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LORENTZ HYPERSURFACES IN E_1^4 SATISFYING $\Delta\vec{H} = \alpha\vec{H}$

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ABSTRACT. A hypersurface M_1^3 in the four-dimensional pseudo-Euclidean space E_1^4 is called a Lorentz hypersurface if its normal vector is space-like. We show that if the mean curvature vector field of M_1^3 satisfies the equation $\Delta\vec{H} = \alpha\vec{H}$ (α a constant), then M_1^3 has constant mean curvature. This equation is a natural generalization of the biharmonic submanifold equation $\Delta\vec{H} = \vec{0}$.

1. Introduction

Let $x : M_r^n \to E_s^m$ be an isometric immersion of an n-dimensional connected submanifold M_r^n of a pseudo-Euclidean space E_s^m. We denote by $\vec{H}, \Delta$ the mean curvature vector field and the Laplace operator of M_r^n respectively, with respect to the induced Riemannian metric. A submanifold of E_s^m is said to have *proper mean curvature vector field* if it satisfies the equation

$$\Delta\vec{H} = \alpha\vec{H} \quad (\alpha \text{ constant}). \tag{1}$$

If $\alpha = 0$ the above equation reduces to $\Delta\vec{H} = \vec{0}$, and the submanifold is called *biharmonic*. Biharmonic submanifolds have been studied by several authors. A well known conjecture of Chen [5] states that *the only biharmonic submanifolds of Euclidean spaces are the minimal submanifolds, that is when $H = 0$.*

Equation (1) was first appeared in [3] where surfaces in E^3 satisfying (1) were classified. In [4], it was shown that a submanifold M of a Euclidean space satisfies (1) if and only if M is biharmonic or of 1-type or a null 2-type. Hypersurfaces in E^4 satisfying (1) with the additional condition of conformal flatness were classified by Garray in [14]. In [10], Defever proved that every hypersurface of E^4 satisfying (1) has constant mean curvature. Other

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results about submanifolds satisfying (1) have been obtained by Chen ([8], [6]), Ekmekci and Yaz [11], and Inoguchi ([15], [16]).

The study of equation (1) for submanifolds in pseudo-Euclidean spaces was originated by Ferrández and Lucas in [12] and [13]. Among other results, they showed that if the minimal polynomial of the shape operator of a hypersurface M_r^{n-1} $(r=0,1)$ in E_1^n is at most of degree two, then M_r^{n-1} has constant mean curvature. Also, in [7] various classification theorems for submanifolds in a Minkowski space–time were obtained. In a recent work, the first two authors and Defever [2] proved that if M_r^3 $(r=0,1,2,3)$ is a nondegenerate hypersurface of the pseudo-Euclidean space E_s^4 satisfying equation (1) and the shape operator is diagonal, then M_r^3 has constant mean curvature.

Even though Chen's conjecture is not true in general for submanifolds in pseudo-Euclidean spaces, there is evidence (see e.g., the main result in [1] and references therein) that the conjecture is in fact true for hypersurfaces in pseudo-Euclidean spaces. It would be reasonable to believe that submanifolds satisfying equation (1) must have constant mean curvature. Towards this direction, in the present article we consider Lorentz hypersurfaces in E_1^4 whose shape operator is not diagonal and we prove the following theorem.

THEOREM. *Let M_1^3 be a nondegenerate Lorentz hypersurface of the 4-dimensional pseudo-Euclidean space E_1^4 satisfying $\Delta\vec{H}=\alpha\vec{H}$. Then M_1^3 has constant mean curvature.*

The headlines of the proof are as follows: we use [9] to express equation $\Delta\vec{H}=\alpha\vec{H}$ as a system of equations

$$S(\nabla H) = -\varepsilon\frac{3H}{2}(\nabla H),$$
$$\Delta H + \varepsilon H \operatorname{tr} S^2 = \alpha H.$$

According to Petrov [19] and Magid [17] the shape operator of a Lorentz hypersurface M_1^3 in E_1^4 can be put in four possible canonical forms. We prove that for each nondiagonal canonical form of the shape operator, the mean curvature of M_1^3 is constant (cf. Propositions 1, 2, 3, and 4). We remark that Propositions 2 and 3 in particular show that M_1^3 is minimal.

2. Preliminaries

Lorentz hypersurfaces in E_1^4. Let M_1^3 be a Lorentz hypersurface of the pseudo-Euclidean space E_1^4. Let $\vec{\xi}$ denote a unit normal vector field with $\langle\vec{\xi},\vec{\xi}\rangle=1$. Denote by ∇ and $\tilde{\nabla}$ the Levi–Civita connections of M_1^3 and E_1^4 respectively. For any vector fields X, Y tangent to M_1^3, the Gauss formula is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\vec{\xi}, \tag{2}$$

where h is the scalar-valued second fundamental form. If we denote by S the shape operator of M_1^3 associated to $\vec{\xi}$, then the Weingarten formula is given by

$$\tilde{\nabla}_X\vec{\xi} = -S(X), \tag{3}$$

where $\langle S(X),Y\rangle = h(X,Y)$. If $H=\frac{1}{3}\operatorname{tr} S$, then the mean curvature vector $\vec{H}=H\vec{\xi}$ is a well defined normal vector field to M_1^3 in E_1^4. The Codazzi equation is given by

$$(\nabla_X S)Y = (\nabla_Y S)X, \tag{4}$$

and the Gauss equation by (cf. [18])

$$R(X,Y)Z = \langle S(Y),Z\rangle S(X) - \langle S(X),Z\rangle S(Y). \tag{5}$$

We assume that the mean curvature vector field satisfies the equation

$$\Delta\vec{H} = \alpha\vec{H}. \tag{6}$$

Condition (6) is equivalent to (cf. [9])

$$\Delta\vec{H} = 2S(\nabla H) + 3H(\nabla H) + \{\Delta H + H\operatorname{tr} S^2\}\vec{\xi} = \alpha\vec{H}. \tag{7}$$

By comparing the vertical and horizontal parts of (7), this is equivalent to the conditions

$$S(\nabla H) = -\frac{3H}{2}(\nabla H), \tag{8}$$

$$\Delta H + H\operatorname{tr} S^2 = \alpha H, \tag{9}$$

where the Laplace operator Δ acting on scalar-valued function f is given by (e.g., [9])

$$\Delta f = -\sum_{i=1}^{3}\epsilon_i(e_ie_if - \nabla_{e_i}e_if). \tag{10}$$

Here, $\{e_1,e_2,e_3\}$ is a local orthonormal frame of $T_pM_1^3$ with $\langle e_i,e_i\rangle=\epsilon_i=\pm 1$.

The shape operator of a hypersurface in E_1^4. Consider the real 4-dimensional vector space $\mathbb{R}^4$ with the standard basis $\{e_i, i=1,\dots,4\}$. Let $\langle\cdot,\cdot\rangle$ denote the indefinite inner product on $\mathbb{R}^4$ whose matrix with respect to the standard basis is $\operatorname{diag}(-1,1,1,1)$. This is called the Lorentz metric on $\mathbb{R}^4$. The space $\mathbb{R}^4$ with this metric is called the 4-*dimensional pseudo-Euclidean space*, and is denoted by E_1^4.

A vector $X\in E_1^4$ is called *time-like*, *space-like*, or *light-like* according to whether $\langle X,X\rangle$ is negative, positive, or zero, respectively. A nondegenerate hypersurface M_r^3 $(r=0,1)$ of the pseudo-Euclidean space E_1^4 can itself be endowed with a Riemannian or a Lorentzian metric structure, according to whether the metric induced on M_r^3 from the Lorentzian metric on E_1^4 is (positive) definite or indefinite. In the former case, a normal vector to M_r^3 is time-like, and in the latter case a normal vector to M_r^3 is space-like.

The shape operator of a Riemannian submanifold is always diagonalizable, but this is not the case for the shape operator of a Lorentzian submanifold. It is known [19, pp. 50–55] that a symmetric endomorphism of a vector space with a Lorentzian inner product can be put into four possible canonical forms. In particular, the matrix representation G of the induced metric on M_1^3 is of Lorentz type, so the shape operator S of M_1^3 can be put into one of the following four forms with respect to frames $\{e_1, e_2, e_3\}$ at $T_pM_1^3$ [17]:

$$\text{(I)} \qquad S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3, \end{pmatrix}, \qquad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{(II)} \qquad S = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{(III)} \qquad S = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{(IV)} \qquad S = \begin{pmatrix} \mu & -\nu & 0 \\ \nu & \mu & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \nu \neq 0.$$

The matrices G for cases (I) and (IV) are with respect to an orthonormal basis of $T_pM_1^3$, whereas for cases (II) and (III) are with respect to a *pseudo-orthonormal basis*. This is a basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$ satisfying $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0$, and $\langle e_1, e_2\rangle = \langle e_3, e_3\rangle = 1$. In [2], the first two authors and Defever proved that every nondegenerate hypersurface M_r^3 ($r = 0, 1, 2, 3$) in E_s^4 ($s = 0, \ldots, 4$) with shape operator of type (I) satisfying (6), has constant mean curvature. In the present work, we study the same problem, where the shape operator has one of the forms (II), (III), and (IV).

3. Proof of the main theorem

In what follows, we assume constant multiplicity and algebraic type for each shape operator. Let M_1^3 be a Lorentz hypersurface in E_1^4 satisfying condition (6), or equivalently relations (8) and (9). We will consider each case for the shape operator S separately.

The shape operator S has the canonical form (II). Suppose that H is not constant.

Since H is not constant $\nabla H \neq \vec{0}$. As the shape operator has the canonical form (II) (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$), then $S(e_1) = \lambda e_1 + e_2$, $S(e_2) = \lambda e_2$, and $S(e_3) = \lambda_3 e_3$. Therefore, by using (8), we conclude that ∇H can be considered either in the direction of e_3, or in the direction of e_2. In the first case, ∇H is space-like (it cannot be time-like as $\langle e_3, e_3\rangle = 1$), and $\lambda_3 = -\frac{3H}{2}$. In the second case, ∇H is light-like, and $\lambda = -\frac{3H}{2}$.

Proposition 1. *Let M_1^3 be a Lorentz hypersurface of the pseudo-Euclidean space E_1^4 satisfying* (6) *with shape operator of type* (II), *and ∇H be space-like. Then M_1^3 has constant mean curvature.*

Proof. We assume that H is not constant and we will end up to a contradiction. Since $\nabla H \neq \vec{0}$, the vectorial equation (8) shows that ∇H is an eigenvector of S with corresponding eigevalue $-\frac{3H}{2}$.

We write $\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k$, we take into account the action of S on the basis $\{e_1, e_2, e_3\}$, and use the Codazzi equations (4). Then the following relations

$$\begin{aligned}
&\langle(\nabla_{e_1} S)e_2, e_1\rangle = \langle(\nabla_{e_2} S)e_1, e_1\rangle, && \langle(\nabla_{e_2} S)e_3, e_3\rangle = \langle(\nabla_{e_3} S)e_2, e_3\rangle,\\
&\langle(\nabla_{e_1} S)e_3, e_3\rangle = \langle(\nabla_{e_3} S)e_1, e_3\rangle, && \langle(\nabla_{e_2} S)e_3, e_2\rangle = \langle(\nabla_{e_3} S)e_2, e_2\rangle,\\
&\langle(\nabla_{e_1} S)e_2, e_3\rangle = \langle(\nabla_{e_2} S)e_1, e_3\rangle, && \langle(\nabla_{e_1} S)e_3, e_2\rangle = \langle(\nabla_{e_3} S)e_1, e_2\rangle,\\
&\langle(\nabla_{e_2} S)e_3, e_1\rangle = \langle(\nabla_{e_3} S)e_2, e_1\rangle
\end{aligned}$$

imply that $\omega_{21}^1 = \omega_{22}^2$, $\omega_{32}^3 = \omega_{31}^3 = \omega_{23}^1 = 0$, $\omega_{12}^3 = \omega_{21}^3$, $e_3(\lambda) = (\lambda_3 - \lambda)\omega_{13}^1$, $e_3(\lambda) = (\lambda_3 - \lambda)\omega_{23}^2$. From the last two equations we obtain that $\omega_{13}^1 = \omega_{23}^2$, as from $\operatorname{tr} S = 3H = 2\lambda + \lambda_3$, it follows that $\lambda = \frac{3H}{4} \neq \lambda_3$.

Further, the conditions

$$\nabla_{e_p}\langle e_1, e_1\rangle = \nabla_{e_p}\langle e_2, e_2\rangle = \nabla_{e_p}\langle e_3, e_3\rangle = \nabla_{e_p}\langle e_1, e_3\rangle = \nabla_{e_p}\langle e_2, e_3\rangle = 0$$

for $p = 1, 2, 3$ imply that $\omega_{p1}^2 = \omega_{p2}^1 = \omega_{p3}^3 = 0$, and $\omega_{p1}^3 = -\omega_{p3}^2$, $\omega_{p2}^3 = -\omega_{p3}^1$. As a consequence, we also obtain that $\omega_{33}^1 = \omega_{33}^2 = \omega_{22}^3 = 0$. Therefore, the covariant derivatives $\nabla_{e_i} e_j$ simplify to the following:

$$\begin{aligned}
&\nabla_{e_1} e_1 = \omega_{11}^1 e_1, \qquad \nabla_{e_1} e_2 = \omega_{12}^2 e_2 + \omega_{12}^3 e_3, \qquad \nabla_{e_1} e_3 = \omega_{13}^1 e_1 + \omega_{13}^2 e_2,\\
&\nabla_{e_2} e_1 = \omega_{21}^3 e_3, \qquad \nabla_{e_2} e_2 = 0, \qquad \nabla_{e_2} e_3 = \omega_{23}^2 e_2,\\
&\nabla_{e_3} e_1 = 0, \qquad \nabla_{e_3} e_2 = \omega_{32}^2 e_2, \qquad \nabla_{e_3} e_3 = 0.
\end{aligned}$$

Next, we construct an orthonormal basis $\{X_1, X_2, X_3\}$ from the pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ such that

$$X_1 = \frac{e_1 + e_2}{\sqrt{2}}, \qquad X_2 = \frac{e_1 - e_2}{\sqrt{2}}, \qquad X_3 = e_3.$$

Then the shape operator S with respect to this new basis takes the form

$$S = \begin{pmatrix} \lambda + \frac{1}{2} & \frac{1}{2} & 0\\ -\frac{1}{2} & \lambda - \frac{1}{2} & 0\\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Note that X_3 is still in the direction of ∇H, and that $\lambda_3 = -\frac{3H}{2}$. Therefore, since $\nabla(H) = X_1(H)X_1 + X_2(H)X_2 + X_3(H)X_3$, then

$$X_1(H) = X_2(H) = 0, \qquad X_3(H) \neq 0. \tag{11}$$

Since M_1^3 is a Lorentz hypersurface, $\operatorname{tr} S = 3H$, $\lambda = \frac{9H}{4}$, and $\operatorname{tr} S^2 = \frac{99H^2}{8}$. By expressing the Laplace operator (10) in terms of the basis $\{X_1, X_2, X_3\}$, equation (9) reduces to

$$\begin{aligned} -\big(X_1X_1(H) - \nabla_{X_1}X_1(H)\big) + \big(X_2X_2(H) - \nabla_{X_2}X_2(H)\big) \\ -\big(X_3X_3(H) - \nabla_{X_3}X_3(H)\big) + H\left(\frac{99H^2}{8}\right) = \alpha H, \end{aligned}$$

which by use of (11) becomes

$$\nabla_{X_1}X_1(H) - \nabla_{X_2}X_2(H) - e_3e_3(H) + \frac{99H^3}{8} = \alpha H. \tag{12}$$

On the other hand, an easy computation shows that

$$\nabla_{X_1}X_1 = \frac{1}{2}[\omega_{11}^1 e_1 + \omega_{11}^3 e_3 + \omega_{12}^2 e_2 + \omega_{12}^3 e_3 + \omega_{21}^3 e_3]$$

and similarly for $\nabla_{X_2}X_2$, thus obtaining

$$\begin{aligned} \nabla_{X_1}X_1(H) &= \frac{1}{2}[\omega_{11}^3 + \omega_{12}^3 + \omega_{21}^3]e_3(H) \quad \text{and} \\ \nabla_{X_2}X_2(H) &= \frac{1}{2}[\omega_{11}^3 - \omega_{12}^3 - \omega_{21}^3]e_3(H). \end{aligned}$$

Hence, equation (12) simplifies to

$$e_3e_3(H) - 2\omega_{12}^3 e_3(H) - \frac{99H^3}{8} = \alpha H. \tag{13}$$

Substituting $\lambda = \frac{9H}{4}$ into $e_3(\lambda) = (\lambda_3 - \lambda)\omega_{13}^1$, we obtain

$$e_3(H) = -\frac{5H}{3}\omega_{13}^1 = \frac{5H}{3}\omega_{12}^3. \tag{14}$$

We evaluate Gauss equation (5) for $\langle R(e_3, e_1)e_2, e_3\rangle$ and equate the left-hand side by using the definition of the curvature tensor to obtain

$$e_3(\omega_{12}^3) = (\omega_{12}^3)^2 - \frac{27H^2}{8}. \tag{15}$$

Applying e_3 on both sides of equation (14) and using (15) we get

$$e_3e_3(H) = \frac{40H}{9}(\omega_{12}^3)^2 - \frac{45H^3}{8}.$$

Substituting this equation to (13) and by use of (14), we obtain

$$\frac{10}{9}(\omega_{12}^3)^2 - 9H^2 = \alpha. \tag{16}$$

Acting now with e_3 on (16) and using expressions (14) and (15) we simultaneously obtain that

$$\frac{20}{9}(\omega_{12}^3)^2 - \frac{225H^2}{6} = 0.$$

Therefore, H must be constant. □

PROPOSITION 2. *Let M_1^3 be a Lorentz hypersurface of the pseudo-Euclidean space E_1^4 with shape operator of type (II) satisfying (6), and ∇H be light-like. Then M_1^3 is minimal.*

Proof. By hypothesis ∇H is along the vector e_2, and $\lambda = -\frac{3H}{2}$. Since $\operatorname{tr} S = 3H$ then $\lambda_3 = 6H$. As the basis $\{e_1, e_2, e_3\}$ is pseudo-orthonormal, it follows that $\nabla(H) = e_2(H)e_1 + e_1(H)e_2 + e_3(H)e_3$. Therefore,

$$e_2(H) = e_3(H) = 0, \qquad e_1(H) \neq 0. \tag{17}$$

By writing $\nabla_{e_i} e_j = \sum_{k=1}^3 \omega_{ij}^k e_k$, we obtain that

$$\begin{aligned} 0 = \nabla_{e_i}\langle e_j, e_k\rangle &= \langle \nabla_{e_i} e_j, e_k\rangle + \langle e_j, \nabla_{e_i} e_k\rangle \\ &= \omega_{ij}^1\langle e_1, e_k\rangle + \omega_{ij}^2\langle e_2, e_k\rangle + \omega_{ij}^3\langle e_3, e_k\rangle \\ &\quad + \omega_{ik}^1\langle e_j, e_1\rangle + \omega_{ik}^2\langle e_j, e_2\rangle + \omega_{ik}^3\langle e_j, e_3\rangle. \end{aligned}$$

By assigning i, j, k any values from $\{1, 2, 3\}$, certain of the ω_{ij}^k vanish, and others satisfy simple relations. In particular we obtain:

$$\nabla_{e_1} e_3 = -\omega_{12}^3 e_1 + \omega_{13}^2 e_2, \qquad \nabla_{e_3} e_1 = \omega_{31}^1 e_1 + \omega_{31}^3 e_3, \tag{18}$$

$$\nabla_{e_2} e_3 = -\omega_{22}^3 e_1 - \omega_{21}^3 e_2, \qquad \nabla_{e_3} e_2 = -\omega_{31}^1 e_2 + \omega_{32}^3 e_3. \tag{19}$$

Using relations (17) we get that $[e_2, e_3](H) = e_2 e_3(H) - e_3 e_2(H) = 0$. Also, since $[e_2, e_3](H) = \nabla_{e_2} e_3(H) - \nabla_{e_3} e_2(H)$ it follows that $\omega_{22}^3 = 0$, so relations (19) simplify to

$$\nabla_{e_2} e_3 = -\omega_{21}^3 e_2, \qquad \nabla_{e_3} e_2 = -\omega_{31}^1 e_2 + \omega_{32}^3 e_3. \tag{20}$$

We use the Codazzi equations to obtain that

$$\langle (\nabla_{e_1} S)e_3, e_3\rangle = \langle (\nabla_{e_3} S)e_1, e_3\rangle, \qquad \langle (\nabla_{e_2} S)e_3, e_3\rangle = \langle (\nabla_{e_3} S)e_2, e_3\rangle,$$

which, by using (18) and (20), imply that

$$e_1(\lambda_3) = \omega_{32}^3 \quad \text{and} \tag{21}$$

$$e_2(\lambda_3) = (\lambda - \lambda_3)\omega_{32}^3, \tag{22}$$

respectively. Using (17) and that $\lambda_3 = 6H$, relation (22) implies that $(\lambda - \lambda_3)\omega_{32}^3 = 0$. If $\omega_{32}^3 = 0$, then from (21) it follows that $e_1(\lambda_3) = 0$, which contradicts (17). If $\lambda = \lambda_3$, then $-\frac{3H}{2} = 6H$, i.e. $H = 0$. □

The shape operator S has the canonical form (III). Suppose that H is not constant.

Then $\nabla H \neq \vec{0}$, and the vectorial equation (8) shows that ∇H is an eigenvector of S with corresponding eigenvalue $-\frac{3H}{2}$. Since the shape operator has the canonical form (III) (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$), then $S(e_1) = \lambda e_1 + e_3$, $S(e_2) = \lambda e_2$, and $S(e_3) = e_2 + \lambda e_3$ (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$). Hence, ∇H is in the direction of e_2, i.e., it is light-like, and $\lambda = -\frac{3H}{2}$. We will prove the following:

PROPOSITION 3. *Let M_1^3 be a Lorentz hypersurface of the pseudo-Euclidean space E_1^4 satisfying* (6), *with shape operator of type* (III) *and ∇H be light-like. Then M_1^3 is minimal.*

Proof. The shape operator S, with respect to the orthonormal basis $\{X_1, X_2, X_3\}$ of $T_pM_1^3$ considered in Proposition 1, takes the form

$$S = \begin{pmatrix} \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \lambda \end{pmatrix}.$$

Since $\operatorname{tr} S = 3H$, it follows that $3\lambda = -\frac{9H}{2} = 3H$, so $H = 0$. □

The shape operator S has the canonical form (IV). Let H be nonconstant.

Since the shape operator, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$, has the canonical form (IV), then $S(e_1) = \mu e_1 + \nu e_2$, $S(e_2) = -\nu e_1 + \mu e_2$, and $S(e_3) = \lambda_3 e_3$. This means that ∇H is in the direction of e_3, i.e., it is space-like.

The following proposition is proved along the same lines as Proposition 1.

PROPOSITION 4. *Let M_1^3 be a Lorentz hypersurface of the pseudo-Euclidean space E_1^4, satisfying* (6), *with shape operator of type* (IV) *and ∇H be space-like. Then M_1^3 has constant mean curvature.*

Proof. We assume that H is not constant and we will end up to a contradiction. Then $\nabla H \neq \vec{0}$ and the vectorial equation (8) shows that ∇H is an eigenvector of S with corresponding eigenvalue $-\frac{3H}{2}$. Then $\lambda_3 = -\frac{3H}{2}$, and

$$e_1(H) = e_2(H) = 0, \qquad e_3(H) \neq 0.$$

From the equation $\operatorname{tr} S = 3H$, it follows that $\mu = \frac{9H}{4}$. Next, we try to obtain simplified expressions for $\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k$. We apply the Codazzi equations (4) for

$$\langle (\nabla_{e_1} S)e_3, e_1 \rangle, \qquad \langle (\nabla_{e_2} S)e_3, e_2 \rangle, \qquad \langle (\nabla_{e_1} S)e_3, e_2 \rangle,$$
$$\langle (\nabla_{e_1} S)e_3, e_3 \rangle, \qquad \langle (\nabla_{e_2} S)e_3, e_3 \rangle$$

and obtain that

$$e_3(H) = -\frac{5H}{3}\omega_{13}^1, \qquad e_3(H) = -\frac{5H}{3}\omega_{23}^2, \qquad e_3(\nu) = -\nu\omega_{13}^1,$$
$$\frac{15H}{4}\omega_{31}^3 + \nu\omega_{32}^3 = 0, \qquad \frac{15H}{4}\omega_{32}^3 - \nu\omega_{31}^3 = 0,$$

respectively. Therefore, $\omega_{13}^1 = \omega_{23}^2$, and since H and ν are not zero, $\omega_{31}^3 = \omega_{32}^3 = 0$. Taking into account the condition $\omega_{ij}^k = -\epsilon_j\epsilon_k\omega_{ik}^j$, the previous relations give $\omega_{33}^1 = \omega_{33}^2 = 0$. Finally, since $[e_1, e_2](H) = 0$, it follows that $\nabla_{e_1}e_2(H) - \nabla_{e_2}e_1(H) = 0$, thus $\omega_{12}^3 = \omega_{21}^3 = 0$.

Next, we use Gauss equation (5) and the definition of the curvature tensor for $\langle R(e_1, e_3)e_1, e_3\rangle$ and $\langle R(e_3, e_2)e_3, e_2\rangle$ to obtain

$$e_3(\omega_{11}^3) = -(\omega_{13}^1)^2 + \frac{27H^2}{8} \quad \text{and} \quad e_3(\omega_{23}^2) = -(\omega_{23}^2)^2 + \frac{27H^2}{8}. \tag{23}$$

Hence, in view of (10), and taking into account the relations $\omega_{11}^3 = -\epsilon_1\epsilon_3\omega_{13}^1 = \omega_{13}^1$, $\omega_{22}^3 = -\epsilon_2\epsilon_3\omega_{23}^2 = -\omega_{23}^2$, and $\omega_{13}^1 = \omega_{23}^2$, equation (9) reduces to

$$e_3e_3(H) + 2\omega_{13}^1 e_3(H) - H\left(\frac{99H^2}{8} - 2\nu^2\right) = \alpha H. \tag{24}$$

Applying e_3 on both sides of equation $e_3(H) = -\frac{5H}{3}\omega_{13}^1$, and using (23) we get

$$e_3e_3(H) = \frac{40H}{9}(\omega_{13}^1)^2 - \frac{45H^3}{8},$$

so equation (24) becomes

$$\frac{10}{9}(\omega_{13}^1)^2 + 2\nu^2 - 18H^2 = \alpha. \tag{25}$$

Acting with e_3 on (25), we obtain

$$\frac{20}{9}(\omega_{13}^1)^2 + 4\nu^2 - \frac{135H^2}{2} = 0.$$

Then the two last equations imply that H must be constant which is a contradiction. □

The theorem stated in the Introduction now follows from Propositions 1, 2, 3, and 4.

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