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NONCOMMUTATIVE EXTRAPOLATION THEOREMS AND APPLICATIONS

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ABSTRACT. In this paper, we prove some noncommutative analogues of Yano's classical extrapolation theorem. Applying one of them to noncommutative martingales, we obtain a maximal inequality for noncommutative martingales from $L\log^2 L$ to L_1. Moreover, the exponent 2 is optimal. We also obtain the noncommutative analogue of the classical theorem of Burkholder and Chow on the iterations of two conditional expectations.

1. Introduction and preliminaries

We start by recalling the classical extrapolation theorem of Yano [Yan51]. Let (Ω, μ) be a probability space, and let T be a bounded sublinear map on $L_p(\Omega)$ for all $1 < p \leq 2$, verifying the following norm estimate

$$\|Tf\|_p \leq \frac{c}{p-1}\|f\|_p \quad \forall f \in L_p(\Omega),$$

where c is a positive constant independent of p and f. Then T can be extrapolated to a bounded map from $L\log L$ into L_1. This theorem was considerably improved and extended. We mention here only the works [JM91] of Jawerth and Milman, where among many other results, the links between Yano's type extrapolation and interpolation theory are fully studied, and the more recent paper [Car00] by Carro, which gives an interesting improvement of Yano's theorem. Let us also point out that Tao [Tao01] proved that the converse to Yano's theorem holds for translation invariant maps.

We investigate in this paper noncommutative analogues of Yano's theorem for maps acting on noncommutative L_p-spaces. Finding such analogues becomes natural after the recent developments on noncommutative martingales and ergodic inequalities. Our starting point is the noncommutative

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Marcinkiewicz type interpolation theorem proved in [JX07] which can be stated as follows: Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful tracial state τ, let $S=(S_n)_{n\geq 0}$ be a sequence of subadditive maps on $L_p^+(\mathcal{M})$ for all $1\leq p\leq\infty$. Assume that S is of type (∞,∞) and weak type $(1,1)$ (see Section 2 below for the definition). Then S is of type (p,p) for any $1<p<\infty$. More precisely, for any $x\in L_p^+(\mathcal{M})$, there exists $a\in L_p^+(\mathcal{M})$ such that

$$S_n(x)\leq a\quad\forall n\geq 0\quad\text{and}\quad \|a\|_p\leq\frac{c}{(p-1)^2}\|x\|_p, \tag{1.1}$$

where c is a constant depending only on S. Our extrapolation theorem asserts that if S satisfies (1.1), then S extends to a sequence of bounded maps from $L\log^2 L(\mathcal{M})$ into $L_1(\mathcal{M})$. Recall that the order $(p-1)^{-2}$ in (1.1) is optimal for filtrations of conditional expectations. Accordingly, we prove that the space $L\log^2 L(\mathcal{M})$ in our extrapolation theorem cannot be replaced by $L\log^r L(\mathcal{M})$ for $r<2$.

A second extrapolation theorem for sublinear maps is proved by similar arguments. The remainder of this paper is devoted to applications. The first one concerns noncommutative maximal ergodic inequalities for factorable maps in Anantharaman's sense [AD06]. Anantharaman proved the noncommutative Rota dilation theorem for these maps. Consequently, one gets the noncommutative maximal ergodic inequalities by virtue of Junge's Doob maximal inequality [Jun02] with a norm estimate as in (1.1). Our first extrapolation theorem then implies a maximal ergodic inequality from $L\log^2 L(\mathcal{M})$ into $L_1(\mathcal{M})$. Accordingly, we obtain an individual ergodic convergence for operators in $L\log^2 L(\mathcal{M})$. These results apply in particular to some free group actions.

The second main application deals with the iterations of two noncommutative conditional expectations. Namely, we prove the noncommutative analogue of a classical theorem due to Burkholder and Chow [BC61].

In the rest of this introduction, we give some necessary preliminaries on noncommutative L_p-spaces. We refer, for instance, to [PX97] for more details.

We will mainly work on semifinite noncommutaitve L_p-spaces, except in the last section on applications. Thus, we confine our attention here to the semifinite case. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. For $0<p\leq\infty$, let $L_p(\mathcal{M})$ be the associated noncommutative L_p-space. We will also need $L\log^r L(\mathcal{M})$ and its dual space, which are Orlicz spaces. More generally, given an Orlicz function ϕ, the Orlicz space $L_\phi(\mathcal{M})$ is defined as the space of all τ-measurable operators x such that $\phi(\frac{|x|}{\lambda})\in L_1(\mathcal{M})$ for some $\lambda>0$. The norm $\|x\|_\phi$ is then defined by

$$\|x\|_\phi=\inf\left\{\lambda>0:\tau\left[\phi\left(\frac{|x|}{\lambda}\right)\right]\leq 1\right\}.$$

If $\phi(t) = t(1+\log^+ t)^r$ for some $r > 0$, we get the space $L\log^r L(\mathcal{M})$, whose norm is often denoted by $\|\cdot\|_{L\log^r L}$. The dual space of $L\log^r L(\mathcal{M})$ is $\exp L^{1/r}(\mathcal{M})$, which is the Orlicz space associated to the function ψ defined by $\psi(t) = \exp(t^{1/r}) - 1$.

Since we will study noncommutative maximal inequalities, we will need the spaces $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_\infty^c)$, $1 \le p \le \infty$. Recall that a sequence $x = (x_n)_{n\ge 0} \subset L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if and only if x can be factored as $x_n = ay_nb$ with $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $(y_n) \subset L_\infty(\mathcal{M})$. We then define

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_n = ay_nb}\Big\{\|a\|_{2p}\sup_n \|y_n\|_\infty \|b\|_{2p}\Big\}.$$

Following [JX07], this norm is symbolically denoted by $\|{\sup_n}^+ x_n\|_p$. Similarly, $L_p(\mathcal{M};\ell_\infty^c)$ is defined by requiring that x can be factored as $x_n = ay_n$ with $a \in L_p(\mathcal{M})$ and a bounded sequence $(y_n) \subset L_\infty(\mathcal{M})$. Recall that a positive sequence $x = (x_n)_n$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if and only if there exists $a \in L_p^+(\mathcal{M})$ such that $x_n \le a$ for all $n \ge 0$. In this case,

$$\Big\|{\sup_n}^+ x_n\Big\|_p = \inf\{\|a\|_p : a \in L_p^+(\mathcal{M}) \text{ s.t. } x_n \le a \ \forall n \ge 0\}.$$

Here and in the rest of the paper, $L_p^+(\mathcal{M})$ denotes the positive cone of $L_p(\mathcal{M})$. We refer to [Jun02] and [JX07] for more information on $L_p(\mathcal{M};\ell_\infty)$.

2. Extrapolation theorems

Throughout this section, $\mathcal{M}$ denotes a von Neumann algebra equipped with a normal faithful finite trace τ. For simplicity, we assume τ is normalized, i.e., $\tau(1) = 1$.

Let $S_n : L_1^+(\mathcal{M}) \to L_1^+(\mathcal{M})$ be a positive map for $n \ge 0$. We suppose that $S = (S_n)$ is subadditive in the following sense $S_n(x+y) \le S_n(x) + S_n(y)$ for $x, y \in L_1^+(\mathcal{M})$ and all $n \ge 0$. Our basic assumption is the following norm estimate on S:

(H) There exist $1 < p_0 \le \infty, c > 0$ and $r > 0$ such that

$$\Big\|{\sup_n}^+ S_n(x)\Big\|_p \le \frac{c}{(p-1)^r}\|x\|_p \quad \forall x \in L_p^+(\mathcal{M}),\ \forall 1 < p \le p_0.$$

Namely, S is of type (p,p) with a constant $c(p-1)^{-r}$.

As already quoted in the Introduction, if S is of weak type $(1,1)$ and type (p_0,p_0), then S satisfies (H) with $r = 2$ (see [JX07]). Recall that the weak type $(1,1)$ of S means that there exists a constant c such that for any $x \in L_1^+(\mathcal{M})$ and any $\lambda > 0$ there is a projection $e \in \mathcal{M}$ such that

$$e(S_n(x))e \le \lambda \quad \forall n \ge 0 \quad \text{and} \quad \tau(e^\perp) \le c\frac{\|x\|_1}{\lambda}.$$

Recall that $s \mapsto \lambda_s(x)$ denotes the distribution of an operator $x \in L_1(\mathcal{M})$. Namely, for $s > 0$

$$\lambda_s(x) = \tau\big[\chi_{(s,\infty)}(|x|)\big],$$

where $\chi_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ corresponding to the interval (s,∞) and $|x| = (x^*x)^{1/2}$.

LEMMA 2.1. *Assume S satisfies* (H). *Let $1 < p \le p_0$ and $x \in L_p^+(\mathcal{M})$. Let $a_x \in L_p^+(\mathcal{M})$ be an operator such that*

$$S_n(x) \le a_x \quad \forall n \quad \textit{and} \quad \|a_x\|_p \le \frac{c}{(p-1)^r}\|x\|_p. \tag{2.1}$$

Then for every $t > 0$

$$\tau[(a_x - t)_+] = \int_t^\infty \lambda_s(a_x)\,ds \le \frac{c^p K t^{1-p}}{(p-1)^r}\|x\|_p^p, \tag{2.2}$$

where $K = \sup_{p>1}\frac{1}{p}(\frac{1}{p-1})^{r(p-1)}$.

Proof. We have

$$\begin{aligned}
\int_t^\infty \lambda_s(a_x)\,ds &= \int_t^\infty s^{1-p}s^{p-1}\lambda_s(a_x)\,ds \\
&\le t^{1-p}\int_t^\infty s^{p-1}\lambda_s(a_x)\,ds \\
&\le \frac{t^{1-p}}{p}\|a_x\|_p^p \\
&\le \frac{c^p t^{1-p}}{p(p-1)^{rp}}\|x\|_p^p.
\end{aligned}$$

It follows that

$$\int_t^\infty \lambda_s(a_x)\,ds \le \frac{c^p K t^{1-p}}{(p-1)^r}\|x\|_p^p.$$

On the other hand,

$$\tau[(a_x - t)_+] = \int_0^\infty \lambda_s[(a_x - t)_+]\,ds = \int_t^\infty \lambda_s(a_x)\,ds.$$

Then we deduce the desired inequality. $\square$

LEMMA 2.2. *Let $x_k \in L_1(\mathcal{M})$ with $x_k^* = x_k$. Then for every $t > 0$ and $c_k \ge 0$ such that $\sum c_k = 1$, we have*

$$\tau\Bigg[\bigg(\sum_{k\ge1} x_k - t\bigg)_+\Bigg] \le \sum_{k\ge1}\tau[(x_k - c_k t)_+].$$

Proof. We have

$$\begin{aligned}
\tau\Big[\Big(\sum_k x_k - t\Big)_+\Big] &= \tau\Big[\Big(\sum_k (x_k - c_k t)\Big)_+\Big] \\
&= \tau\Big[\frac{1}{2}\Big(\Big|\sum_k (x_k - c_k t)\Big| + \sum_k (x_k - c_k t)\Big)\Big] \\
&\le \sum_k \tau\Big(\frac{|x_k - c_k t|}{2}\Big) + \sum_k \tau\Big(\frac{x_k - c_k t}{2}\Big) \\
&= \sum_k \tau\Big(\frac{|x_k - c_k t| + x_k - c_k t}{2}\Big) \\
&= \sum_k \tau[(x_k - c_k t)_+].
\end{aligned}$$

□

THEOREM 2.3. *Let $S=(S_n)$ satisfy* (H). *Then for $x \in L\log^r L(\mathcal{M})$ with $x \ge 0$ we have*

$$\Big\|\sup_n{}^+ S_n(x)\Big\|_1 \le c' \|x\|_{L\log^r L},$$

where c' is a constant depending only on p_0, c and r in (H).

Proof. Fix a positive operator x in $L\log^r L(\mathcal{M})$. We can decompose x as

$$x = \sum_{k\in\mathbb{Z}} xe_k = \sum_{k\le 0} xe_k + \sum_{k\ge 1} xe_k \stackrel{\text{def}}{=} x_0 + \sum_{k\ge 1} x_k,$$

where $e_k = \chi_{(2^k,2^{k+1}]}(x)$. Let $p_k = 1 + \frac{1}{k\log 2}$ for $k \ge 1$. Applying Lemma 2.1 to x_k, we find $a_k \in L^+_{p_k}(\mathcal{M})$ satisfying the properties there. In particular,

$$\begin{aligned}
\tau[(a_k - 2^{-k})_+] &\le \frac{c^{p_k} K 2^{k(p_k-1)}}{(p_k-1)^r}\|x_k\|_{p_k}^{p_k} \\
&= K 2^{\frac{1}{\log 2}} c^{1+\frac{1}{k\log 2}} k^r (\log 2)^r \tau(x_k^{p_k}) \\
&\le c' k^r \tau(x_k^{p_k}).
\end{aligned}$$

Similarly, we also find a corresponding majorant $a_0 \in L^+_{p_0}(\mathcal{M})$ for x_0. Set $a = \sum_{k\ge 0} a_k$. Then we have $S_n(x) \le a$ for all $n \ge 0$. On the other hand, by Lemma 2.2,

$$\begin{aligned}
\|a\|_1 &\le \|a_0\|_1 + \Big\|\sum_{k\ge 1} a_k\Big\|_1 \\
&\le \|a_0\|_{p_0} + \int_0^1 \lambda_s\Big(\sum_{k\ge 1} a_k\Big)\, ds + \int_1^\infty \lambda_s\Big(\sum_{k\ge 1} a_k\Big)\, ds \\
&\le \frac{c}{(p_0-1)^r}\|x_0\|_{p_0} + 1 + \tau\Big[\Big(\sum_{k\ge 1} a_k - 1\Big)_+\Big]
\end{aligned} \tag{2.3}$$

$$\leq \frac{2c}{(p_0-1)^r} + 1 + c' \sum_{k\geq 1} k^r \tau(x_k^{p_k})$$
$$\leq c' + c' \sum_{k\geq 1} k^r \tau(x_k^{p_k}).$$

Since

$$x_k^{p_k} \leq (2^{k+1} e_k)^{1+\frac{1}{k\log 2}} \leq 2^{k+1} 2^{\frac{2}{\log 2}} e_k \leq 2^{1+\frac{2}{\log 2}} x_k,$$

we deduce that

$$\|a\|_1 \leq c' + 2^{1+\frac{2}{\log 2}} c' \sum_{k\geq 1} k^r \tau(x_k)$$
$$\leq c' + c'\tau\left(\sum_{k\geq 1} x_k \log^r x_k\right)$$
$$\leq c' \|x\|_{L\log^r L}.$$

Therefore, the theorem is proved. □

Using similar arguments, we can prove an extrapolation theorem for a single sublinear map on $L_p(\mathcal{M})$. For our purpose, the sublinearity can be defined in the following general sense.

Definition 2.4. A map T on $L_p(\mathcal{M})$ is called *sublinear* if it satisfies the following conditions:

- $|T(\lambda x)| = |\lambda||T(x)|$ for all $\lambda \in \mathbb{C}$ and $x \in L_p(\mathcal{M})$;
- For any pair (x, y) in $L_p(\mathcal{M})$, there exist two contractions u and v in $\mathcal{M}$ such that $|T(x+y)| \leq u|T(x)|u^* + v|T(y)|v^*$.

By virtue of the famous inequality of Akemann, Anderson and Pederson [AAP82], we see that any linear map is sublinear.

Theorem 2.5. *Let T be a sublinear map on $L_p(\mathcal{M})$ for any $1 < p \leq p_0$. Assume that there exist two positive constants c and r such that*

$$\|T(x)\|_p \leq \frac{c}{(p-1)^r} \|x\|_p \quad \forall x \in L_p(\mathcal{M}).$$

Then T extends to a bounded map from $L\log^r L(\mathcal{M})$ into $L_1(\mathcal{M})$.

Proof. Since any operator in $L\log^r L(\mathcal{M})$ is a linear combination of four positive ones, it suffices to consider the positive operators in $L\log^r L(\mathcal{M})$. Fix such an operator x. Decompose x as in the proof of Theorem 2.3, i.e.,

$$x = \sum_{k\leq 0} x e_k + \sum_{k\geq 1} x e_k = x_0 + \sum_{k\geq 1} x_k.$$

We can also assume that the series above is a finite sum for it converges in $L\log^r L(\mathcal{M})$. Using the sublinearity of T, we find a sequence of contractions

$(u_k)_{k\geq 0} \subset \mathcal{M}$ such that

$$\Big|T\Big(\sum_{k\geq 0} x_k\Big)\Big| \leq \sum_{k\geq 0} u_k |T(x_k)| u_k^*.$$

Therefore,

$$\begin{aligned}\|T(x)\|_1 &\leq \big\|u_0|T(x_0)|u_0^*\big\|_1 + \Big\|\sum_{k\geq 1} u_k|T(x_k)|u_k^*\Big\|_1 \\ &\leq \|T(x_0)\|_{p_0} + 1 + \int_1^\infty \lambda_s\Big[\sum_{k\geq 1} u_k|T(x_k)|u_k^*\Big]\,ds.\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}\int_1^\infty \lambda_s\Big[\sum_{k\geq 1} u_k|T(x_k)|u_k^*\Big]\,ds &= \tau\Big[\Big(\sum_{k\geq 1} u_k|T(x_k)|u_k^* - 1\Big)_+\Big] \\ &\leq \sum_{k\geq 1}\tau\big[\big(u_k|T(x_k)|u_k^* - 2^{-k}\big)_+\big] \\ &= \sum_{k\geq 1}\int_{2^{-k}}^\infty \lambda_s[u_k|T(x_k)|u_k^*]\,ds \\ &\leq \sum_{k\geq 1}\int_{2^{-k}}^\infty \lambda_s[T(x_k)]\,ds.\end{aligned}$$

Now let p_k be as in the proof of Theorem 2.3. Then by assumption and as in the proof of Lemma 2.1, we deduce

$$\int_{2^{-k}}^\infty \lambda_s[T(x_k)]\,ds \leq \frac{c^{p_k}K2^{k(p_k-1)}}{(p_k-1)^r}\tau(x_k^{p_k}) \leq c'k^r\tau(x_k).$$

Then we conclude the proof as before. □

3. The martingale case

In this section, $\mathcal{M}$ still denotes a von Neumann algebra with a normal faithful normalized trace τ. Let $(\mathcal{M}_n)$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$. Let $(\mathbb{E}_n)$ be the associated sequence of trace preserving conditional expectations. As usual, each $\mathbb{E}_n$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_n)$ $\forall 1 \leq p < \infty$. The extended map is still denoted by $\mathbb{E}_n$. By Junge's noncommutative Doob maximal inequality [Jun02], we have that for $1 < p \leq \infty$

$$\Big\|{\sup_n}^+ \mathbb{E}_n(x)\Big\|_p \leq \delta_p\|x\|_p \quad \forall x \in L_p(\mathcal{M}), \tag{3.1}$$

where $\delta_p \leq c(p-1)^{-2}$ with a universal constant c.

THEOREM 3.1. *Let* $(\mathbb{E}_n)$ *be as above.*

(i) *We have*

$$\Big\|\sup_n{}^+\mathbb{E}_n(x)\Big\|_1 \le c\|x\|_{L\log^2 L} \quad \forall x\in L\log^2 L(\mathcal{M}), x\ge 0.$$

(ii) *The exponent* 2 *in the inequality above cannot be replaced by any* $0<r<2$. *More precisely, if there is a constant* c *such that*

$$\Big\|\sup_n{}^+\mathbb{E}_n(x)\Big\|_1 \le c\|x\|_{L\log^r L} \quad \forall x\in L\log^r L(\mathcal{M}), x\ge 0,$$

then $r\ge 2$.

Proof. The first part immediately follows from Theorem 2.3. Thus, it remains to prove (ii). Suppose that there is a $c>0$, such that

$$\Big\|\sup_n{}^+\mathbb{E}_n(x)\Big\|_1 \le c\|x\|_{L\log^r L}.$$

Then by duality, for any finite sequence $(a_n)_{n\ge1}$ in $L_\infty^+(\mathcal{M})$,

$$\Big\|\sum_{n\ge1}\mathbb{E}_n a_n\Big\|_{\exp(L^{1/r})} \le c\Big\|\sum_{n\ge1}a_n\Big\|_\infty. \tag{3.2}$$

Recall that $\exp(L^{1/r})(\mathcal{M})$ is the Orlicz space associated to the function

$$\psi(t)=\exp(t^{1/r})-1.$$

Let $\psi_2(t)=\psi(t^2)$. By Kadison's Cauchy–Schwarz inequality and (3.2), for any finite sequence $(a_n)\subset L_\infty^+(\mathcal{M})$, we have

$$\begin{aligned}\Big\|\Big(\sum|\mathbb{E}_n a_n|^2\Big)^{1/2}\Big\|_{\psi_2} &= \Big\|\sum|\mathbb{E}_n a_n|^2\Big\|_{\psi}^{1/2} \le \Big\|\sum_n \mathbb{E}_n(|a_n|^2)\Big\|_{\psi}^{1/2}\\ &\le \sqrt{c}\Big\|\sum_n|a_n|^2\Big\|_\infty^{1/2} = \sqrt{c}\Big\|\Big(\sum_n|a_n|^2\Big)^{1/2}\Big\|_\infty.\end{aligned} \tag{3.3}$$

Now we specialize to matrix algebras. Namely, let $\mathcal{M}=\mathbb{M}_m$ for an arbitrary large integer m, where $\mathbb{M}_m$ denotes the algebra of $m\times m$ matrices, equipped with the usual trace Tr (which becomes normalized if we wish). For $n\le m$, $\mathbb{M}_n$ is viewed as the subalgebra of $\mathbb{M}_m$ at the upper left corner. Note that $\mathbb{M}_n$ does not contain the unit of $\mathbb{M}_m$. However, there still exists a natural conditional expectation $\widetilde{\mathbb{E}}_n$ from $\mathbb{M}_m$ to $\mathbb{M}_n$, which is defined by $\widetilde{\mathbb{E}}_n=e_nxe_n$, where e_n projects a vector in ℓ_2^m into its first n coordinates. Thus, we still have a finite increasing filtration $(\widetilde{\mathbb{E}}_n)_{1\le n\le m}$ of conditional expectations. The only difference is that $\widetilde{\mathbb{E}}_n$ is no longer faithful. However, using the arguments in [JX03], we can easily make $\widetilde{\mathbb{E}}_n$ faithful. Then we deduce that (3.4) holds for $\widetilde{\mathbb{E}}_n$. Consequently, for any $a_n\in\mathbb{M}_m$,

$$\Big\|\Big(\sum_{n=1}^m|\widetilde{\mathbb{E}}_n(a_n)|^2\Big)^{1/2}\Big\|_{\psi_2} \le \sqrt{c}\Big\|\Big(\sum_{n=1}^m|a_n|^2\Big)^{1/2}\Big\|_\infty.$$

We are now in a position of using the lower triangular projection T as in [JX05]. Recall that T is defined by

$$(T(x))_{ij} = \begin{cases} x_{ij}, & \text{if } j \le i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a \in \mathbb{M}_m$, and let a_n denote the matrix whose nth row is that of a and all others are zero. It is clear that

$$\left(\sum_{n=1}^{m} |a_n|^2 \right)^{1/2} = |a|.$$

On the other hand, one easily sees that $\widetilde{\mathbb{E}}_n(a_n)$ is the matrix whose nth row is that of Ta and all other rows are zero. Then we deduce

$$\|Ta\|_{\psi_2} \le c\|a\|_\infty. \tag{3.4}$$

Consider the Hilbert matrix $h = (h_{ij})$:

$$h_{ij} = \begin{cases} (i-j)^{-1}, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

It is well known that (cf. e.g., [KP70])

$$\|h\|_\infty \le c \quad \text{and} \quad \|Th\|_\infty \approx \log m. \tag{3.5}$$

Thus by (3.4), $\|Th\|_{\psi_2} \le c$. Therefore, there is a constant γ such that

$$\mathrm{Tr}\left[\sum_{n \ge 1} \frac{1}{n!} \left(\frac{|Th|}{\gamma} \right)^{2n/r} \right] \le 1.$$

Thus, $\mathrm{Tr}(|Th|^{2n/r}) \le \gamma^{2n/r} n!$, so $\|Th\|_{2n/r} \le \gamma(n!)^{r/(2n)}$. Since $\|Th\|_{2n/r} \approx \|Th\|_\infty$, if $\frac{2n}{r} \approx \log m$, then

$$\|Th\|_\infty \le \gamma(n!)^{r/(2n)} \approx \gamma n^{r/2} \le \gamma(\log m)^{r/2}.$$

From (3.5), we have $\log m \le \gamma(\log m)^{r/2}$, so $r \ge 2$. □

We end this section by a remark on applications of Theorem 2.5. Using it, we can give new proofs of some inequalities in [Ran02]. For instance, let us consider martingale transforms. Let $\alpha = (\alpha_n)_n \subset \mathbb{C}$ be a bounded sequence, and let T_α be the associated martingale transform:

$$T_\alpha(x) = \sum_{n \ge 0} \alpha_n d_n x_n,$$

where $d_n = \mathbb{E}_n - \mathbb{E}_{n-1}$ (with $\mathbb{E}_{-1} = 0$). It is proved in [Ran02] that for any $1 < p < \infty$,

$$\|T_\alpha(x)\|_p \le \frac{cp^2}{p-1} \sup_{n \ge 0} |\alpha_n| \|x\|_p \quad \forall x \in L_p(\mathcal{M}).$$

Moreover, T_α is also of weak type $(1,1)$, which implies that $T_\alpha : L\log L(\mathcal{M}) \to L_1(\mathcal{M})$ is bounded. This latter result is also a consequence of Theorem 2.5.

4. Applications

4.1. Noncommutative Rota theorem. In this subsection, we consider general von Neumann algebras. To simplify terminology, we call a noncommutative probability space a pair $(\mathcal{M},\varphi)$, where $\mathcal{M}$ is a von Neumann algebra and φ is a normal faithful state on $\mathcal{M}$.

Let $(\mathcal{M}_0,\varphi_0)$ and $(\mathcal{M}_1,\varphi_1)$ be two noncommutative probability spaces. A Markov operator $T:(\mathcal{M}_0,\varphi_0)\to(\mathcal{M}_1,\varphi_1)$ is a normal unital completely positive map. T is (φ_1,φ_0)-preserving if $\varphi_1\circ T=\varphi_0$ and $T\circ\sigma_t^{\varphi_0}=\sigma_t^{\varphi_1}\circ T$ for all $t\in\mathbb{R}$, where σ_t^{φ} denotes the modular automorphism group of a state φ. All maps in what follows are assumed to be φ-preserving in this sense.

Definition 4.1. Let $T:(\mathcal{M}_0,\varphi_0)\to(\mathcal{M}_1,\varphi_1)$ be a (φ_1,φ_0)-preserving Markov operator. We call T *factorable* if there exist a noncommutative probability space $(\mathcal{M}_2,\varphi_2)$ and two normal unital homomorphisms $V:(\mathcal{M}_0,\varphi_0)\to(\mathcal{M}_2,\varphi_2)$, $U:(\mathcal{M}_2,\varphi_2)\to(\mathcal{M}_1,\varphi_1)$ such that V and U are respectively, (φ_2,φ_0)- and (φ_1,φ_2)-preserving and $T=U\circ V$.

Recall that for a φ-preserving Markov operator T on $(\mathcal{M},\varphi)$, there exists a unital completely positive map T^* such that

$$\varphi(T^*(b)a)=\varphi(bT(a)).$$

The following noncommutative Rota dilation theorem for factorable maps is proved in [AD06].

Theorem 4.2. *Let T be a factorable operator on $(\mathcal{M},\varphi)$. Then*

$$T^{*n}T^n=\mathbb{E}\circ\mathbb{E}_n,$$

where $\mathbb{E}_n$ and $\mathbb{E}$ are normal faithful conditional expectations on $\mathcal{M}$ and $(\mathbb{E}_n)$ is decreasing.

We will use Haagerup noncommutative L_p-spaces in the type III case (see [Haa79]). Thus, for $\mathcal{M}$ as above its noncommutative L_p-spaces are denoted by $L_p(\mathcal{M})$, $1\le p\le\infty$. The noncommutative maximal martingale and ergodic inequalities remain true for the type III von Neumann algebras (see [Jun02] and [JX07]). Using Junge's Doob maximal inequality, we obtain

Corollary 4.3. *Let T be a factorable operator on $(\mathcal{M},\varphi)$.*

(i) *For $1<p<\infty$, we have*

$$\Big\|{\sup_n}^+ T^{*n}T^n(x)\Big\|_p\le\frac{c}{(p-1)^2}\|x\|_p\quad\forall x\in L_p(\mathcal{M}),$$

where c is a universal constant.

(ii) *If additionally T is φ-symmetric (i.e., $\varphi(T(y)^*x) = \varphi(y^*T(x))$), then*

$$\Big\|\sup_n{}^+ T^n(x)\Big\|_p \leq \frac{c}{(p-1)^2}\|x\|_p \quad \forall x \in L_p(\mathcal{M}).$$

We specialize to the case of a tracial state. Applying Theorem 3.1 and Corollary 4.3, we obtain the following theorem.

THEOREM 4.4. *Let T be a factorable map on $(\mathcal{M},\varphi)$ with φ tracial.*

(i) *For $x \in L\log^2 L(\mathcal{M})$,*

$$\Big\|\sup_n{}^+ T^{*n}T^n(x)\Big\|_1 \leq c\|x\|_{L\log^2 L}.$$

(ii) *If additionally T is φ-symmetric,*

$$\Big\|\sup_n{}^+ T^n(x)\Big\|_1 \leq c\|x\|_{L\log^2 L}.$$

As a consequence of this theorem, we deduce the corresponding individual ergodic theorem. Let us first recall the relevant notions of individual convergence in the noncommutative setting. We restrict ourselves for the moment to the case where the state φ is tracial. A sequence $(x_n) \subset L_p(\mathcal{M})$ with $1 \leq p \leq \infty$ is said to converge to $x \in L_p(\mathcal{M})$ almost uniformly (in short a.u.) if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\varphi(e^{\perp}) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty}\|(x_n - x)e\|_\infty = 0.$$

In a similar way, the bilateral almost uniform convergence is defined by requiring that $\lim\|e(x_n - x)e\|_\infty = 0$. In the proof of the following corollary, we will use the space $L_p(\mathcal{M}; c_0)$. It is the subspace of $L_p(\mathcal{M};\ell_\infty)$ consisting of all sequences $(x_n)_{n\geq 0}$ which admit a factorization as follows: there are $a, b \in L_{2p}(\mathcal{M})$ and $(y_n) \subset \mathcal{M}$ such that $x_n = a y_n b$ and $\lim_{n\to\infty}\|y_n\|_\infty = 0$. The subspace $L_p(\mathcal{M}; c_0^c)$ of $L_p(\mathcal{M};\ell_\infty^c)$ is defined similarly.

COROLLARY 4.5. *Let T be a factorable operator on $(\mathcal{M},\varphi)$.*

(i) *For $x \in L\log^2 L(\mathcal{M})$, $T^{*n}T^n(x)$ converges b.a.u. to $\mathbb{E}\circ\mathbb{E}_\infty(x)$.*

(ii) *Moreover, if T is φ-symmetric, for $x \in L\log^2 L(\mathcal{M})$, $T^n(x)$ converges b.a.u. to $F(x)$, where F is the projection onto $\{x \in L\log^2 L(\mathcal{M}) : T(x) = x\}$.*

Proof. (i) Due to [AD06], we know that for $p > 1$ and $x \in L_p(\mathcal{M})$, $T^{*n}T^n(x)$ converges b.a.u. to $\mathbb{E}\circ\mathbb{E}_\infty(x)$. We can suppose that $\mathbb{E}\circ\mathbb{E}_\infty(x) = 0$. Since $L_2(\mathcal{M})$ is dense in $L\log^2 L(\mathcal{M})$, then for $x \in L\log^2 L(\mathcal{M})$, there are $(x_k)_{k\geq 1} \subset L_2(\mathcal{M})$ such that $\|x_k - x\|_{L\log^2 L} \to 0$ as $k \to \infty$. By the first part, we know that for every k, $(T^{*n}T^n(x_k))_n \in L_2(\mathcal{M}; c_0) \subset L_1(\mathcal{M}; c_0)$. Therefore, by Corollary 4.3, we have $(T^{*n}T^n(x))_n \in L_1(\mathcal{M}; c_0)$. So from [JX07], $T^{*n}T^n(x)$ converges b.a.u. to 0.

(ii) By [JX07], we can know that $T^{2n}(x)$ converges b.a.u. to $F(x)$ for $x \in L_p(\mathcal{M}), p > 1$. Using similar arguments, we get part (ii) of the proof. □

4.2. Maximal ergodic theorems for free group actions. In this subsection, $\mathcal{M}$ will be a von Neumann algebra with a normal tracial state τ. We will consider noncommutative ergodic theorems for free group actions. We will follow Bufetov [Buf02] and refer to [NS94] for history and more references in the commutative case. Many of Bufetov's results were extended to the noncommutative setting by Anantharaman [AD06]. We also refer to [Hu08] for a different approach which is modeled on Nevo–Stein's arguments [NS94].

Let $\mathbb{F}_m$ be a free group on m generators $\{l_1, \dots, l_m\}$. Set

$$I = \{-m, \dots, -1, 1, \dots, m\}.$$

For $1 \le i \le m$, let T_i be a normal unital automorphism of $\mathcal{M}$ corresponding to the generator l_i such that $\tau \circ T_i = \tau$ and we set $T_{-i} = T_i^{-1}$. Let Ω_I be the set of all finite words over the alphabet I, that is,

$$\Omega_I = \{\omega = \omega_1\omega_2 \cdots \omega_n | \omega_i \in I, 1 \le i \le n, n \in \mathbb{N}\}.$$

Let $P = (p_{ij})_{i,j\in I}$ be a Markov matrix and $(p_i)_{i\in I}$ a stationary distribution with $p_i > 0$ for $i \in I$. Denote by $|\omega|$ the length of a word ω. For $\omega = \omega_1 \cdots \omega_n$, set

$$p(\omega) = p_{\omega_1\omega_2} \cdots p_{\omega_{n-1}\omega_n}, \qquad \mu(\omega) = p_{\omega_1} p(\omega)$$

and define

$$T_\omega = T_{\omega_1} T_{\omega_2} \cdots T_{\omega_n}.$$

Consider the maps

$$\sigma_n = \sum_{|\omega|=n} \mu(\omega) T_\omega$$

and their Cesaro averages

$$U_n = \frac{1}{n+1} \sum_{k=0}^{n} \sigma_k.$$

Set $\mathcal{N} = \ell_\infty^{2m}(\mathcal{M}) = \{\tilde{y} = (y_i)_{i\in I} : y_i \in \mathcal{M}\}$ and define a normal unital positive map

$$\mathbf{P}(\tilde{y})_i = \sum_j p_{ij} T_i(y_j).$$

Given $x \in \mathcal{M}$, let $\tilde{x} = (x_i)_{i\in I} \in \mathcal{N}$ with $x_i = x$ for all $i \in I$. Then

$$\mathbf{P}^n(\widetilde{x})_i = \frac{1}{p_i} \sum_{|\omega|=n, \omega_1=i,} \mu(\omega) T_\omega(x).$$

The adjoint of $\mathbf{P}$ is given by

$$\mathbf{P}^*(\tilde{y})_i = \frac{1}{p_i}\sum_{j\in I} p_j p_{ji} T_{-j}(y_j).$$

Consider a unitary operator

$$V(\tilde{y})_i = T_i(y_{-i}).$$

Then $V\mathbf{P}^*V = \mathbf{P}$. Therefore,

$$U_n(x) = \frac{1}{n+1}\sum_{i\in I}\sum_{k=0}^{n} p_i \mathbf{P}^k(\tilde{x})_i.$$

From [JX07], [Hu08], we know that

$$\Big\|{\sup_n}^+ U_n(x)\Big\|_p \le \Big\|{\sup_n}^+ \frac{1}{n+1}\sum_{k=0}^{n}\mathbf{P}^k(\tilde{x})\Big\|_p \le \frac{c}{(p-1)^2}\|\tilde{x}\|_p \le \frac{c_m}{(p-1)^2}\|x\|_p.$$

Therefore, Theorem 2.3 and an argument similar to that in Corollary 4.5 yield the following theorem.

THEOREM 4.6. *Let U_n be as above. Then for $x \in L\log^2 L(\mathcal{M})$,*

$$\Big\|{\sup_n}^+ U_n(x)\Big\|_1 \le c_m\|x\|_{L\log^2 L}$$

and $U_n(x)$ converges bilaterally almost uniformly.

In fact, $\mathbf{P}$ is factorable (see [AD06]), $\mathbf{P}^{*n}\mathbf{P}^n = \mathbb{E}\circ\mathbb{E}_n$, where $\mathbb{E}_n$ and $\mathbb{E}$ are normal faithful conditional expectations and $(\mathbb{E}_n)$ is decreasing. By the noncommutative Doob's inequality, we have

$$\Big\|{\sup_n}^+ \mathbf{P}^{*n}\mathbf{P}^n(\tilde{y})\Big\|_p \le \Big\|{\sup_n}^+ \mathbb{E}_n(\tilde{y})\Big\|_p \le \frac{c}{(p-1)^2}\|\tilde{y}\|_p.$$

Then by Theorem 2.5, for $\tilde{y} \in L\log^2 L(\mathcal{N})$,

$$\Big\|{\sup_n}^+ \mathbf{P}^{*n}\mathbf{P}^n(\tilde{y})\Big\|_1 \le \Big\|{\sup_n}^+ \mathbb{E}_n(\tilde{y})\Big\|_1 \le c\|\tilde{y}\|_{L\log^2 L}.$$

Let us now consider the special case where $p_i = \frac{1}{2m}$ and $p_{ij} = \frac{1}{2m-1}$ for $i, j \in I$. Then

$$\sigma_n(x) = \frac{1}{2m(2m-1)^{n-1}}\sum_{|\omega|=n} T_\omega(x).$$

Accordingly,

$$\mathbf{P}^n(\tilde{x})_i = \frac{1}{(2m-1)^{n-1}}\sum_{|\omega|=n,\omega_1=i} T_\omega(x)$$

for $x \in \mathcal{M}$, $\tilde{x} = (x_i)_{i\in I} \in \mathcal{N}$ with $x_i = x$ for every $i \in I$. In this special situation, we have

$$\mathbf{P}^{2n-1} = \frac{2m-1}{2m-2} V\mathbf{P}^{*n}\mathbf{P}^n - \frac{1}{2m-2} V\mathbf{P}^{*n-1}\mathbf{P}^{n-1}.$$

Then, for $\tilde{y} \in L\log L^2(\mathcal{N})$,

$$\begin{aligned}\Big\|\sup_n{}^+ \mathbf{P}^{2n-1}(\tilde{y})\Big\|_1 &\le \frac{2m-1}{2m-2}\Big\|\sup_n{}^+ \mathbf{P}^{*n}\mathbf{P}^n(\tilde{y})\Big\|_1 \\ &\quad + \frac{1}{2m-2}\Big\|\sup_n{}^+ \mathbf{P}^{*n-1}\mathbf{P}^{n-1}(\tilde{y})\Big\|_1 \\ &\le \frac{m}{m-1} c\|\tilde{y}\|_{L\log^2 L} \le c\|\tilde{y}\|_{L\log^2 L}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Big\|\sup_n{}^+ \mathbf{P}^{n}(\tilde{y})\Big\|_1 &\le \Big\|\sup_n{}^+ \mathbf{P}^{2n}(\tilde{y})\Big\|_1 + \Big\|\sup_n{}^+ \mathbf{P}^{2n-1}(\tilde{y})\Big\|_1 \\ &\le c\|\mathbf{P}(\tilde{y})\|_{L\log^2 L} + c\|\tilde{y}\|_{L\log^2 L} \\ &\le c\|\tilde{y}\|_{L\log^2 L}.\end{aligned} \tag{4.1}$$

THEOREM 4.7. *In this special setting, for $x \in L\log^2 L(\mathcal{M})$,*

$$\Big\|\sup_n{}^+ \sigma_n(x)\Big\|_1 \le c_m\|x\|_{L\log^2 L}$$

and $\sigma_{2n}(x)$ converges bilaterally almost uniformly to $F(x)$, where F is the projection onto $\{x \in L\log^2 L(\mathcal{M}) : \sigma_1^2(x) = x\}$.

Proof. By inequality (4.2),

$$\begin{aligned}\Big\|\sup_n{}^+ \sigma_n(x)\Big\|_1 &\le \frac{1}{2m}\sum_{i\in I}\Big\|\sup_n{}^+ \mathbf{P}^n(\tilde{x})_i\Big\|_1 \le \Big\|\sup_n{}^+ \mathbf{P}^n(\tilde{x})\Big\|_1 \\ &\le c\|\tilde{x}\|_{L\log^2 L} \le c_m\|x\|_{L\log^2 L}.\end{aligned}$$

The rest of the proof is similar to the one of Corollary 4.5. □

4.3. Group von Neumann algebra. Let G be a discrete group, and let $\mathrm{VN}(G)$ denote its group von Neumann algebra. We are interested here in the case where $G = \mathbb{F}_m$ is a free group on m generators $\{l_1, \dots, l_m\}$. Let $|\cdot|$ denote the associated length function on $\mathbb{F}_m$. For $t \ge 0$, let T_t be the Herz–Schur multiplier defined by $\exp(-t|\cdot|)$. T_t is a normal completely positive trace preserving unital map on $\mathrm{VN}(\mathbb{F}_m)$. $(T_t)_{t\ge 0}$ is called the free Poisson semigroup of $\mathbb{F}_m$. It is proved in [JLMX06] that each T_t admits a Rota type dilation. On the other hand, each T_t can be extended to the L_p-space. Still denote it by T_t. Consequently, we have for any t,

$$\Big\|\sup_n{}^+ T_t^n(x)\Big\|_p \le \frac{c}{(p-1)^2}\|x\|_p,$$

where c is a universal constant. Therefore, by Theorem 2.3,

$$\Big\|{\sup_n}^{+} T_t^n(x)\Big\|_1 \le c\|x\|_{L\log^2 L}. \tag{4.2}$$

THEOREM 4.8. *Let $(T_t)_{t\ge 0}$ be the free Poisson semigroup. Then for $x \in L\log^2 L(VN(\mathbb{F}_m))$,*

$$\Big\|{\sup_t}^{+} T_t(x)\Big\|_1 \le c\|x\|_{L\log^2 L}.$$

Proof. To prove this inequality, we only need to consider $T_t(x)$ for all t in a dense subset of $(0,\infty)$, for instance, the subset $\{2^{-m}n, m, n\in\mathbb{N}\}$. Using Proposition 2.1 from [JX07] and inequality (4.2), we have

$$\begin{aligned}\Big\|{\sup_{n,m}}^{+} T_{n2^{-m}}(x)\Big\|_1 &= \sup_m \Big\|{\sup_{n,1\le k\le m}}^{+} T_{n2^{-k}}(x)\Big\|_1\\ &= \sup_m \Big\|{\sup_n}^{+} T_{n2^{-m}}(x)\Big\|_1\\ &= \sup_m \Big\|{\sup_n}^{+} T^n_{2^{-m}}(x)\Big\|_1\\ &\le c\|x\|_{L\log^2 L}.\end{aligned}$$

Then we deduce the desired inequality. □

THEOREM 4.9. *Let $\{T_t\}_{t\ge 0}$ be the free Poisson semigroup.*

(i) *When $t\to\infty$, $T_t(x)$ converges b.a.u. to $\tau(x)\mathbf{1}$ for $x\in L\log^2 L(VN(\mathbb{F}_m))$.*
(ii) *When $t\to 0$, $T_t(x)$ converges b.a.u. to x for $x\in L\log^2 L(VN(\mathbb{F}_m))$.*

Proof. (i) In [JX07], we know that for all $x\in L_p(\mathrm{VN}(\mathbb{F}_n))$ $(p>1)$, $T_t(x)$ converges bilaterally almost uniformly to $F(x)$, where F is the projection onto $\{x\in L_p(\mathrm{VN}(\mathbb{F}_n)) : T_t(x)=x\ \forall t\ge 0\}$. It is easy to see that $T_t(y)=y$ for all t if and only if $y=c\mathbf{1}$ for some $c\in\mathbb{C}$. Therefore, $F(x)=\tau(x)\mathbf{1}$. By the same discussion as in Corollary 4.5, for $x\in L\log^2 L(\mathrm{VN}(\mathbb{F}_n))$, $T_t(x)$ converges b.a.u. to $\tau(x)\mathbf{1}$.

(ii) We know when $t\to 0$, $T_t(x)$ converges b.a.u. to x for $x\in L^p(\mathrm{VN}(\mathbb{F}_n))$ (see [JX07]). $T_t(x)$ converges b.a.u. to x for $x\in L\log^2 L(\mathrm{VN}(\mathbb{F}_n))$ by the same way as in Corollary 4.5. □

The previous theorem is also valid for some other semigroups considered in [JLMX06], for instance, the free Uhlenbeck–Ornstein semigroup and more generally, the q-Uhlenbeck–Ornstein semigroups.

4.4. Conditional expectation. We first recall the classical Burkholder–Chow theorem on the iterations of two conditional expectations. Let $\mathbb{E}$ and $\mathbb{F}$ be the conditional expectations on a probability space $(\Omega,\mathfrak{F},P)$ relative to two σ-subalgebras $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\mathfrak{F}$, respectively. Then for any $f\in L_2(\Omega)$

$$\Big\|\sup_n |(\mathbb{E}\mathbb{F})^n(f)|\Big\|_2 \le c\|f\|_2.$$

Moreover, $(\mathbb{E}\mathbb{F})^n(f)$ converges to $\mathbb{E}\wedge\mathbb{F}(f)$ almost everywhere and in norm, where $\mathbb{E}\wedge\mathbb{F}$ denotes the conditional expectation relative to $\mathfrak{F}_1\cap\mathfrak{F}_2$. Our goal here is to prove the noncommutative analogue of Burkholder–Chow theorem. In the sequel, $\mathcal{M}$ will denote a von Neumann algebra equipped with a normal faithful state φ. If $\mathbb{E}$ is a φ-preserving normal faithful conditional expectation on $\mathcal{M}$, $\mathbb{E}$ automatically extends to a contractive projection on $L_p(\mathcal{M})$ for every $1\le p<\infty$ (see the discussion in Section 4.1). We start with the following lemma.

LEMMA 4.10. *Let T and S be two factorable maps on $(\mathcal{M},\varphi)$. Then $T\circ S$ is also factorable.*

Proof. There are $(\mathcal{N}_T,\psi_T)$, $(\mathcal{N}_S,\psi_S)$ and the normal unital homomorphisms $i_0:(\mathcal{M},\varphi)\to(\mathcal{N}_T,\psi_T)$, $i_1:(\mathcal{M},\varphi)\to(\mathcal{N}_T,\psi_T)$, $j_0:(\mathcal{M},\varphi)\to(\mathcal{N}_S,\psi_S)$, $j_1:(\mathcal{M},\varphi)\to(\mathcal{N}_S,\psi_S)$ such that $T=i_0^*\circ i_1$ and $S=j_0^*\circ j_1$. Therefore, $T\circ S=i_0^*\circ i_1\circ j_0^*\circ j_1$. By [AD06], $i_1\circ j_0^*$ is factorable. Thus, there are $(\mathcal{N},\psi)$ and the normal unital homomorphisms $k_0:(\mathcal{N}_T,\psi_T)\to(\mathcal{N},\psi)$, $k_1:(\mathcal{N}_S,\psi_S)\to(\mathcal{N},\psi)$ such that $i_1\circ j_0^*=k_0^*\circ k_1$. Thus $T\circ S=(k_0\circ i_0)^*\circ k_1\circ j_1$, so $T\circ S$ is factorable. ☐

COROLLARY 4.11. *Let $\mathbb{E}$ and $\mathbb{F}$ be φ-preserving normal faithful conditional expectations on $\mathcal{M}$. Then $\mathbb{E}\mathbb{F}\mathbb{E}$ is factorable.*

Proof. This is an immediate consequence of the preceding lemma for state preserving normal conditional expectations are factorable. ☐

Recall that $L_p(\mathcal{M};\ell_\infty)$ is the dual space of $L_{p'}(\mathcal{M};\ell_1)$ if $p'<\infty$, where p' is the index conjugate to p (cf. [Jun02]). Given $1\le p\le\infty$, $L_p(\mathcal{M};\ell_1)$ is defined as the space of all sequences $x=(x_n)_{n\ge0}$ which admit a factorization of the following form: $\exists u_{kn},v_{kn}\in L_{2p}(\mathcal{M})$ such that

$$x_n=\sum_{k\ge0}u_{kn}^*v_{kn}\quad\forall n\ge0$$

and

$$\sum_{k,n\ge0}u_{kn}^*u_{kn}\in L_p(\mathcal{M}),\qquad\sum_{k,n\ge0}v_{kn}^*v_{kn}\in L_p(\mathcal{M}),$$

where these series are required to be convergent in $L_p(\mathcal{M})$ (if $p=\infty$, then both series are convergent relative to the w^*-topology). We then define the norm of $L_p(\mathcal{M};\ell_1)$:

$$\|x\|_{L_p(\mathcal{M};\ell_1)}=\inf\left\{\left\|\sum_{k,n\ge0}u_{kn}^*u_{kn}\right\|_p^{\frac12}\left\|\sum_{k,n\ge0}v_{kn}^*v_{kn}\right\|_p^{\frac12}\right\},$$

where the infimum runs over all factorization as above. In the following proof, the subspaces $L_p(\mathcal{M};\ell_\infty^n)$ and $L_p(\mathcal{M};\ell_1^n)$ will be used. Recall that $L_p(\mathcal{M};\ell_\infty^n)$ is the subspace of $L_p(\mathcal{M};\ell_\infty)$ consisting of all finite sequences

$(x_0, x_1, \ldots, x_{n-1}, 0, \ldots)$. $L_p(\mathcal{M}; \ell_1^n)$ is defined in a similar way. The following result is certainly known to experts.

LEMMA 4.12. *Let T be a completely positive contractive map on $(\mathcal{M}, \varphi)$ with $\varphi \circ T \leq \varphi$ and $\sigma_t^\varphi \circ T = T \circ \sigma_t^\varphi$.*

(i) *For $1 \leq p \leq \infty$,*

$$\|T(x)\|_{L_p(\mathcal{M};\ell_\infty)} \leq \|x\|_{L_p(\mathcal{M};\ell_\infty)} \quad \forall x = (x_n) \in L_p(\mathcal{M}; \ell_\infty)$$

and for $2 \leq p \leq \infty$,

$$\|T(x)\|_{L_p(\mathcal{M};\ell_\infty^c)} \leq \|x\|_{L_p(\mathcal{M};\ell_\infty^c)} \quad \forall x = (x_n) \in L_p(\mathcal{M}; \ell_\infty^c).$$

(ii) *For $1 \leq p \leq \infty$,*

$$\|T(x)\|_{L_p(\mathcal{M};c_0)} \leq \|x\|_{L_p(\mathcal{M};c_0)} \quad \forall x = (x_n) \in L_p(\mathcal{M}; c_0)$$

and for $2 \leq p \leq \infty$,

$$\|T(x)\|_{L_p(\mathcal{M};c_0^c)} \leq \|x\|_{L_p(\mathcal{M};c_0^c)} \quad \forall x = (x_n) \in L_p(\mathcal{M}; c_0^c).$$

Proof. (i) Let $x = (x_n) \in L_p(\mathcal{M}; \ell_\infty)$ with $1 < p \leq \infty$. For any $y = (y_n) \in L_{p'}(\mathcal{M}; \ell_1)$, $\|y\|_{L_{p'}(\mathcal{M};\ell_1)} \leq 1$, we have

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \leq \|x\|_{L_p(\mathcal{M};\ell_\infty)} \|T^*(y)\|_{L_{p'}(\mathcal{M};\ell_1)}. \tag{4.3}$$

Since T is completely positive, then we can find a representation π and a bounded linear operator V such that

$$\|V\| \leq \|T^*\|^{1/2}, \qquad T^*(a) = V^*\pi(a)V \quad \forall a \in L_p(\mathcal{M}).$$

Therefore, for $u_{k,n}, v_{k,n} \in L_{2p'}(\mathcal{M})$ in the factorization of y,

$$T^*(y_n) = \sum_{k\geq 0} V^*\pi(u_{kn}^* v_{kn})V = \sum_{k\geq 0} V^*\pi(u_{kn}^*)\pi(v_{kn})V.$$

Thus, by the contraction of T,

$$\Big\| \sum_{k,n\geq 0} V^*\pi(u_{kn}^*)\pi(u_{kn})V \Big\|_{p'} \leq \Big\| \sum_{k,n\geq 0} \pi(u_{kn}^*)\pi(u_{kn}) \Big\|_{p'}.$$

From the inequality above and the definition of $L_p(\mathcal{M}; \ell_1)$, we know that $\|T^*(y)\|_{L_{p'}(\mathcal{M};\ell_1)} \leq \|y\|_{L_{p'}(\mathcal{M};\ell_1)} \leq 1$. Therefore, by duality and from (4.3),

$$\|T(x)\|_{L_p(\mathcal{M};\ell_\infty)} \leq \|x\|_{L_p(\mathcal{M};\ell_\infty)} \quad \forall x \in L_p(\mathcal{M}; \ell_\infty), 1 < p \leq \infty.$$

For the case $p = 1$, we have $L_1(\mathcal{M}; \ell_\infty^n)^* = L_\infty(\mathcal{M}; \ell_1^n)$ (see [JX07]). A same discussion then shows that $\|T(x)\|_{L_1(\mathcal{M};\ell_\infty)} \leq \|x\|_{L_1(\mathcal{M};\ell_\infty)}$.

From the Cauchy–Schwarz inequality and the fact that

$$x = (x_n) \in L_p(\mathcal{M}; \ell_\infty^c) \quad \text{if and only if} \quad x^*x = (x_n^* x_n) \in L_{p/2}(\mathcal{M}; \ell_\infty),$$

we have for $x = (x_n) \in L_p(\mathcal{M}; \ell_\infty^c)$ with $2 \leq p \leq \infty$

$$\|T(x)\|_{L_p(\mathcal{M};\ell_\infty^c)} \leq \|x\|_{L_p(\mathcal{M};\ell_\infty^c)}.$$

(ii) In consequence, we know (ii) similarly. □

With the help of the preceding lemma, we obtain the following theorem.

THEOREM 4.13. *Let $\mathbb{E}$ and $\mathbb{F}$ be φ-preserving normal faithful conditional expectations on $\mathcal{M}$.*

(i) *For $1<p<\infty$, we have*

$$\left\|\sup_n{}^+(\mathbb{E}\mathbb{F})^n(x)\right\|_p \leq \frac{c}{(p-1)^2}\|x\|_p \quad \forall x\in L_p(\mathcal{M}),$$

where c is a universal constant.

(ii) *If additionally φ is tracial, then*

$$\left\|\sup_n{}^+(\mathbb{E}\mathbb{F})^n(x)\right\|_1 \leq c\|x\|_{L\log^2 L} \quad \forall x\in L\log^2 L(\mathcal{M}).$$

Proof. (i) Set $T=\mathbb{F}\mathbb{E}\mathbb{F}$. T is a state preserving completely positive map on $(\mathcal{M},\varphi)$. From Corollary 4.11, T is factorable. Moreover, T is φ-symmetric. Therefore, T satisfies the noncommutative Rota's dilation property. Thus, for $1<p<\infty$,

$$\left\|\sup_n{}^+T^n(x)\right\|_p \leq \frac{c}{(p-1)^2}\|x\|_p \quad \forall x\in L_p(\mathcal{M}).$$

So by Lemma 4.12, we have

$$\begin{aligned}\left\|\sup_n{}^+(\mathbb{E}\mathbb{F})^n(x)\right\|_p &= \left\|\sup_n{}^+\mathbb{E}T^{n-1}(x)\right\|_p\\ &\leq \left\|\sup_n{}^+T^{n-1}(x)\right\|_p\\ &\leq \frac{c}{(p-1)^2}\|x\|_p.\end{aligned}$$

(ii) This follows from (i) and Theorem 2.3. □

Theorem 4.13 allows us to deduce the corresponding individual ergodic theorem. To this end, we first recall the almost sure convergence for operators in Haagerup noncommutative L_p-spaces. As a normal faithful state φ on $\mathcal{M}$, φ corresponds to a positive element in $L_1(\mathcal{M})$. In the sequel, this element will be denoted by D, which is called the density of φ in $L_1(\mathcal{M})$. A sequence $(x_n)\subset L_p(\mathcal{M})$ with $p<\infty$ is said to converge to $x\in L_p(\mathcal{M})$ almost surely (in short a.s.) if for every $\varepsilon>0$ there is a projection $e\in\mathcal{M}$ and a family $(a_{n,k})$ in $\mathcal{M}$ such that

$$\varphi(e^\perp)<\varepsilon, \qquad x_n-x=\sum_k\left(a_{n,k}D^{\frac{1}{p}}\right) \quad\text{and}\quad \lim_{n\to\infty}\left\|\sum_k(a_{n,k}e)\right\|_\infty=0.$$

Similarly, the bilateral almost sure (in short b.a.s.) convergence is defined by the symmetric injection of $\mathcal{M}$ into $L_p(\mathcal{M}): a\mapsto D^{1/(2p)}aD^{1/(2p)}$.

THEOREM 4.14. *Let $\mathbb{E}$ and $\mathbb{F}$ be φ-preserving normal faithful conditional expectations on $\mathcal{M}$.*

(i) *If $1<p\leq 2$, $(\mathbb{E}\mathbb{F})^n(x)$ converges b.a.s. to $(\mathbb{E}\wedge\mathbb{F})(x)$ for $x\in L_p(\mathcal{M})$.*

(ii) *If* $2 < p < \infty$, $(\mathbb{E}\mathbb{F})^n(x)$ *converges a.s. to* $(\mathbb{E} \wedge \mathbb{F})(x)$ *for* $x \in L_p(\mathcal{M})$.
(iii) *If* $p = \infty$, $(\mathbb{E}\mathbb{F})^n(x)$ *converges a.u. to* $(\mathbb{E} \wedge \mathbb{F})(x)$ *for* $x \in \mathcal{M}$.
(iv) *If additionally* φ *is tracial*, $(\mathbb{E}\mathbb{F})^n(x)$ *converges b.a.u. to* $(\mathbb{E} \wedge \mathbb{F})(x)$ *for* $x \in L\log^2 L(\mathcal{M})$.

Proof. Set $T = \mathbb{F}\mathbb{E}\mathbb{F}$. Then T is a completely positive unital φ-preserving map on $\mathcal{M}$. So T extends to a contraction on $L_p(\mathcal{M})$ for every $p \geq 1$. Moreover, T is symmetric, i.e., self-adjoint as an operator on $L_2(\mathcal{M})$. By [JX07], we have

- if $1 < p \leq 2$, $(T^n(x) - F(x))_n \in L_p(\mathcal{M}; c_0)$ for $x \in L_p(\mathcal{M})$;
- if $2 < p < \infty$, $(T^n(x) - F(x))_n \in L_p(\mathcal{M}; c_0^c)$ for $x \in L_p(\mathcal{M})$,

where F is the projection onto $\{x \in L_p(\mathcal{M}) : T(x) = x\}$. For $x \in \mathcal{M}$, we have $((T^n(x) - F(x))D^{1/p})_n \in L_p(\mathcal{M}; c_0^c)$ with $p > 2$. Therefore, $T^n(x)$ converges a.u. to $F(x)$ for $x \in \mathcal{M}$.

On the other hand, it is clear that $\mathbb{E}(L_2(\mathcal{M})) \cap \mathbb{F}(L_2(\mathcal{M})) = \{x \in L_2(\mathcal{M}) : T(x) = x\}$. Therefore,

$$\mathbb{E}(L_p(\mathcal{M})) \cap \mathbb{F}(L_p(\mathcal{M})) = \{x \in L_p(\mathcal{M}) : T(x) = x\}, \quad 1 \leq p \leq \infty.$$

That is $F = \mathbb{E} \wedge \mathbb{F}$. In the following, we only consider the case $1 < p \leq 2$. From Lemma 4.12,

$$\begin{aligned}\Big\| \sup_{m \leq n \leq k}{}^{+} \mathbb{E}T^n(x) - \mathbb{E} \wedge \mathbb{F}(x) \Big\|_p \\ \leq \Big\| \sup_{m \leq n \leq k}{}^{+} T^n(x) - \mathbb{E} \wedge \mathbb{F}(x) \Big\|_p \to 0 \quad \text{as } n, k \to \infty.\end{aligned}$$

That is $(\mathbb{E}T^n(x) - \mathbb{E} \wedge \mathbb{F}(x))_n \in L_p(\mathcal{M}; c_0)$ for $x \in L_p(\mathcal{M})$. So $(\mathbb{E}\mathbb{F})^n(x)$ converges bilaterally almost surely to $\mathbb{E} \wedge \mathbb{F}(x)$ for $x \in L_p(\mathcal{M})$. In the same way, we can deal with the other cases.

Finally, if additionally φ is tracial, by Corollary 4.5, a similar discussion shows that $(\mathbb{E}\mathbb{F})^n(x)$ converges b.a.u. to $(\mathbb{E} \wedge \mathbb{F})(x)$ for $x \in L\log^2 L(\mathcal{M})$. $\square$

REMARK 4.15. The case $p = 2$ of Theorem 4.14 was also proved by Jajte (see [Jaj91, Theorem 4.2.2]).

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