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HOMOGENEOUS STRUCTURES ON REAL AND COMPLEX HYPERBOLIC SPACES

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Abstract. The connected groups acting by isometries on either the real or the complex hyperbolic spaces are determined. A Lie-theoretic description of the homogeneous Riemannian, respectively Kähler, structures of linear type on these spaces is then found. On both spaces, examples that are not of linear type are given.

1. Introduction

A tensorial approach to homogeneous Riemannian manifolds was introduced by Ambrose and Singer [2]. Tricerri and Vanhecke [16] studied these ideas in depth and decomposed the space $\mathcal{T}$ of such tensors into three components $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ (direct sum). The space $\mathcal{T}_1$ is characterised by the fact that it is the space of sections of a vector-bundle whose fibre dimension grows linearly with that of the base manifold.

Substantial results on homogeneous Riemannian structures on the real hyperbolic space and its related homogeneous descriptions have been obtained by Tricerri and Vanhecke (e.g., [16]), Pastore ([11, Th. 2], [13, Sect. 3], [12, Sect. 3]) and Pastore and Verrocca [14, Props. 2.2, 3.1]. One main result of [16] is that nontrivial homogeneous structures in $\mathcal{T}_1$ can only be realized on the real hyperbolic space $\mathbb{R}\mathrm{H}(n)$.

Subsequently, similar results were obtained ([6], [5]) for the complex and quaternionic cases. One considers homogeneous Kähler or homogeneous quaternionic Kähler manifolds, and then one examines the similar decomposition giving spaces analogous to $\mathcal{T}_1$. In these cases, one finds several subspaces with the linear growth property. It is proved that if a nontrivial homogeneous

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structure is of linear type, i.e., belongs to the sum of these spaces, then the geometry is that of the complex, respectively quaternionic, hyperbolic space, and that the tensor is of a special type.

As is well-known, the same underlying Riemannian manifold M can admit many homogeneous tensors. Different tensors may describe M as different homogeneous spaces G/H. One open question in [16, p. 55] was the determination of all homogeneous structures on $\mathbb{R}\mathrm{H}(n)$ for $n \geq 3$. In the present paper, we will demonstrate how results of Witte [17] can be used to write down all the pairs G and H which can then in principle be used to determine all the possible corresponding tensors. The zero tensor corresponds to G being the full (connected) isometry group $\mathrm{SO}(n+1)$ (as is the case for any symmetric space), whereas Tricerri and Vanhecke proved that the tensors of type $\mathcal{T}_1$ come from the description of $\mathbb{R}\mathrm{H}(n)$ as a solvable manifold.

The same idea was used in [5] to tackle the quaternionic case and to describe the isometry group when the tensor is of linear type. In this paper, we go on to show how the techniques apply to the situation for $\mathbb{R}\mathrm{H}(n)$ and $\mathbb{C}\mathrm{H}(n)$. In the latter case, this provides a purely Lie-theoretic approach to the construction of the structure of linear type found in [6].

2. Preliminaries

2.1. Some conventions. Throughout the paper, sums of vector spaces, algebras, bundles, etc. are direct. We will denote by $\Lambda^{1,0}$ the standard representation of the unitary group $U(n)$ on $\mathbb{C}^n$, and by $[\![\Lambda^{1,0}]\!]$ the corresponding real representation.

We shall follow Tricerri–Vanhecke's conventions for the curvature tensor of a Riemannian manifold and that of $\mathbb{R}\mathrm{H}(n)$:

$$\begin{aligned} R_{XY}Z &= \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z, \\ R^{\mathbb{R}\mathrm{H}(n)}_{XY}Z &= c\big(g(X,Z)Y - g(Y,Z)X\big), \quad c<0. \end{aligned} \tag{2.1}$$

Similarly, we take

$$\begin{aligned} R^{\mathbb{C}\mathrm{H}(n)}_{XY}Z = \frac{c}{4}\big(&g(X,Z)Y - g(Y,Z)X \\ &+ g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\big), \quad c<0. \end{aligned} \tag{2.2}$$

2.2. Homogeneous Riemannian and Kähler structures. Let (M,g) be a connected, simply-connected, complete Riemannian manifold. Ambrose and Singer [2] gave a characterisation for (M,g) to be homogeneous in terms of a $(1,2)$ tensor field S, usually called a homogeneous Riemannian structure (Tricerri and Vanhecke [16]). If ∇ denotes the Levi–Civita connection and R its curvature tensor, then one introduces the torsion connection $\widetilde{\nabla} = \nabla - S$ which satisfies the Ambrose–Singer equations

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0.$$

The manifold (M,g) above admits a homogeneous Riemannian structure if and only if it is a homogeneous Riemannian manifold.

In particular, the necessary condition is given as follows. Fix a point $p\in M$ and let $\mathfrak{m}=T_pM$. Writing $\widetilde{R}$ for the curvature tensor of $\widetilde{\nabla}$, we can consider the holonomy algebra $\mathfrak{h}$ of $\widetilde{\nabla}$ as the Lie subalgebra of skew-symmetric endomorphisms of $(\mathfrak{m},g_p)$ generated by the operators $\widetilde{R}_{XY}$, where $X,Y\in\mathfrak{m}$. Then, according to Nomizu [10] (see also [2], [16]), a Lie bracket is defined on the vector space

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m} \tag{2.3}$$

by

$$\begin{cases} [U,V]=UV-VU, & U,V\in\mathfrak{h},\\ [U,X]=U(X), & U\in\mathfrak{h}, X\in\mathfrak{m},\\ [X,Y]=S_XY-S_YX+\widetilde{R}_{XY}, & X,Y\in\mathfrak{m}. \end{cases} \tag{2.4}$$

One calls $(\mathfrak{g},\mathfrak{h})$ the reductive pair associated to the homogeneous Riemannian structure S. The connected, simply-connected Lie group $\widetilde{G}$ whose Lie algebra is $\mathfrak{g}$ acts transitively on M via isometries. The kernel Γ of this action is a discrete normal subgroup of $\widetilde{G}$, and $G=\widetilde{G}/\Gamma$ acts effectively on M. The stabilizer $H=\operatorname{stab}_G(p)$ is a connected subgroup with Lie algebra $\mathfrak{h}$. Thus, $M\equiv G/H$ and the Riemannian metric g corresponds to an invariant metric on G/H.

Homogeneous Riemannian structures are sections of $T^*M\otimes\mathfrak{so}(M)$, where $\mathfrak{so}(M)$ is the bundle of endomorphisms that preserve the metric g infinitesimally, i.e., $\mathfrak{so}(M)$ consists of A such that $g(AX,Y)+g(X,AY)=0$ for all vector fields X,Y. Decomposing under the action of $O(n)$, Tricerri and Vanhecke [16] showed that there are 8 classes of homogeneous Riemannian structures. The three primitive classes are denoted by $\mathcal{T}_1\cong\Gamma(TM)$, $\mathcal{T}_2$ and $\mathcal{T}_3\cong\Gamma(\Lambda^3TM)$; the class of linear type is $\mathcal{T}_1$. We write simply $\mathcal{T}_{i+j}$ for the class $\mathcal{T}_i+\mathcal{T}_j$. Note that a homogeneous structure that belongs to $\mathcal{T}_i$ at some point of M has the same type at all other points.

One recovers S from (2.3) as follows. The Levi–Civita connection ∇ is given (Besse [4, p. 183]) by

$$2g(\nabla_{B^*}C^*,D^*)=-\{g([B,C]^*,D^*)+g(B^*,[C,D]^*)+g(C^*,[B,D]^*)\},$$

where for $B\in\mathfrak{g}$, B^* denotes the vector field with one-parameter group $g\mapsto\exp(tB)g$ ($g\in G$, $t\in\mathbb{R}$). Note that $[B^*,C^*]=-[B,C]^*$. The homogeneous tensor is now given by $S=\nabla-\widetilde{\nabla}$, where $\widetilde{\nabla}$ is the canonical connection $\widetilde{\nabla}$. The latter is uniquely determined [16, p. 20] by its value at $eH\in G/H$, where one has $\widetilde{\nabla}_{B^*}C^*=-[B,C]^*_{\mathfrak{m}}$. Indeed $\widetilde{\nabla}$ is the connection for which every left-invariant tensor on G/H is parallel [9, p. 192]. Working at eH, we now have

$$2g(S_B C, D) = g([B,C],D) - g([C,D],B) + g([D,B],C). \tag{2.5}$$

REMARK 2.1. Two homogeneous structures S_1, S_2 are equivalent if there is an isometry φ of (M,g) such that $S_2 = \varphi_*^{-1}\varphi^* S_1$. In the Ambrose–Singer picture, this corresponds to the existence of a Lie algebra homomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ mapping $\mathfrak{h}_1 \to \mathfrak{h}_2$ and $\mathfrak{m}_1 \to \mathfrak{m}_2$, with $\psi|_{\mathfrak{m}_1}$ a linear isometry [16, Theorem 2.4].

If (M,g,J) is a Kähler manifold, the isometries considered in the definition are also required to preserve the complex structure J and this leads to the condition $\widetilde{\nabla} J = 0$. In this case, homogeneous Kähler structures were classified by Abbena and Garbiero [1] into 16 classes, corresponding to spaces invariant under the action of $U(n)$. Using work of Sekigawa [15], one considers the bundle $T^*M \otimes \mathfrak{u}(M)$ where, with the usual notations, $\mathfrak{u}(M) = \{A \in \mathfrak{so}(M) : AJ = JA\}$. The four primitive classes are denoted by $\mathcal{K}_1, \dots, \mathcal{K}_4$, and, denoting $\mathcal{K}_i + \mathcal{K}_j$ simply by $\mathcal{K}_{i+j}$, the class of linear type is $\mathcal{K}_{2+4} \cong \Gamma(TM + TM)$.

2.3. Witte's Theorem on cocompact groups. We consider transitive (isometric) actions on noncompact Riemannian symmetric spaces M. For this section, let G be the component of the identity of the isometry group of M, and assume that G is semi-simple. Then $M = G/K$ with K compact. We are particularly interested in, $\mathbb{R}\mathrm{H}(n) = \mathrm{SO}(n,1)/O(n)$ and $\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/S(U(n) \times U(1))$. A group T acts transitively on M only if $T\backslash G/K$ is a point. Since K is compact, this implies that T is a nondiscrete cocompact subgroup of the semi-simple group G.

Witte [17] gives a classification of nondiscrete cocompact subgroups for a connected, semisimple Lie group G with finite centre as follows (cf. Goto and Wang [8]). Start with a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of G [7, pp. 190–192]. Decompose $\mathfrak{g}$ with respect to the action of $\mathfrak{a}$ as $\mathfrak{g} = \mathfrak{g}_0 + \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda$, where Σ is the set of roots corresponding to $\mathfrak{a}$. Choose a system Θ of simple roots in Σ. Write Σ^+ for the set of positive roots with respect to Θ and let Ψ be a subset of Θ. Let $[\Psi]$ denote the set of roots in Σ that are linear combinations of elements of Ψ. A standard parabolic subalgebra $\mathfrak{p}(\Psi)$ of $\mathfrak{g}$ is defined by $\mathfrak{p}(\Psi) = \mathfrak{l}(\Psi) + \mathfrak{n}(\Psi)$, where $\mathfrak{l}(\Psi) = \mathfrak{g}_0 + \sum_{\lambda\in[\Psi]} \mathfrak{g}_\lambda$ and $\mathfrak{n}(\Psi) = \sum_{\mu\in\Sigma^+\setminus[\Psi]} \mathfrak{g}_\mu$, are respectively, reductive and nilpotent. The first can be decomposed as $\mathfrak{l}(\Psi) = \mathfrak{l} + \mathfrak{e} + \mathfrak{a}(\Psi)$, with $\mathfrak{l}$ semi-simple with all factors of noncompact type, $\mathfrak{e}$ compact reductive, and $\mathfrak{a}$ the noncompact part of the centre of $\mathfrak{l}(\Psi)$. The decomposition $P(\Psi)^0 = LEAN$,

$$\mathfrak{p}(\Psi) = \mathfrak{l} + \mathfrak{e} + \mathfrak{a}(\Psi) + \mathfrak{n}(\Psi),$$

is referred to as the refined Langlands decomposition of the parabolic subgroup $P(\Psi)$ in [17] (cf. [8]), and one has the following theorem.

THEOREM 2.2 (Witte [17]). *Let L_r be a connected normal subgroup of L and F_r a connected closed subgroup of EA. Then there is a closed cocompact subgroup of G contained in $P(\Psi)$ with identity component L_rF_rN. Moreover, every nondiscrete cocompact subgroup of G arises in this way.*

3. Real hyperbolic space

We consider now the real hyperbolic space $\mathbb{R}\mathrm{H}(n)$, $n>1$, with curvature (2.1) (cf. Tricerri and Vanhecke [16, Ch. 5]).

The usual homogeneous description of $\mathbb{R}\mathrm{H}(n)$ is as $\mathbb{R}\mathrm{H}(n)=\mathrm{SO}(n,1)/O(n)$, where $\mathrm{SO}(n,1)$ is its full group of isometries. In this case, the homogeneous tensor S vanishes and the manifold is symmetric.

To discuss the other homogeneous structures, we take $\mathrm{SO}(n,1)$ as the set of determinant 1 matrices preserving the bilinear form $\mathrm{diag}(\mathrm{Id}_{n-1}, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$. The Iwasawa decomposition is then $\mathrm{SO}(n,1)=O(n)AN$, with

$$\begin{aligned}\mathfrak{so}(n)&=\left\{\begin{pmatrix}B&v&v\\-v^T&0&0\\-v^T&0&0\end{pmatrix} : B\in\mathfrak{so}(n-1), v\in\mathbb{R}^{n-1}\right\},\\ \mathfrak{a}=\mathbb{R}A_0,\qquad \mathfrak{n}&=\left\{\begin{pmatrix}0&0&v\\-v^T&0&0\\0&0&0\end{pmatrix} : v\in\mathbb{R}^{n-1}\right\},\end{aligned}\tag{3.1}$$

where $A_0=\mathrm{diag}(0,\dots,0,1,-1)$.

3.1. The solvable description. The first alternative description of the real hyperbolic space $\mathbb{R}\mathrm{H}(n)$ is as the Lie group AN, which may be identified with $\mathbb{R}_{>0}\mathbb{R}^{n-1}$ and multiplication

$$(x_1,\dots,x_n)(y_1,\dots,y_n)=(x_1y_1,x_1y_2+x_2,\dots,x_1y_n+x_n).$$

The Lie algebra structure of $\mathfrak{a}+\mathfrak{n}$ is given by

$$\left[\begin{pmatrix}0&0&v\\-v^T&x&0\\0&0&-x\end{pmatrix},\begin{pmatrix}0&0&w\\-w^T&y&0\\0&0&-y\end{pmatrix}\right]=\begin{pmatrix}0&0&xw-yv\\yv^T-xw^T&0&0\\0&0&0\end{pmatrix}.$$

Let us show how a homogeneous structure $S\in\mathcal{T}_1$ corresponds to the homogeneous description of $\mathbb{R}\mathrm{H}(n)$ as the group AN. Note that Tricerri and Vanhecke [16] proved that the condition $S\in\mathcal{T}_1$ implies that (M,g) is isometric to $\mathbb{R}\mathrm{H}(n)$. With the notation of [16], we have that

$$S_XY=g(X,Y)\xi-g(\xi,Y)X$$

for some nonzero $\xi\in\mathfrak{X}(M)$. The torsion connection is $\widetilde{\nabla}=\nabla-S$. From the Ambrose–Singer equations (2.4), we have $\widetilde{\nabla}\xi=0$, so $\|\xi\|$ is constant, and

$$\widetilde{R}_{XY}Z=R_{XY}Z-R^S_{XY}Z,\tag{3.2}$$

where

$$R^S_{XY}Z = S_{S_XY - S_YX}Z - S_X(S_YZ) + S_Y(S_XZ).$$

Using the concrete expression for S, one gets

$$R^S_{XY}Z = -\|\xi\|^2\big(g(X,Z)Y - g(Y,Z)X\big) = c' R^{\mathbb{R}\mathrm{H}(n)}_{XY} Z,$$

where $c' = -\|\xi\|^2/c$ by (2.1). Since $\widetilde{\nabla}\xi = 0$, we have $\widetilde{R}\xi = 0$. But $\widetilde{R}\xi = (1-c')R^{\mathbb{R}\mathrm{H}(n)}\xi$ and $R^{\mathbb{R}\mathrm{H}(n)}$ has nonzero sectional curvature on all planes, so $c' = 1$, $\|\xi\|^2 = -c$ and $\widetilde{R} \equiv 0$. Hence, the holonomy of $\widetilde{\nabla}$ is trivial and this structure of type $\mathcal{T}_1$ thus gives a description of $\mathbb{R}\mathrm{H}(n)$ as a group. According to (2.4), the Lie algebra structure is given by

$$[X,Y] = -g(\xi,Y)X + g(\xi,X)Y,$$

so $[\xi,X] = \|\xi\|^2X$, $[X,Y] = 0$, for $X,Y \in \xi^\perp$.

Conversely, consider the group AN. Note that $\mathfrak{a} + \mathfrak{n}$ has $\mathfrak{n}$ as its derived algebra, at that the elements acting as $+1$ on $\mathfrak{n}$ are of the form $A_0 + X$. Using the equivalence of Remark 2.1, we may thus assume that the splitting $\mathfrak{a} + \mathfrak{n}$ is orthogonal and $g(V,V) = \|v\|^2$, where

$$V = \begin{pmatrix} 0 & 0 & v \\ -v^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $k = g(A_0, A_0)$. Then $g([B,C],D) = g(B,\xi)g(C,D) - g(C,\xi)g(B,D)$, with $\xi = A_0/\sqrt{k}$. Using (2.5), we get

$$g(S_BC,D) = g(D,\xi)g(B,C) - g(C,\xi)g(B,D),$$

which is of class $\mathcal{T}_1$ and g is real hyperbolic with $c = -1/k$, since $R = R^S$. We thus see that homogeneous structures of type $\mathcal{T}_1$ correspond to the homogeneous description $\mathbb{R}\mathrm{H}(n) = AN$, as claimed in Tricerri and Vanhecke [16]. Furthermore, these are the only homogeneous Riemannian structures carried by AN.

3.2. Other homogeneous descriptions. To describe the other homogeneous structures on $\mathbb{R}\mathrm{H}(n)$, we need Witte's Theorem 2.2. Up to conjugation, the only maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{so}(n,1)$ is $\mathfrak{a} = \mathrm{span}\{A_0\}$. Its set of roots is $\{\pm\lambda\}$, $\lambda(A_0) = 1$, and $\Theta = \{\lambda\}$ is a system of simple roots. There are only two choices for Ψ, either empty or equal to all of Θ. The corresponding refined Langlands decompositions read

$$\mathfrak{p}(\Theta) = \mathfrak{so}(n,1) + \{0\} + \{0\} + \{0\}, \qquad \mathfrak{p}(\varnothing) = \{0\} + \mathfrak{so}(n-1) + \mathfrak{a} + \mathfrak{n},$$

where $\mathfrak{n}$ is as in (3.1). For the first parabolic subalgebra, we have that the cocompact subgroup is either all of $\mathrm{SO}(n,1)$ or discrete. As for the second decomposition, we consider connected subgroups F_r of $EA = \mathrm{SO}(n-1)\mathbb{R}$. As $E \le K$, the group $G = F_rN$ acts transitively on $\mathbb{R}\mathrm{H}(n) = \mathrm{SO}(n,1)/K$ if and only if the projection $F_r \to EA = \mathrm{SO}(n-1)\mathbb{R} \to A = \mathbb{R}$ is surjective. We thus have the following theorem.

THEOREM 3.1. *The connected groups acting transitively on* $\mathbb{R}\mathrm{H}(n)$ *are the full isometry group* $\mathrm{SO}(n,1)$ *and the groups* $G = F_r N$, *where* N *is the nilpotent factor in the Iwasawa decomposition of* $\mathrm{SO}(n,1)$ *and* F_r *is a connected closed subgroup of* $\mathrm{SO}(n-1)\mathbb{R}$ *with nontrivial projection to* $\mathbb{R}$.

Given a group G acting transitively on $\mathbb{R}\mathrm{H}(n)$ with stabilizer H, determination of the corresponding tensor S depends on a choice of complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. Considering the maximal case with $\mathfrak{g}$, the normalizer $\mathfrak{so}(n-1)+\mathbb{R}+\mathfrak{n}$ of $\mathbb{R}+\mathfrak{n}$ and $\mathfrak{h} = \mathfrak{so}(n-1)$ there will be families of choices of complements $\mathfrak{m}$ in (2.3) and hence families of homogeneous structures if either

$$\text{(a)} \quad \mathfrak{so}(n-1) \cong \mathbb{R} \quad \text{or} \quad \text{(b)} \quad \mathfrak{so}(n-1) \cong \mathfrak{n}$$

as vector spaces. Case (a) occurs when $n=3$ and corresponds to the homogeneous description $\mathrm{SO}(2)AN/\mathrm{SO}(2)$. Case (b) occurs when $n=4$: $\mathfrak{so}(3) \cong \mathbb{R}^3$. Interestingly, this is exactly the case when $\mathbb{R}\mathrm{H}(4) = \mathbb{H}\mathrm{H}(1)$ (note that $\mathfrak{so}(3) = \mathrm{Im}\,\mathbb{H}$). This gives rise to Tricerri–Vanhecke's example [16, p. 89] of a $\mathcal{T}_{1+3}$ structure, see also Bérard–Bergery [3]. This may be seen as follows. For $\lambda \in \mathbb{R}$, let

$$\mathfrak{m}_\lambda = \left\{ \begin{pmatrix} \lambda(b\times) & b & b \\ -b^T & a & 0 \\ -b^T & 0 & -a \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{R}^3 \right\},$$

where

$$b\times = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

is the matrix given by the operation of taking the cross product with $b = (b_1, b_2, b_3)^T$ in $\mathbb{R}^3$. Now let A_0 be as before and let V_b be the typical element of $\mathfrak{m}_\lambda$ defined by $a=0$. Then $[A_0, V_b]_{\mathfrak{m}_\lambda} = V_b$, $[V_b, V_c]_{\mathfrak{m}_\lambda} = 2\lambda V_{b\times c}$, since the top right entry determines the projection to $\mathfrak{m}_\lambda \subset \mathfrak{so}(3) + \mathfrak{m}_\lambda$. For $B \in \mathfrak{m}_\lambda$, write $B = b_0 A_0 + V_b$, $b_0 \in \mathbb{R}$. Then $g([B,C],D) = b_0\langle c,d\rangle - c_0\langle b,d\rangle + 2\lambda \det(bcd)$, so

$$2g(S_B C, D) = -2c_0\langle b,d\rangle + 2d_0\langle b,c\rangle + 2\lambda\det(bcd).$$

The two first summands constitute a tensor of type $\mathcal{T}_1$, and the last summand one of type $\mathcal{T}_3$, as claimed.

Other families of homogeneous structures arise by taking a subgroup of $\mathrm{SO}(n-1)$ whose representation on $\mathbb{R}^{n-1}$ includes a copy of the adjoint representation. Thus, for any connected compact G, let $n = \dim G + 1$. Then G acts on $\mathbb{R}^{n-1} \cong \mathfrak{g}$ preserving the Killing form and hence preserving some inner product. This realizes G as a subgroup of $\mathrm{SO}(n-1)$ and the homogeneous space $\mathbb{R}\mathrm{H}(n) = GAN/G$ will have nontrivial choices of complements.

It is these nonstandard choices of complements that give rise to new homogeneous tensors S. For example, consider the solvable description $\mathbb{R}\mathrm{H}(n) = AN$. As a computation shows, the normalizer of AN in $G = KAN$ has Lie algebra $\mathfrak{so}(n-1) + \mathfrak{a} + \mathfrak{n}$. Extending AN to any subgroup of the normalizer means that we can still use $\mathfrak{a} + \mathfrak{n}$ as an ad-invariant complement, and the computation of S remains unchanged from Section 3.1.

4. Complex hyperbolic space

We now consider the complex hyperbolic space $\mathbb{C}\mathrm{H}(n)$ with constant holomorphic curvature c, see the convention (2.2).

4.1. Transitive actions. Viewed as the symmetric space $\mathrm{SU}(n,1)/S(U(n)\times U(1))$, the space $\mathbb{C}\mathrm{H}(n)$ has $S \equiv 0$. The group has an Iwasawa decomposition $\mathrm{SU}(n,1) = KAN$. As $\mathbb{C}\mathrm{H}(n) \equiv AN$, this gives a second homogeneous description of the quaternionic hyperbolic space, in this case as a Lie group.

To explore all possible groups acting transitively, we now compute the Iwasawa decomposition, considering $\mathrm{SU}(n,1)$ as the complex matrices that are unitary with respect to the bilinear form $B = \operatorname{diag}(\mathrm{Id}_{n-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$. The Lie algebra of $\mathrm{SU}(n,1)$ is then given by

$$\begin{aligned}\mathfrak{su}(n,1) &= \{C \in M_{n+1}(\mathbb{C}) : \overline{C}^T B + BC = 0, \operatorname{tr} C = 0\} \\ &= \left\{ \begin{pmatrix} \alpha & v_1 & v_2 \\ -\bar{v}_2^T & z & ib \\ -\bar{v}_1^T & ib' & -\bar{z} \end{pmatrix} : \begin{array}{l} \alpha \in \mathfrak{u}(n-1), v_1, v_2 \in \mathbb{C}^{n-1}, \\ z \in \mathbb{C}, b, b' \in \mathbb{R}, \\ z - \bar{z} + \operatorname{tr}\alpha = 0 \end{array} \right\}.\end{aligned}$$

For the Iwasawa decomposition, consider

$$\mathfrak{u}(n) = \left\{ \begin{pmatrix} \alpha & v & v \\ -\bar{v}^T & ia & ia \\ -\bar{v}^T & ia & ia \end{pmatrix} \right\}, \qquad \mathfrak{u}(1) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib & -ib \\ 0 & -ib & ib \end{pmatrix} \right\},$$

where $\alpha \in \mathfrak{u}(n-1)$, $a, b \in \mathbb{R}$. Then the Lie algebra of K is given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} \alpha & v & v \\ -\bar{v}^T & i(a+b) & i(a-b) \\ -\bar{v}^T & i(a-b) & i(a+b) \end{pmatrix} : \begin{array}{l} \alpha \in \mathfrak{u}(n-1), \\ v \in \mathbb{C}^{n-1}, a, b \in \mathbb{R}, \\ 2i(a+b) + \operatorname{tr}\alpha = 0 \end{array} \right\}.$$

That is, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1))$ and $K = S(U(n)U(1))$.

We now apply Witte's construction, Section 2.3. Up to conjugation, $\mathfrak{su}(n,1)$ contains a unique maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a} = \operatorname{span}_{\mathbb{R}}\{A_0\}$, with $A_0 = \operatorname{diag}(0,\dots,0,1,-1)$. The corresponding set of roots is $\Sigma = \{\pm\lambda, \pm 2\lambda\}$, where $\lambda(A_0) = 1$, and $\Theta = \{\lambda\}$ is a system of simple roots with positive root system $\Sigma^+ = \{\lambda, 2\lambda\}$. Then there are only two choices for Ψ, either empty or equal to all of Θ. The resulting parabolic subalgebras $\mathfrak{p}(\Psi)$ have the following refined Langlands decompositions:

$$\mathfrak{p}(\Theta) = \mathfrak{su}(n,1) + \{0\} + \{0\} + \{0\},$$

$$\mathfrak{p}(\varnothing) = \{0\} + \mathfrak{s}\big(\mathfrak{u}(n-1) + \mathfrak{u}(1)\big) + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2,$$

where

$$\mathfrak{s}\big(\mathfrak{u}(n-1)+\mathfrak{u}(1)\big) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & ia & 0 \\ 0 & 0 & ia \end{pmatrix} : \begin{array}{l} \alpha \in \mathfrak{u}(n-1), a \in \mathbb{R}, \\ 2ia + \operatorname{tr}\alpha = 0 \end{array} \right\}, \qquad \mathfrak{a} = \mathbb{R}A_0,$$

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0 & 0 & v \\ -\bar{v}^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \in \mathbb{C}^{n-1} \right\}, \qquad \mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ib \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{R} \right\},$$

the last being the $+1$ and $+2$-eigenspaces of $\operatorname{ad} A_0$. (The centralizer is $\mathfrak{g}_0 = \mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))+\mathfrak{a}$.) The Iwasawa decomposition is $\mathrm{SU}(n,1) = KAN$, where N has Lie algebra $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$.

For the first Langlands refined decomposition, Witte's Theorem 2.2 tells us that for a cocompact G then G^0 is either all of $\mathrm{SU}(n,1)$ or it is trivial. Thus, the only transitive action coming from $\Psi = \varnothing$ is that of the full isometry group $\mathrm{SU}(n,1)$ on $\mathbb{C}\mathrm{H}(n)$.

In the second case, given a subgroup F_r of $S(U(n-1)U(1))\mathbb{R}$ that is closed and connected, we get a corresponding cocompact subgroup. To get a transitive action on $\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/S(U(n)U(1))$, it is necessary and sufficient that the projection $F_r \to S(U(n-1)U(1))\mathbb{R} \to \mathbb{R}$ be surjective. According to Theorem 2.2, G is then F_rN. We thus have the following.

THEOREM 4.1. *The connected groups acting transitively on* $\mathbb{C}\mathrm{H}(n)$ *are the full isometry group* $\mathrm{SU}(n,1)$ *and the groups* $G = F_rN$, *where* N *is the nilpotent factor in the Iwasawa decomposition of* $\mathrm{SU}(n,1)$ *and* F_r *is a connected closed subgroup of* $S(U(n-1)U(1))\mathbb{R}$ *with nontrivial projection to* $\mathbb{R}$.

4.2. The solvable description is not of linear type. The simplest choice in Theorem 4.1 is $F_r = A$, this is then the description of $\mathbb{C}\mathrm{H}(n)$ as the solvable group AN. One may determine the homogeneous type for this solvable description as follows.

Set

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, for any generic element $V \in \mathfrak{n}_1$, we have

$$\begin{gathered} [A_0, X] = 2X, \qquad [A_0, V] = V, \qquad [X, V] = 0, \\ [V_1, V_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{v}_1^T v_2 + \bar{v}_2^T v_1 \\ 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{4.1}$$

Here, we see the solvable Lie algebra structure of $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2$.

To find Kähler structures, we first determine the possible invariant symplectic forms. Consider the dual splitting $\mathfrak{g}^* = \mathfrak{a}^* + \mathfrak{n}_1^* + \mathfrak{n}_2^*$. Let $a_0 \in \mathfrak{a}^*$ be the dual element to A_0, and let $x \in \mathfrak{n}_2^*$ be dual to X. Extending these to left-invariant forms on AN, we compute exterior derivatives via $da_0(B^*, C^*) = -a_0([B,C]^*)$. This gives, for any $v \in \mathfrak{n}_1^*$,

$$da_0 = 0, \qquad dv = -a_0 \wedge v, \qquad dx = -2(a_0 \wedge x + \omega_1),$$

where ω_1 is the nondegenerate two-form on $\mathfrak{n}_1$ determined by (4.1). It follows, that the invariant closed two-forms on AN are generated by $a_0 \wedge x + \omega_1$ and $a_0 \wedge \mathfrak{n}_1^*$. Therefore, any invariant symplectic form may be written as

$$\omega = \lambda\bigl(a_0 \wedge (x+v) + \omega_1\bigr) \tag{4.2}$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$ and $v \in \mathfrak{n}_1^*$.

If g is a left-invariant metric, let $\tilde{\mathfrak{n}}_1$ be the orthogonal complement of $\mathfrak{n}_2$ in $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$, and let $\tilde{\mathfrak{a}}$ be the orthogonal complement of $\mathfrak{n}_2$ in $\mathfrak{a} + \mathfrak{n}_2$. Then there is a Lie algebra isomorphism $\psi : \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2 \to \tilde{\mathfrak{a}} + \tilde{\mathfrak{n}}_1 + \mathfrak{n}_2$ respecting the direct sum decompositions. Replacing g by $\psi^* g$, we may assume $\mathfrak{n}_2$ is orthogonal to $\mathfrak{a} + \mathfrak{n}_1$, by Remark 2.1.

Now suppose we have a left-invariant Kähler structure (g, J, ω) on $G = AN$. Our convention is that $\omega(A,B) = g(A, JB)$. Then we may assume g satisfies the orthogonality of the previous paragraph and that ω is given by (4.2). Now $X^\flat = g(X,X)x$ and $JX^\flat = g(JX, \cdot) = \omega(\cdot, X) = \lambda a_0$. In particular, $a_0 \wedge x$ is of type $(1,1)$. As J is integrable, we have that $d(x + iJx)$ has no $(0,2)$-component. However, $d(x+iJx) = dx = -2(a_0 \wedge x + \omega_1)$ which is real and so must be of type $(1,1)$. Thus, $\omega_1 \in \Lambda^{1,1}$ and equation (4.2), then implies that $a_0 \wedge v \in \Lambda^{1,1}$ too. Concretely, $a_0 \wedge v = Ja_0 \wedge Jv$, but the latter is proportional to $x \wedge Jv$ which is only in $a_0 \wedge \mathfrak{n}_1^*$ when $v = 0$. We conclude that $v = 0$ in (4.2) and that the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2$ is orthogonal with $\mathfrak{n}_1$ J-invariant. Now $(\mathfrak{n}_1, J, \omega_1)$ is linearly isomorphic to the standard Kähler structure on $\mathbb{C}^{n-1}$, and this extends to a Lie algebra automorphism of $\mathfrak{g}$, so J is equivalent to $\mathbf{v} \mapsto i\mathbf{v}$ on

$$V = \begin{pmatrix} 0 & 0 & \mathbf{v} \\ -\bar{\mathbf{v}}^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We may thus assume $g(V,V) = \lambda \|\mathbf{v}\|^2$, and putting $g(A_0, A_0) = \mu$ we find $g(X,X) = 1/\mu$ and that we have a Kähler structure. Now computing S as in Section 3 and taking $\xi = A_0/\sqrt{\mu}$ gives (at p)

$$\begin{aligned} g(S_B C, D) = \mu^{-1/2}\bigl(&g(B,C)g(D,\xi) - g(B,D)g(C,\xi) \\ &+ g(B,JC)g(JD,\xi) - g(B,JD)g(JC,\xi) \\ &- g(C,JD)g(JB,\xi) \\ &+ g(JB,\xi)\{g(C,\xi)g(JD,\xi) - g(JC,\xi)g(D,\xi)\}\bigr). \end{aligned} \tag{4.3}$$

The first and second lines are a tensor in $\mathcal{K}_{2+4}$ (for $\theta_1 = \theta_2$ in the notation of [6]). The third line is also a tensor in $\mathcal{K}_{2+4}$ (this time for $\theta_1 = -\theta_2$). The fourth line is a tensor in $\mathcal{K}_{3+4} \cong \Gamma(\llbracket S^{2,1}M \rrbracket)$. One can easily conclude that $S \in \mathcal{K}_{2+3+4}$ at p. Now, if a connected, simply-connected, and complete Kähler manifold (M, g, J) admits a homogeneous Kähler structure S, then, as a consequence of Sekigawa's Theorem [15] (see also [1], [6]), M is homogeneous. Also the type of S is determined by its value at p. Hence, the previous tensor uniquely determines a homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(n)$ belonging to $\mathcal{K}_{2+3+4}$, and we have proved the following.

PROPOSITION 4.2. *Any homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(n) \equiv AN$ with trivial holonomy lies in the class $\mathcal{K}_{2+3+4}$ and has S given by* (4.3). *In particular, there are nontrivial homogeneous structures on $\mathbb{C}\mathrm{H}(n)$ that are not of linear type.*

REMARK 4.3. As a computation shows, the normalizer $\mathcal{N}$ of AN in $G = KAN$ has Lie algebra

$$\mathfrak{N} = \left\{ C = \begin{pmatrix} \alpha & 0 & v \\ -\bar{v}^T & a & b \\ 0 & 0 & -\bar{a} \end{pmatrix} : \begin{array}{l} \alpha \in \mathfrak{u}(n-1), \\ a \in \mathbb{C}, b \in \operatorname{Im}\mathbb{C}, \\ \operatorname{tr} C = 0 \end{array} \right\},$$

so we can write

$$\mathfrak{N} = \mathfrak{s}\big(\mathfrak{u}(n-1) + \mathfrak{u}(1)\big) + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2.$$

We recall that $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1))$. Extending AN to any subgroup of the normalizer means that we can still use $\mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2$ as an ad-invariant complement, and the computation of S remains unchanged.

4.3. Structures of linear type. For $n > 1$, we will find the group theoretic description of the homogeneous structures on $\mathbb{C}\mathrm{H}(n)$ of linear type, i.e., in $\mathcal{K}_{2+4}$. Write $\omega(X, Y) = g(X, JY)$. According to [1], [6], such a structure is given by a tensor

$$S_X Y = g(X, Y)\xi - g(\xi, Y)X + \omega(\xi, Y)JX - \omega(X, Y)J\xi,$$

where $\xi \in \mathfrak{X}(M)$ is nonzero and satisfies $\widetilde{\nabla}\xi = 0$, i.e., $\nabla\xi = S\xi$, hence ξ has constant length.

As before, $\widetilde{R}$ is given via (3.2). Using the explicit form of S and (2.2), we have

$$\begin{aligned} R^S_{XY} Z &= \|\xi\|^2 \big(g(Y, Z)X + \omega(Y, Z)JX - g(X, Z)Y - \omega(X, Z)JY\big) \\ &\qquad + 2\omega(X, Y)\big(\omega(\xi, Z)\xi + g(Z, \xi)J\xi\big) \\ &= c' R^{\mathbb{C}\mathrm{H}(n)}_{XY} Z + 2\omega(X, Y)\big(-\|\xi\|^2 JZ + \omega(\xi, Z)\xi + g(Z, \xi)J\xi\big) \end{aligned}$$

with $c' = -4\|\xi\|^2/c$. Now $0 = \widetilde{R}\xi = (1 - c')R^{\mathbb{C}\mathrm{H}(n)}\xi$, but $R^{\mathbb{C}\mathrm{H}(n)}$ has nonzero holomorphic sectional curvature, so we must have $c' = 1$, $\|\xi\|^2 = -c/4$ and

$$\widetilde{R}_{XY} Z = 2\omega(X, Y)\big(\|\xi\|^2 JZ + \omega(Z, \xi)\xi - g(Z, \xi)J\xi\big).$$

Thus, $\widetilde{R}_{XY}$ acts as zero on $\mathbb{C}\xi$ and as $2\|\xi\|^2\omega(X,Y)J$ on $U=(\mathbb{C}\xi)^\perp$. We see that $\widetilde{R}$ has holonomy $\mathfrak{u}(1)$ with representation $T_pM=\mathbb{R}^2+U$, where $\mathbb{R}^2$ is spanned by ξ and $J\xi$, and $\mathcal{J}=-(1/2\|\xi\|^4)\widetilde{R}_{\xi J\xi}$ acts as J on the factor U. The corresponding homogeneous manifold G/H has

$$\mathfrak{h}=\mathfrak{u}(1), \qquad \mathfrak{g}=\mathfrak{h}+T_pM,$$

and from (2.4) the remaining Lie brackets in $\mathfrak{g}$ are

$$\begin{aligned}[X,Y]&=S_XY-S_YX+\widetilde{R}_{XY}\\ &=g(\xi,X)Y-g(\xi,Y)X-\omega(\xi,X)JY+\omega(\xi,Y)JX\\ &\quad -2\omega(X,Y)J\xi+\widetilde{R}_{XY}\end{aligned}$$

for $X,Y\in T_pM$. Writing $L_0=J\xi-\|\xi\|^2\mathcal{J}$, this gives

$$\begin{aligned}&[Z_1,Z_2]=-2\omega(Z_1,Z_2)L_0, \qquad [\xi,Z]=\|\xi\|^2Z,\\ &[\xi,J\xi]=2\|\xi\|^2L_0, \qquad [J\xi,Z]=\|\xi\|^2JZ\end{aligned} \tag{4.4}$$

for $Z,Z_1,Z_2\in U$.

By Theorem 4.1, this Lie algebra must be that of a subgroup $G=F_rN$ of $S(U(n-1)U(1))\mathbb{R}N$, where F_r has nontrivial projection to $\mathbb{R}$. We now find this identification. Our holonomy algebra $\mathfrak{h}$ is isomorphic to $\mathfrak{u}(1)$, so the group G has Lie algebra

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m}=\mathfrak{u}(1)+\tilde{\mathfrak{a}}+\mathfrak{n}_1+\mathfrak{n}_2.$$

Here, $\mathfrak{h}+\tilde{\mathfrak{a}}$ is the two-dimensional Lie algebra of F_r and the factor $\mathfrak{a}$ projects nontrivially to $\mathfrak{a}$.

Note that $\mathfrak{h}+\tilde{\mathfrak{a}}$ is a subalgebra of the reductive Lie algebra $\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))+\mathfrak{a}$, so is reductive. As it is 2-dimensional, $\mathfrak{h}+\tilde{\mathfrak{a}}$ must be Abelian. Since $\mathcal{J}$ is the generator of the infinitesimal holonomy $\mathfrak{h}$ and the full holonomy algebra of $\mathbb{C}\mathrm{H}(n)$ is $\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$, one has $\mathcal{J}\in\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$. Now $\mathcal{J}$ acts trivially on $\mathfrak{h}+\tilde{\mathfrak{a}}+\mathfrak{n}_2$ and effectively on $\mathfrak{n}_1$. So in the splitting $T_pM=\mathbb{R}^2+U$, U corresponds to $\mathfrak{n}_1$ and $\mathbb{R}^2\subset\mathfrak{h}+\tilde{\mathfrak{a}}+\mathfrak{n}_2$. Equation (4.4) implies that for $Z\in U$ we have $[Z,JZ]=2g(Z,Z)L_0$, so $L_0\in\mathfrak{n}_2$. Also (4.4) implies $\xi\in\mathfrak{h}+\tilde{\mathfrak{a}}+\mathfrak{n}_2\setminus(\mathfrak{h}+\mathfrak{n}_2)$ has only real eigenvalues on $\mathfrak{g}$, so we have $\tilde{\mathfrak{a}}=\mathfrak{a}$ and $\xi=\|\xi\|^2A_0+sL_0\in\mathfrak{a}+\mathfrak{n}_2$, $s\in\mathbb{R}$. Now there is a Lie algebra automorphism ψ of $\mathfrak{g}=\mathfrak{h}+\mathfrak{a}+\mathfrak{n}_1+\mathfrak{n}_2$ which is the identity on $\mathfrak{h}+\mathfrak{n}_1+\mathfrak{n}_2$ and has $\psi(A_0)=A_0+sL_0$. By Remark 2.1, we may thus take $\xi=\|\xi\|^2A_0$.

In the notation of Section 4.2, we may write $L_0=tX$ for some nonzero $t\in\mathbb{R}$. We may obtain $t>0$ by replacing $(J,\mathcal{J})$ by $(-J,-\mathcal{J})$ if necessary. Then using the automorphism of $\mathfrak{g}=\mathfrak{h}+\mathfrak{a}+\mathfrak{n}_1+\mathfrak{n}_2$ that acts as $(1,1,1/\sqrt{t},1/t)$ on these subspaces, we may ensure $L_0=X$, and hence $J\xi=X+\|\xi\|^2\mathcal{J}$.

Now equation (4.1) has $[V_1, V_2] = 2\langle v_1, iv_2\rangle X$. Comparing this with (4.4), gives $J = -i$. It follows that $\mathcal{J} = \frac{i}{n+1}\operatorname{diag}(-2\operatorname{Id}_{n-1}, (n-1)\operatorname{Id}_2)$. The corresponding complement is

$$\mathfrak{m} = \mathfrak{a} + \mathfrak{n}_1 + \mathbb{R}(X + \|\xi\|^2\mathcal{J}). \tag{4.5}$$

We thus have the following theorem.

THEOREM 4.4. *The complex hyperbolic space* $\mathbb{C}\mathrm{H}(n)$ *admits a nonvanishing homogeneous Kähler structure of linear type, which can be realized as a homogeneous space* G/H *with* $G = HAN \subset S(U(n-1)U(1))\mathbb{R}N$, $H \cong U(1)$ *and* ad*-invariant complement described in* (4.5).

Note that the above structure is realized by the homogeneous Kähler manifold given in [6, pp. 92–93]. In the Siegel domain model,

$$D = \{(z = x + iy, v^1, \dots, v^{n-1}) \in \mathbb{C}^n : y - \|v\|^2 > 0\},$$

ξ is proportional to $\partial/\partial y$.

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