\overline{Q(x)})$ and $I(x) = \operatorname{Im}(P(x)\overline{Q(x)})$ and observe that these are real polynomials, not both vanishing identically. Now calculate the left-hand side of (3.11) writing $k = k_r + ik_i$ and $l = l_r + il_i$. Noting that the coefficients of $\sin 2cx$ and $\cos 2cx$ must vanish because P, Q, R and I are all polynomials, this leads to

$$k_i|P(x)|^2 + l_i|Q(x)|^2 = 0, \tag{3.12}$$

$$(k_r - l_r)R(x) - (k_i + l_i)I(x) = 0, \tag{3.13}$$

$$(k_r - l_r)I(x) + (k_i + l_i)R(x) = 0. \tag{3.14}$$

Inspection of (3.13) and (3.14) yields $k_r = l_r$ and $k_i = -l_i$. Hence, $l = \overline{k}$ and k must be nonreal, otherwise H would be constant. Now (3.12) shows that, for real z,

$$P(z)\overline{P(\overline{z})} = Q(z)\overline{Q(\overline{z})}, \tag{3.15}$$

and in fact this holds on the whole plane, as both sides are polynomials in z. Since P and Q have no common zeroes, it follows that z_0 is a zero of P if and only if $\overline{z_0}$ is a zero of Q of equal multiplicity. Therefore,

$$\overline{P(\overline{z})} = \beta Q(z)$$

for some β and (3.15) gives that $|\beta| = 1$. Using the fact that $\overline{\beta^{1/2}} = \beta^{-1/2}$ allows us to assume that $\beta = 1$ by replacing P and Q by $P_1 = \beta^{1/2}P$ and $Q_1 = \beta^{1/2}Q$ and relabeling.

By writing $k = \overline{l} = A + \lambda i$ and using (3.1), equation (3.8) now becomes (1.2).

4. Proof of Theorem 1—Part two

In this section, f is assumed to be given by (1.2) where α, λ and A are real, $\alpha\lambda \neq 0$, $c > 0$ and P is a polynomial with zeroes $a_1, \ldots, a_N$ (repeated to multiplicity) such that $a_j \neq \overline{a_k}$. We aim to prove that f and f'' have only finitely many nonreal zeroes and poles, and that the equation $f'(z) = \alpha$ has at most $2N$ solutions, counting with multiplicities. We show further that all but finitely many of the zeroes of f' are real if and only if either $0 < \lambda c/\alpha < 1$ or condition (1.3) is satisfied.

Together with the result established in the previous section, this completes the proof of Theorem 1.

It will be useful to write $Q(z) = \overline{P(\overline{z})}$ and to differentiate (1.2) to obtain

$$f' - \alpha = 2i\lambda \frac{P'Q - PQ' + 2icPQ}{(Pe^{icz} + Qe^{-icz})^2} \tag{4.1}$$

and

$$f'' = \frac{p_0(z)e^{icz} + p_1(z)e^{-icz}}{(Pe^{icz} + Qe^{-icz})^3}, \tag{4.2}$$

where p_0 and p_1 are polynomials. Since f'' is a real function it is easily seen that $p_1(z) = \overline{p_0(\overline{z})}$.

The assertion that the equation $f'(z) = \alpha$ has at most $2N$ solutions is proved simply by observing that the numerator of the right-hand side of (4.1) is a polynomial of degree $2N$.

From (1.2), we see that if z_0 is a pole of f then z_0 satisfies

$$P(z_0)e^{icz_0} + \overline{P(\overline{z_0})}e^{-icz_0} = 0,$$

and if z_1 is a zero of f then z_1 satisfies

$$(\alpha z_1 + A + i\lambda)P(z_1)e^{icz_1} + (\alpha z_1 + A - i\lambda)\overline{P(\overline{z_1})}e^{-icz_1} = 0.$$

Similarly, from (4.2) we see that if z_2 is a zero of f'' then z_2 satisfies

$$p_0(z_2)e^{icz_2} + \overline{p_0(\overline{z_2})}e^{-icz_2} = 0.$$

Therefore, the fact that f and f'' have only finitely many nonreal zeroes and poles follows from the next lemma.

Lemma 6. *If $p(z) \not\equiv 0$ is a polynomial, then $F(z) = p(z)e^{iz} + \overline{p(\overline{z})}e^{-iz}$ has only finitely many nonreal zeroes.*

Proof. For real x,

$$F(x) = 2\,\mathrm{Re}(p(x))\cos x - 2\,\mathrm{Im}(p(x))\sin x. \tag{4.3}$$

Let m be a large positive or negative integer. If $\mathrm{Re}(p(x)) \not\equiv 0$, then (4.3) shows that $F(x)$ changes sign over the interval $[m\pi, (m+1)\pi]$. Otherwise, $\mathrm{Im}(p(x)) \not\equiv 0$ and $F(x)$ changes sign over $[(m-\frac{1}{2})\pi, (m+\frac{1}{2})\pi]$. In either case, denoting by $n(t)$ the number of nonreal zeroes of F in $\{z : |z| \leq t\}$, we have that $n(t, 1/F) \geq n(t) + 2t/\pi - O(1)$. Hence,

$$\int_0^r \frac{n(t)}{t}\,dt \leq T(r,F) - \frac{2r}{\pi} + O(\log r) = O(\log r), \quad r \to \infty,$$

using the fact that $T(r,F) = 2r/\pi + O(\log r)$ because $p(z)e^{iz}$ is large where $\overline{p(\overline{z})}e^{-iz}$ is small. This implies that $n(t)$ is bounded and so F has finitely many nonreal zeroes. □

Lemma 7. *All but finitely many of the zeroes of f' are real if and only if either $0 < \lambda c/\alpha < 1$ or condition (1.3) holds.*

Proof. Define the real functions

$$g_1 = \frac{P}{Q}e^{2icz} + \frac{Q}{P}e^{-2icz} \tag{4.4}$$

and

$$\begin{aligned} g_2 &= \frac{2\lambda i}{\alpha}\left(\frac{Q'}{Q} - \frac{P'}{P} - 2ic\right) - 2 \\ &= \frac{4\lambda c}{\alpha} - 2 + \frac{4\lambda}{\alpha}\sum_{j=1}^{N}\frac{\mathrm{Im}\,a_j}{(z-a_j)(z-\overline{a_j})}. \end{aligned} \tag{4.5}$$

Then by (4.1)

$$f' = \frac{\alpha PQ}{(Pe^{icz} + Qe^{-icz})^2}(g_1 - g_2)$$

so that f' and $g_1 - g_2$ have the same zeroes with finitely many exceptions. To see this, note that $g_1(z) = -2$ at a zero of $Pe^{icz} + Qe^{-icz}$, but that $g_2(z) = -2$ only finitely often.

Fix an analytic branch of $\log(P/Q)$ on the simply-connected domain

$$D = \{z : |z| > R, \operatorname{Im} z < 1\},$$

where R is large, and choose a real number ϕ such that

$$\varepsilon(z) = \phi - i\log\left(\frac{P(z)}{Q(z)}\right) = o(1) \quad \text{as } z \to \infty \text{ in } D. \tag{4.6}$$

The function $\varepsilon(z)$ is real on the real axis; hence for a large positive or negative integer n we can find a real number x_n such that $2cx_n + \varepsilon(x_n) = n\pi + \phi$. Using (4.4) and (4.6), we can now write

$$g_1(z) = 2\cos\bigl(2cz - \phi + \varepsilon(z)\bigr), \quad z \in D, \tag{4.7}$$

$$g_1(x_n) = 2(-1)^n \quad \text{and} \quad x_n \sim \frac{n\pi + \phi}{2c} \quad \text{as } n \to \pm\infty. \tag{4.8}$$

Assume now that either $0 < \lambda c/\alpha < 1$ or condition (1.3) holds. Then (4.5) gives $|g_2(x)| < 2$ for all large real x and so (4.8) shows that $g_1 - g_2$ changes sign over $[x_n, x_{n+1}]$. Therefore, $g_1 - g_2$ has at least $4ct/\pi - O(1)$ real zeroes in $\{z : |z| \le t\}$, and the same is true of f'. Using (4.1), we calculate

$$T(r, f') = 2T(r, Pe^{icz} + Qe^{-icz}) + O(\log r) = \frac{4cr}{\pi} + O(\log r), \quad r \to \infty,$$

using the fact that Pe^{icz} is large where Qe^{-icz} is small and vice versa. By an argument similar to that used at the end of the proof of Lemma 6, this is sufficient to show that all but finitely many of the zeroes of f' are real.

We tackle the proof of the converse in two cases.

(i) Suppose first that either $\lambda c/\alpha < 0$ or $\lambda c/\alpha > 1$. Then by (4.5) and (4.7) for real x of large absolute value, we have $|g_1(x)| \le 2$ and $|g_2(x)| > 2$. Therefore, both $g_1 - g_2$ and f' have only finitely many real zeroes. Hence, f' has infinitely many nonreal zeroes as it cannot have only finitely many zeroes and α-points in the plane.

(ii) Now suppose instead that $\lambda c = \alpha$ but that (1.3) fails to hold. Then $g_2(x) > 2$ either for all large positive x or for all large negative x. For such x, we have $|g_1(x)| \le 2$ by (4.7). Hence, $g_1 - g_2$ either has only finitely many positive zeroes, or only finitely many negative zeroes.

Using (4.5) and (4.8) gives that

$$g_1(x_{2n}) - 2 = 0 \quad \text{and} \quad g_2(z) - 2 = o(1) \quad \text{as } z \to \infty,$$

and we see from (4.7) that $|g_1 - 2|$ is bounded away from zero on a small circle about x_{2n}. Hence, it follows from Rouché's Theorem that $g_1 - g_2$ has at least one zero near each point x_{2n}, for n sufficiently large. Combining this with the result of the previous paragraph shows that $g_1 - g_2$ has infinitely many nonreal zeroes, and the same is true of f'. $\square$

5. Proof of Theorem 2

The following lemma is the key to the proof of Theorem 2.

LEMMA 8. *Let F be meromorphic such that all but finitely many of the zeroes and poles of F are real, and $F(z) = 1$ only finitely often. If F has infinitely many multiple poles, then F is real.*

Proof. The order of F does not exceed one by Lemma 1. Hence, we can write

$$F(z) = \frac{h(z)P_1(z)e^{iaz}}{k(z)P_2(z)},$$

where h and k are real entire functions of order at most one with only real zeroes and no common zeroes; the polynomials P_1 and P_2 have no real zeroes; and a is a real constant. Furthermore, there exists an unbounded real sequence (x_n) of multiple zeroes of k. Since F takes the value 1 only finitely often we can also write

$$F(z) = 1 + \frac{P_3(z)e^{ibz}}{g(z)P_2(z)}, \tag{5.1}$$

where g is a real entire function with only real zeroes, P_3 is a polynomial and b is a real constant. Equating these two expressions for $F(z)$ yields

$$h(z)P_1(z) - k(z)P_2(z)e^{-iaz} = \frac{k(z)P_3(z)e^{i(b-a)z}}{g(z)}. \tag{5.2}$$

Observe that if the right-hand side of (5.2) vanishes then either k or P_3 must vanish, but that the left-hand side cannot vanish at a zero of k. Hence,

$$h(z)P_1(z) - k(z)P_2(z)e^{-iaz} = CP_3(z)e^{Dz} \tag{5.3}$$

for some constants C and D. Note that $C \neq 0$ as otherwise $F(z) \equiv 1$. Evaluating (5.3) and its derivative at each of the points x_n gives

$$h(x_n)P_1(x_n) = CP_3(x_n)e^{Dx_n} \tag{5.4}$$

and

$$h'(x_n)P_1(x_n) + h(x_n)P_1'(x_n) = C\big(P_3'(x_n) + DP_3(x_n)\big)e^{Dx_n},$$

which lead to

$$\frac{h'(x_n)}{h(x_n)} + \frac{P_1'(x_n)}{P_1(x_n)} = \frac{P_3'(x_n)}{P_3(x_n)} + D.$$

Therefore, D must be real because h is a real function and $P_j'(x_n)/P_j(x_n) \to 0$ as $|x_n| \to \infty$. From (5.2) and (5.3) we see that

$$k(z)e^{i(b-a)z} = Cg(z)e^{Dz},$$

so that $Ce^{i(a-b)z}$ is real for all real z. Thus, $a = b$ and C is real. Now (5.4) shows that $P_1(x_n)/P_3(x_n)$ is real for every x_n and therefore P_1/P_3 is a real function. Dividing equation (5.3) by P_3 gives that the function P_2e^{-iaz}/P_3 is real and hence (5.1) shows that F must also be real. □

Let the function g be as in the hypothesis of Theorem 2. Assume first that g has infinitely many poles and apply Lemma 8 with $F = g'/\beta$. This gives that on the real axis g/β has constant imaginary part and it follows immediately that we have case (i) of the theorem.

Now suppose instead that g has only finitely many poles. By Lemma 1, the order of g' is at most one and it follows that

$$g'(z) - \beta = R_1(z)e^{icz}$$

for some rational function $R_1 \not\equiv 0$. We show next that c is real. Suppose not, then $g'(x)$ tends to either β or infinity as real $x \to \pm\infty$ and so g' must have finitely many real zeroes. But then g' takes each of the values 0, β and ∞ only finitely often, implying that g' is rational and hence $c = 0$.

Repeated integration by parts now shows that we have case (ii) of the theorem.

Finally, the assertion about the zeroes of g'' follows from Theorem 1 in case (i) and by straightforward differentiation in case (ii).

6. Proof of Theorem 4

Let g be as in the hypothesis and assume without loss of generality that $\alpha = 1$. We begin with a simple estimate of the logarithmic derivative.

LEMMA 9. *Let F be a meromorphic function of order at most ρ with all but finitely many of its zeroes and poles real. Let $\delta > 0$ and $\eta > 0$. Then*

$$\left|\frac{F'(z)}{F(z)}\right| = o(|z|^{\rho-1+\eta}) \quad \text{as } z \to \infty, \delta < |\arg z| < \pi - \delta.$$

Proof. Let z be such that $|z| = r$ and $\delta < |\arg z| < \pi - \delta$. The differentiated Poisson–Jensen formula [13, p. 65] gives

$$\left|\frac{F'(z)}{F(z)}\right| \leq \frac{4}{r}\big(m(2r, F) + m(2r, 1/F)\big) + \sum_{|z_j|<2r} \frac{2}{|z - z_j|},$$

where the z_j are the zeroes and poles of F repeated according to multiplicity. For the finitely many nonreal z_j, we have $|z - z_j|^{-1} = O(r^{-1})$ as $r \to \infty$, while for the real z_j we have $|z-z_j| \geq |\operatorname{Im} z| \geq r\sin\delta$. Therefore, as $r\to\infty$,

$$\left|\frac{F'(z)}{F(z)}\right| \leq o(r^{\rho-1+\eta}) + \frac{2}{r\sin\delta}\big(n(2r,F)+n(2r,1/F)\big) = o(r^{\rho-1+\eta}). \qquad \square$$

By Lemma 1, the order of g is at most one. Hence, taking $0<\varepsilon_1<\varepsilon/4$ and $\eta>0$ both small and applying Lemma 9 gives that

$$\left|\frac{g'(z)}{g(z)}\right| = o(|z|^\eta) \quad \text{as } z\to\infty, \varepsilon_1 < |\arg z| < \pi-\varepsilon_1. \tag{6.1}$$

Define $\sigma\in(1,2)$ by

$$\sigma = 1+\frac{\lambda\sin(\varepsilon/2)}{8}, \tag{6.2}$$

where $\lambda=\lambda(g)$ is the lower order of g. Applying [16, Lemma 5] to $g-1$, we can find a small positive constant m and a set J of lower logarithmic density greater than $1/\sigma$ such that if $r\in J$ is large and F_r is a subinterval of $[0,2\pi]$ of length m then

$$\int_{F_r}\left|\frac{rg'(re^{i\theta})}{g(re^{i\theta})-1}\right| d\theta \leq \frac{\delta(1,g)}{4}T(r,g).$$

By the definition of deficiency, for large $r\in J$ there exists z_0 with $|z_0|=r$ and

$$\log|g(z_0)-1| \leq -\frac{\delta(1,g)}{2}T(r,g).$$

It follows that g is near 1 on any arc of angular measure m with z_0 as one endpoint. In particular, because g is real and ε_1 is small we can find, for large $r\in J$, an arc

$$\Omega(r) \subseteq A(r) = \{z : |z|=r, 2\varepsilon_1 < \arg z < \pi-2\varepsilon_1\}$$

of angular measure $m/2$ on which

$$\log|g(z)-1| < -c_1T(r,g), \tag{6.3}$$

denoting by $c_1,c_2,\ldots$ positive constants not depending on r.

It is now claimed that we can choose by induction a sequence (r_k) in J satisfying $2r_k<r_{k+1}<r_k^\sigma$. Otherwise, there exists a large $r_k\in J$ such that $(2r_k,r_k^\sigma)\cap J=\emptyset$. Taking l such that $1/\sigma<l<\underline{\operatorname{logdens}} J$ then leads to the following contradiction

$$l\log r_k^\sigma < \int_{[1,r_k^\sigma]\cap J}\frac{dt}{t} \leq \int_1^{2r_k}\frac{dt}{t} = (1/\sigma)\log r_k^\sigma + \log 2.$$

We deduce immediately that

$$\bigcup_{k=1}^\infty (r_k, r_k^\sigma) \text{ contains all large } r. \tag{6.4}$$

Define two sequences of arcs by $\Omega_k = \Omega(r_k)$ and $A_k = A(r_k)$. Applying Lemma 9 to $g-1$ gives that on Ω_k we have $|g'/(g-1)| = o(r_k^\eta)$ as $r_k \to \infty$. Hence, for $z \in \Omega_k$, using (6.3) twice yields

$$\begin{aligned}\log\left|\frac{g'(z)}{g(z)}\right| &\le \log|g'(z)| + o(1)\\ &\le \log|g(z)-1| + O(\log r_k) < -c_2 T(r_k, g), \quad r_k \to \infty.\end{aligned}$$

We show next that a similar bound holds on the whole of the arc A_k. To do this, let $D_k = \{z : r_k/2 < |z| < 2r_k,\ \varepsilon_1 < \arg z < \pi - \varepsilon_1\}$ and note that by conformal invariance

$$\omega(z, \Omega_k, D_k \setminus \Omega_k) > c_3, \quad z \in A_k.$$

Thus, using (6.1) and the above, the Two Constants Theorem now gives

$$\log\left|\frac{g'(z)}{g(z)}\right| < -c_4 T(r_k, g), \quad z \in A_k. \tag{6.5}$$

Let

$$\begin{aligned}S_k &= \{z : r_k < |z| < r_k^2,\ 2\varepsilon_1 < \arg z < \pi - 2\varepsilon_1\},\\ S_k' &= \{z : r_k < |z| < r_k^\sigma,\ \varepsilon < \arg z < \pi - \varepsilon\}.\end{aligned}$$

LEMMA 10. *For large k, the harmonic measure of the arc A_k satisfies*

$$\omega(z, A_k, S_k) \ge \frac{1}{2\pi r_k^{4(\sigma-1)/\sin(\varepsilon/2)}} = \frac{1}{2\pi r_k^{\lambda/2}}, \quad z \in S_k'.$$

The proof of Lemma 10 is simply an application of the following lemma that goes back to Nevanlinna.

LEMMA 11 ([2, Lemma E]). *Let D be a domain bounded by a Jordan curve $\mathcal{C}$ consisting of a Jordan arc $\mathcal{A}$ and its complement $\mathcal{B}$ in $\mathcal{C}$. Let Γ be a rectifiable curve in D joining a point $a \in \mathcal{A}$ to a point in $\mathcal{B}$. Let z be a point on Γ and let $\rho_{\mathcal{B}}(z)$ denote the distance of z from $\mathcal{B}$. Then*

$$\omega(z, \mathcal{A}, D) \ge \frac{1}{2\pi} \exp\left\{-4\int_a^z \frac{|d\zeta|}{\rho_{\mathcal{B}}(\zeta)}\right\},$$

where the integral is taken along Γ.

Proof of Lemma 10. The equality in the statement of the result simply follows from (6.2).

Let r_k be large, $\zeta \in S_k'$ and let w be a nearest point to ζ of $\mathcal{B} = \partial S_k \setminus A_k$. Then either $\arg w = 2\varepsilon_1$ or $\arg w = \pi - 2\varepsilon_1$. Using the fact that $\varepsilon - 2\varepsilon_1 > \varepsilon/2$ it follows that

$$\rho_{\mathcal{B}}(\zeta) = |\zeta - w| \ge |\zeta| \sin(\varepsilon/2).$$

For $z \in S_k'$, choose the path $\Gamma(t) = te^{i \arg z}$ for $t \in [r_k, r_k^2]$. Then applying Lemma 11 yields

$$\omega(z, A_k, S_k) \geq \frac{1}{2\pi} \exp\left\{-4 \int_{r_k}^{|z|} \frac{dt}{t \sin(\varepsilon/2)}\right\} = \frac{1}{2\pi} \left(\frac{r_k}{|z|}\right)^{4/\sin(\varepsilon/2)},$$

which gives the required result upon noting that $|z| < r_k^\sigma$. □

Using (6.1), (6.5) and Lemma 10, the Two Constants Theorem now gives that, for $z \in S_k'$,

$$\log\left|\frac{g'(z)}{g(z)}\right| \leq \frac{-c_4 T(r_k, g)}{2\pi r_k^{\lambda/2}} + O(\log r_k) < -c_5 r_k^{\lambda/4}, \quad r_k \to \infty. \tag{6.6}$$

Pick a point $z_k \in \Omega_k$ for each k. For large k, there are no zeroes or poles of g in S_k and so for $z \in S_k'$ we can write

$$g(z) = g(z_k) \exp\left(\int_{z_k}^{z} \frac{g'(w)}{g(w)}\, dw\right) = 1 + o(1), \quad k \to \infty,$$

using (6.3) and (6.6). By (6.4), if z is large and $\varepsilon < \arg z < \pi - \varepsilon$ then $z \in S_k'$ for some k which tends to infinity with z. Hence, by the above,

$$g(z) \sim 1 \quad \text{as } z \to \infty,\ \varepsilon < \arg z < \pi - \varepsilon.$$

Since g is real this completes the proof.

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