Illinois Journal of Mathematics
Volume 53, Number 1, Spring 2009, Pages 349–364
S 0019-2082

SYZYGIES OF SEMI-REGULAR SEQUENCES

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ABSTRACT. A semi-regular sequence in $S = k[x_1, \ldots, x_n]$ is a sequence of polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ which satisfy a certain generic condition. Suppose that $I \subset S$ is generated by such a semi-regular sequence and let ρ be the Castelnuovo–Mumford regularity of S/I. We show that a minimal free resolution of S/I is isomorphic to the Koszul complex on $f_1, \ldots, f_r$ in degrees $\leq \rho - 2$. If a common numerical condition is satisfied, then this isomorphism also holds in degree $\rho - 1$. Therefore, the Betti diagram of S/I and the Betti diagram of the Koszul complex always agree in rows $\leq \rho - 2$; we can sometimes determine that they also agree in row $\rho - 1$. We also give a partial converse, that if the Betti diagram of S/I agrees with the diagram of the Koszul complex except in possibly the last two rows, then I can be generated by a (not necessarily minimal) semi-regular sequence.

1. Introduction

In this paper, we consider the free resolutions of ideals generated by semi-regular sequences of homogeneous polynomials in a polynomial ring. Such ideals are interesting since, in a sense to be made precise later, most sequences of polynomials are conjectured to be semi-regular. So, on the one hand such ideals are quite common and thus worth understanding as well as we can. On the other hand, the conjecture that most sequences are semi-regular is a long-standing difficult problem, and it is our hope that a better understanding of the homological properties of semi-regular sequences may help.

The main result of this paper is a structure theorem for the minimal graded free resolution of an ideal generated by a semi-regular sequence in a polynomial

Received September 22, 2008.
2000 *Mathematics Subject Classification.* Primary 13D02. Secondary 13D40.

ring, generalizing the famous theorem that an ideal generated by a regular sequence is minimally resolved by a Koszul complex. We also show that among ideals of maximal height, those generated by semi-regular sequences are characterized by resolutions of this type, just as the Koszul complex characterizes regular sequences.

DEFINITION 1.1. Let $S=k[x_1,\dots,x_n]$ and let I be a homogeneous ideal. A nonzero form $f\in S_d$ is called *semi-regular* on S/I if the multiplication maps $(S/I)_{a-d}\xrightarrow{f}(S/I)_a$ are linear maps of maximal rank for all a. A sequence of forms $f_1,\dots,f_r$ in S with degrees $d_1,\dots,d_r$ is called a *semi-regular sequence* if f_i is semi-regular on $S/(f_1,\dots,f_{i-1})$ for all $i=1,\dots,r$.

Notice that a regular sequence is a semi-regular sequence, since all of the multiplication maps are injective, and thus of maximal rank. Here are some other easy examples of semi-regularity.

If S/I is Artinian and the highest degree in which S/I is nonzero is ρ, then any nonzero form of degree greater than ρ is semi-regular on S/I. This is because for every multiplication map either the domain or the codomain has dimension 0, so that the multiplication map has maximal rank. A form of degree exactly ρ is semi-regular on R/I if and only if it is not in I. In this case, the only multiplication map for which neither the domain nor the codomain has dimension 0 is the map from degree 0 to degree ρ. In degree 0, R/I has dimension 1, while in degree ρ, R/I has dimension at least 1, so that semi-regularity requires that this multiplication map be injective, which is the case if and only if the form is not in I.

Using these examples, it is easy to make very uninteresting semi-regular sequences that are as long as we want, so long as we may choose the degrees of the forms. What is more interesting is that for any finite sequence $d_1,\dots,d_r$ of nonnegative integers it is a conjecture, essentially due to Fröberg, that most sequences of forms $f_1,\dots,f_r$ of degrees $d_1,\dots,d_r$ are semi-regular sequences.

Precisely, consider the set of sequences $\{f_1,\dots,f_r\,|\,\deg(f_i)=d_i\}=S_{d_1}\times\cdots\times S_{d_r}$ as an affine space whose coordinates describe the coefficients of the polynomials in the sequence. Then the conjecture is the following.

CONJECTURE. *The set of semi-regular sequences in $S_{d_1}\times\cdots\times S_{d_r}$ is a nonempty Zariski-open set.*

Fröberg's Conjecture [3] is actually a statement about the Hilbert series of ideals generated by generic sequences of forms, but is equivalent to the conjecture above. See [6] for several equivalent conjectures. Fröberg's conjecture is proven in the following cases, which also imply the conjecture above. In the list below, n is the number of variables in the polynomial ring and r is the number of forms.

(1) $r\le n$: In this case, the conjecture is a statement about regular sequences.
(2) $n\le 2$ [3].

(3) $n = 3$ [1].
(4) $r = n+1$ and k has characteristic 0. In this case, the sequence $x_1^{d_1}, \dots, x_n^{d_n}$, $(x_1 + \cdots + x_n)^{d_{n+1}}$ is a semi-regular sequence [7].

Our main results, informally stated, are:

(1) A minimal free resolution of S/I for I generated by a semi-regular sequence is isomorphic to the Koszul complex on the generators of I in degrees $\le \rho - 2$. In the language of Macaulay 2, this say that the Betti diagram of an ideal generated by a semi-regular sequence looks like a Koszul diagram except possibly in the last two rows.
(2) A common numerical condition on the degree sequence $d_1, \dots, d_r$ implies that a minimal free resolution of S/I is isomorphic to the Koszul complex on the generators of I in degrees $\le \rho - 1$. In the language of Macaulay 2, this says that a common numerical condition on the degree sequence $d_1, \dots, d_r$ implies that the Betti diagram of an ideal generated by a semi-regular sequence looks like a Koszul diagram except possibly in the last row.
(3) If S/I is Artinian and the second term in a minimal free resolution of S/I has the same number of generators in each degree $\le p - 2$, then I is generated by a semi-regular sequence. Restated, this says that for S/I Artinian, if the Betti diagram of S/I looks like a Koszul diagram except possibly in the last two rows, then I is generated by a semi-regular sequence.

It should be noted that the results in this paper are related to work of Migliore and Miró-Roig. In particular, in [5], they calculate the graded Betti numbers of quotients of semi-regular almost complete intersections when the dimension of S is 3, when the dimension of S is 4, and a numerical condition on the degrees has been met, and when the dimension of S is even and all forms have the same degree. They also give the graded Betti numbers of quotients of semi-regular almost complete intersections in several other interesting and important cases. In [4], Migliore and Miró-Roig give a proof that the Betti diagram of an ideal generated by a semi-regular sequence looks like a Koszul diagram except possibly in the last two rows. This is our Theorem 3.6 (see Remark 3.7). It should be noted that the method of proof utilized by Migliore and Miró-Roig is entirely different from that which we employ. Chandler is also doing related work, in a more geometric setting.

We wish to offer our thanks to several people who contributed to the completion of this paper. David Moulton helped on the proof of Proposition 3.10. We also discussed the implications of this example with Juan Migliore. Haverford College graciously provided funds for housing so that we could work on this paper together.

Finally, we wish to dedicate the paper to Tony Geramita on the occasion of his 65th birthday. He has been an encouragement and inspiration to both authors.

2. Notation, background and conventions

Throughout this paper, $S = k[x_1, \ldots, x_n]$, with the natural grading. All polynomials are forms (i.e., homogeneous), all S-modules (including ideals) are graded, and all elements and homomorphisms of modules are homogeneous. We use k to denote the base field of S and n always denotes the number of variables in S. The number r always denotes the number of forms in a sequence of interest in S. Finally, $\mathfrak{m}$ is the graded maximal ideal generated by the variables.

If I is an ideal in S, then S/I has a minimal graded free resolution

$$\cdots \to F_1 \to F_0 \to S/I,$$

where the F_i are free S-modules. The i,jth graded Betti number of S/I is $\beta_{i,j}(S/I) = \dim_k \operatorname{Tor}_i(S/I, S/\mathfrak{m})_j$, which is also equal to the number of degree j basis elements of F_i. These numbers are our main object of study in this paper.

By Hilbert's Syzygy theorem, $\beta_{i,j}(S/I) = 0$ for $i > n$. Clearly, $F_0 = S$ so that $\beta_{0,0} = 1$, and $\beta_{0,j} = 0$ for $j \neq 0$. The minimality of the resolution implies that $\beta_{i,j}(S/I) = 0$ for $j < i$. The Castelnuovo–Mumford regularity $\rho(S/I)$, or simply ρ when the context is clear, is the maximum value of j such that $\beta_{i,i+j}(S/I) \neq 0$ for some i. If S/I is Artinian, then $\rho(S/I)$ is also equal to the maximum value of j such that $(S/I)_j \neq 0$.

The Poincaré series is $P_{S/I}(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{\infty} \beta_{i,j} s^i t^j$, the generating series of the graded Betti numbers. Notice that the Poincaré series is a polynomial. We will often refer to the Betti diagram notation used in the computer algebra package Macaulay 2. The Betti diagram of S/I is a table with $\rho + 1$ rows and $n+1$ columns where the i,jth entry, counting from zero, is $\beta_{i,i+j}(S/I)$.

Let $f_1, \ldots, f_r$ be a sequence of forms of degrees $d_1, \ldots, d_r$. For each integer $i \geq 0$, let K_i be the free S-module with basis κ_σ indexed by the order i subsets $\sigma \subset \{1, \ldots, r\}$. Let the degree of σ be $\sum_{h \in \sigma} d_h$. The Koszul complex is defined by

$$\cdots \to K_2 \to K_1 \to K_0,$$

where if $\sigma = \{\sigma_1 < \sigma_2 < \cdots < \sigma_i\}$ has order $i > 0$ then the image of κ_σ in K_{i-1} is $\sum_{h=1}^{i} (-1)^{i+h} f_{\sigma_h} \kappa_{\sigma - \sigma_h}$. The Koszul complex is a minimal free resolution of S/I if and only if $f_1, \ldots, f_r$ is a regular sequence.

In fact, it is only necessary to look at the Poincaré series, or the Betti diagram, to see if I is generated by a regular sequence. Let $K(d_1, \ldots, d_r) = \prod_{i=1}^{r} (1 + s t^{d_i})$. Then I is generated by a regular sequence of degrees $d_1, \ldots, d_r$ if and only if $P_{S/I}(s,t) = K(d_1, \ldots, d_r)$.

The Hilbert series of S/I is $H_{S/I}(t) = \sum_{i=0}^{\infty} \dim_k(S/I)_i t^i$. The Hilbert series and the Poincaré series are related by $(1-t)^n H_{S/I}(t) = P_{S/I}(-1,t)$. In particular, the Hilbert series of S/I where I is generated by a regular sequence of forms of degrees $d_1, \dots, d_r$ is

$$\frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n}.$$

DEFINITION 2.1. Given a series $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \mathbb{Z}$ for all i, let $|\sum_{i=0}^{\infty} a_i t^i|$ be the series $\sum_{i=0}^{\infty} b_i t^i$ where

$$b_i = \begin{cases} a_i & \text{if } a_j > 0 \text{ for all } 0 \le j \le i, \\ 0 & \text{otherwise.} \end{cases}$$

The Hilbert series of S/I where I is generated by a semi-regular sequence of forms of degrees $d_1, \dots, d_r$ is

$$\left| \frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n} \right|.$$

Fröberg's Conjecture is that a generic sequence of forms of degrees $d_1, \dots, d_r$ has this Hilbert series. We will work do a lot of work with the coefficients of this series, and we get stronger results when the coefficients satisfy the following special condition.

DEFINITION 2.2. The nonnegative integers $n, d_1, \dots, d_r$ satisfy the *special numerical condition* if the first nonpositive coefficient of the series

$$\frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n}$$

is equal to 0.

3. The main theorem

In this section, we prove the structure theorem for the minimal free resolution of an ideal generated by a semi-regular sequence. We need a few lemmas before proving the main theorem.

DEFINITION 3.1. If M is a graded module and τ is an integer, then let $M_{(\tau)}$ be the submodule of M generated by the elements of M of degrees less than or equal to τ.

Notice that given a set of minimal generators for M, the subset consisting of those of degrees less than or equal to τ is a set of minimal generators for $M_{(\tau)}$. Conversely, any set of minimal generators for $M_{(\tau)}$ is a subset of a set of minimal generators for M.

LEMMA 3.2. *Let $\phi : M \to N$ be a degree zero homomorphism of modules such that $\phi(M) \subseteq \mathfrak{m}N$. Then $\phi(M_{(\tau+1)}) \subseteq N_{(\tau)}$ for any integer τ.*

Proof. Let m be an element of M of degree less than or equal to $\tau + 1$. Choose a minimal system of generators for N. Then $\phi(m)$ may be written as an S-linear combination of these generators, where the nonzero coefficients have positive degree. Thus, the generators appearing with nonzero coefficients are all of degree less than the degree of m, and thus of degree less than or equal to τ. This shows that $\phi(m) \in N_{(\tau)}$, which is enough to prove the lemma. $\square$

This allows us to make the following definition for complexes.

DEFINITION 3.3. If $F_\bullet$ is a complex of S-modules with differential δ such that $\delta_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$, then for any integer τ let $F_\bullet^{(\tau)}$ be the complex with $F_i^{(\tau)} = (F_i)_{(\tau+i)}$, with differentials equal to the restriction of δ.

LEMMA 3.4. *Let $\phi : M \to N$ be a module homomorphism inducing an isomorphism $M_{(\tau)} \cong N_{(\tau)}$. If $F_\bullet \to M$ and $G_\bullet \to N$ are minimal free resolutions of M and of N, then any lifting of ϕ to a homomorphism from $F_\bullet$ to $G_\bullet$ induces an isomorphism $F_\bullet^{(\tau)} \to G_\bullet^{(\tau)}$.*

REMARK 3.5. We will often use Lemma 3.4 for $F_\bullet$ and $G_\bullet$ minimal free resolutions of S/I and $S/(I : mf)$, with $I_d = (I : f)_d$ for $d \leq \tau$. In this context, we conclude that $F_\bullet^{(\tau-1)} \cong G_\bullet^{(\tau-1)}$, since

$$\bar{F}_\bullet \to I \to 0 = \cdots \to F_2 \to F_1 \to I \to 0$$

and

$$\bar{G}_\bullet \to (I : f) \to 0 = \cdots \to G_2 \to G_1 \to (I : f) \to 0$$

are minimal free resolutions of I and $(I : f)$, respectively, and we can apply the lemma to these shorter resolutions.

Proof of Lemma 3.4. We proceed by induction on the projective dimension of M. If the M has projective dimension 0, then M is free, so that $M_{(\tau)}$ and $N_{(\tau)}$ are isomorphic free modules. It follows that $(G_0)_{(\tau)}$ is isomorphic to $N_{(\tau)}$ so that G_1 is generated in degrees greater than $\tau + 1$. Because $G_\bullet$ is a minimal resolution, it follows that G_i is generated in degrees greater than $\tau + i$, so that the complex $G_\bullet^{(\tau)}$ is simply $0 \to (G_0)_{(\tau)} \to 0$. Given a lifting of ϕ, we have

$$\begin{tikzcd}
0 \arrow[r] & (F_0)_{(\tau)} \arrow[r] \arrow[d] & M_{(\tau)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (G_0)_{(\tau)} \arrow[r] & N_{(\tau)} \arrow[r] & 0.
\end{tikzcd}$$

The horizontal arrows and the right-hand arrow are isomorphisms, so the left hand arrow is as well.

Suppose that the projective dimension of M is greater than 0, let K be the kernel of the map $F_0 \to M$, and let L be the kernel of the map $G_0 \to N$. Any lifting of ϕ to a homomorphism $F_\bullet \to G_\bullet$ takes K to L, as one can see from an easy diagram chase. Moreover, we claim that this map induces an isomorphism $K_{(\tau+1)} \cong L_{(\tau+1)}$. This is because K and L are the images of F_1 and G_1. By Lemma (3.2), the image of $(F_1)_{(\tau+1)}$ and of $(G_1)_{(\tau+1)}$ are in $(F_0)_{(\tau)}$ and $(G_0)_{(\tau)}$, respectively, and it is clear that ϕ induces an isomorphism $(F_0)_{(\tau)} \cong (G_0)_{(\tau)}$ (because any set of minimal generators for $M_{(\tau)}$ is mapped by ϕ to a set of minimal generators for $N_{(\tau)}$). Since K has projective dimension one less than that of M, we apply induction to this case, proving the lemma. $\square$

We can now prove the main theorem.

THEOREM 3.6. *Let $I \subset S$ be an ideal generated by a semi-regular sequence $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$, let $F_\bullet \to S/I$ be a minimal free resolution of S/I, let ρ be the Castelnuovo–Mumford regularity of S/I and let $K_\bullet$ be the Koszul complex on $f_1, \ldots, f_r$. Then $F_\bullet^{(\rho-2)} \cong K_\bullet^{(\rho-2)}$. Furthermore, if $n > r$ or the sequence $n, d_1, \ldots, d_r$ satisfies the special numerical condition, then $F_\bullet^{(\rho-1)} \cong K^{(\rho-1)}$.*

Proof. If $r \le n$, then $f_1, \ldots, f_r$ is a regular sequence and $F_\bullet \cong K_\bullet$. So, we may assume that $r > n$ and proceed by induction on r.

Let $\tilde{I} = (f_1, \ldots, f_{r-1})$, $f = f_r$, $d = d_r$, $G_\bullet \to S/\tilde{I}$ be a minimal free resolution and $L_\bullet$ be the Koszul complex on $f_1, \ldots, f_{r-1}$. Note that because $r-1 \ge n$, $S/\tilde{I}$ is Artinian. Let $\tilde{\rho}$ be the Castelnuovo–Mumford regularity of $S/\tilde{I}$. By induction, we know that $G_\bullet^{(\tilde{\rho}-2)} \cong L_\bullet^{(\tilde{\rho}-2)}$.

Note that we may assume $d \le \tilde{\rho}+1$, because otherwise $I = \tilde{I}$ and we are finished. Likewise, we may take $d > 0$, because $d = 0$ implies that $I = S$, which would complete the proof.

Let $\varepsilon = 1$ if the special numerical condition is satisfied, and $\varepsilon = 0$ otherwise. We must show that $F_\bullet^{(\rho+\varepsilon-2)} \cong K_\bullet^{(\rho+\varepsilon-2)}$.

First, we will show that $\tilde{\rho} \ge \rho + \varepsilon$. It is clear that $\tilde{\rho} \ge \rho$ because $S/\tilde{I}$ is Artinian and $\tilde{I} \subseteq I$. If $\varepsilon = 1$, then in

$$\frac{(1-t^d)\prod_{i=1}^{r-1}(1-t^{d_i})}{(1-t)^n} \tag{1}$$

the coefficients of $t^0, \ldots, t^\rho$ are all positive while the coefficient of $t^{\rho+1}$ is zero. So we have to show that the coefficients of $1, t, \ldots, t^{\rho+1}$ in

$$\frac{\prod_{i=1}^{r-1}(1-t^{d_i})}{(1-t)^n} \tag{2}$$

are positive. Note that the coefficients of $t^{\rho+1}$ and $t^{\rho+1-d}$ in (2) must be equal (since we are assuming the coefficient of $t^{\rho+1}$ in (1) is zero), so it is actually enough to show that the coefficients of $1, t, \ldots, t^\rho$ in (2) are positive. Because $\tilde{\rho} \ge \rho$, this follows immediately. We conclude that $\tilde{\rho} \ge \rho + \varepsilon$.

Now, we show that multiplication by f from $(S/\tilde{I})_{j-d}$ to $(S/\tilde{I})_j$ is injective for all $j \le \rho+\varepsilon$. As f is semi-regular modulo $\tilde{I}$, we only need to show that $\dim(S/\tilde{I})_{j-d} \le \dim(S/\tilde{I})_j$ for $j \le \rho+\varepsilon$. If $j \le \rho$ then we have a strict inequality, since then $\dim(S/I)_j = \dim(S/\tilde{I})_j - \dim(S/\tilde{I})_{j-d}$ and this number is positive. If $\varepsilon = 1$, then in fact $\dim(S/\tilde{I})_{\rho+1-d} = \dim(S/\tilde{I})_{\rho+1}$ as we have already seen.

Thus, $(\tilde{I} : f)_j = \tilde{I}_j$ for $j \le \rho+\varepsilon$. Consider the short exact sequence

$$0 \to S/(\tilde{I} : f)(-d) \to S/\tilde{I} \to S/I \to 0.$$

We will use this sequence to produce the minimal free resolution $F_\bullet$ of S/I as a summand of a mapping cone. See [2] for details about mapping cones constructions.

Let $H_\bullet \to S/(\tilde{I} : f)$ be a minimal free resolution. Then it follows by Remark 3.5 that $G_\bullet^{(\rho+\varepsilon-1)} \cong H_\bullet^{(\rho+\varepsilon-1)}$. Since $\tilde{\rho} \ge \rho+\varepsilon$, we now have that

$$H_\bullet^{(\rho+\varepsilon-2)} \cong G_\bullet^{(\rho+\varepsilon-2)} \cong L_\bullet^{(\rho+\varepsilon-2)}.$$

We may lift the multiplication by f map $S/(\tilde{I} : f)(-d) \to S/\tilde{I}$ to a map on complexes $\phi : H_\bullet(-d) \to G_\bullet$ in such a way that the restriction to

$$H(-d)_\bullet^{(\rho+\varepsilon-2)} \to G_\bullet^{(\rho+\varepsilon-2)}$$

is itself realized as multiplication by f via the isomorphism with

$$L(-d)_\bullet^{(\rho+\varepsilon-2)} \to L_\bullet^{(\rho+\varepsilon-2)}.$$

Let $M_\bullet$ be the mapping cone of $\phi : H_\bullet(-d) \to G_\bullet$. Then $M_\bullet$ is a free resolution of S/I, so that $M_\bullet \cong F_\bullet \oplus T_\bullet$, where T is a trivial complex. (See Theorem 20.2 in [2]). Note also that the mapping cone of $L(-d)_\bullet \to L_\bullet$ is the Koszul complex $K_\bullet$. (Using a mapping cone construction is a common method for inductively constructing a Koszul complex.) Here, $M_i = G_i \oplus H_{i-1}(-d)$ with differential $\delta_M : (g,h) \mapsto (\delta_G(g) + \phi(h), -\delta_H(h))$. By the isomorphisms above, $M_\bullet^{(\rho+\varepsilon-2)} \cong K_\bullet^{(\rho+\varepsilon-2)}$. Since $\delta_{M,i}((M_i)_{(\rho+\varepsilon-2+i)}) \subseteq \mathfrak{m}(M_{i-1})_{(\rho+\varepsilon-2+i-1)}$ we have that $(M_i)_{(\rho+\varepsilon-2+i)} \cap T_i = 0$, so that $(M_i)_{(\rho+\varepsilon-2+i)} \subseteq F_i$, and hence $(M_i)_{(\rho+\varepsilon-2+i)} = (F_i)_{(\rho+\varepsilon-2+i)}$, which is to say that $F_\bullet^{(\rho+\varepsilon-2)} \cong M_\bullet^{(\rho+\varepsilon-2)} \cong K_\bullet^{(\rho+\varepsilon-2)}$ as required. □

Remark 3.7. The theorem implies that the coefficient of $s^i t^j$ in the Poincaré polynomial of S/I and the coefficient of $s^i t^j$ in $\prod_{i=1}^r (1 + st^{d_i})$ agree whenever $j \le \rho+\varepsilon-2+i$, where $\varepsilon = 1$ if the special numerical condition is satisfied and 0 otherwise. The i,jth entry of the Betti diagram of S/I is the coefficient of $s^i t^j$ in the twisted Poincaré polynomial $\tilde{P}_{S/I}(s,t) = P_{S/I}(s/t, t)$. In terms of the twisted Poincaré polynomial, Theorem 3.6 implies that the $\tilde{P}_{S/I}(s,t)$ and $\prod(1 + st^{d_i-1})$ agree in t degrees less than or equal to $\rho-2+\varepsilon$, that is that the Betti diagram of S/I and the diagram of the Koszul complex agree in rows $\le \rho-2+\varepsilon$.

EXAMPLE 3.8. Suppose we let $S = k[x_1, x_2, x_3, x_4]$, and let $I = (f_1, \ldots, f_6)$ be the ideal generated by the forms

$$\begin{aligned} f_1 &= x_2^2 + x_3x_4, \\ f_2 &= x_1^2 + x_2x_3, \\ f_3 &= x_3^3 - x_3^2x_4, \\ f_4 &= x_2x_3x_4 + x_4^3, \\ f_5 &= x_1x_2x_3 + x_3^2x_4, \\ f_6 &= x_1x_4^2. \end{aligned}$$

To show that these f_i form a semi-regular sequence, we only need to confirm that $H_{S/(f_1,\ldots,f_i)}(t) = |(1 - t^{d_i})H_{S/(f_1,\ldots,f_{i-1})}(t)|$ for all $i = 1, \ldots, 6$. This is easily verified with Macaulay 2. The final calculation yields

$$\begin{aligned} H_{S/I}(t) &= \left| \prod_{i=0}^{6} \frac{(1 - t^{d_i})}{(1 - t)^4} \right| = |1 + 4t + 8t^2 + 8t^3 + 0t^4 - 12t^5 + \cdots| \\ &= 1 + 4t + 8t^2 + 8t^3. \end{aligned}$$

Note that the Castelnuovo–Mumford regularity is $\rho = 3$, and the coefficient of $t^4 = t^{\rho+1}$ is zero, so by Theorem 3.6, $\tilde{P}_{S/I}(s, t)$ agrees with

$$\prod_{i=1}^{6}(1 + st^{d_i - 1}) = 1 + 2st + (4s + s^2)t^2 + p(s, t)t^3$$

in t degrees ≤ 2 where $p(s, t)$ is some polynomial in s and t. The Betti diagram of S/I is therefore:

total:	1	6	21	24	8
0:	1	.	.	.	.
1:	.	2	.	.	.
2:	.	4	1	.	.
3:	.	.	$\beta_{2,5}$	$\beta_{3,6}$	$\beta_{4,7}$,

where $\beta_{2,5}$, $\beta_{3,6}$, and $\beta_{4,7}$ have yet to be determined.

Recall that $(1 - t)^n H_{S/I}(t) = P_{S/I}(-1, t)$. For the twisted Poincaré series, we have $(1-t)^n H_{S/I}(t) = \tilde{P}(-t, t) = \sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \beta_{i,j} t^j$, so that the alternating sums along the southwest to northeast diagonals of the Betti diagram are determined by the Hilbert series. In our particular example,

$$\begin{aligned} (1 + 4t + 8t^2 + 8t^3)(1 - t)^4 &= 1 - 2t^2 - 4t^3 + t^4 + 20t^5 - 24t^6 + 8t^7 \\ &= 1 - 2t^2 - 4t^3 + t^4 + \beta_{2,5}t^5 - \beta_{3,6}t^6 + \beta_{4,7}t^7, \end{aligned}$$

and we then easily complete the Betti diagram:

total:	1	6	21	24	8
0:	1	.	.	.	.
1:	.	2	.	.	.
2:	.	4	1	.	.
3:	.	.	20	24	8.

The proof of the following corollary is a generalization of this example.

COROLLARY 3.9. *Let $r \geq n$ and let ρ be the degree of the polynomial*

$$\left| \frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n} \right|.$$

If $n, d_1, \dots, d_r$ satisfy the special numerical criterion, then all ideals generated by semi-regular sequences of degrees $d_1, \dots, d_r$ have the same graded Betti numbers.

The special numerical condition is not as rare as one might think. Indeed, it is quite common in the almost complete intersection case—the case in which $r = n+1$. To prove this, we first need some basic properties of the Hilbert Series of an Artinian complete intersection. The properties in the proposition below are well known except possibly for the explicit determination of the coefficients that attain the maximum value in the Hilbert series. We are grateful to David Moulton for his help in finding this proof.

PROPOSITION 3.10. *Let $d_1, \dots, d_{n+1}$ be positive integers with $d_1 \leq d_2 \leq \cdots \leq d_{n+1}$. Let $\Delta = \sum_{i=1}^{n+1}(d_i - 1)$, $\delta = \sum_{i=1}^{n}(d_i - 1)$, and $\mu = \min\{\delta, \lfloor \frac{\Delta}{2} \rfloor\}$. Let*

$$H(t) = \prod_{i=1}^{n+1}(1 + t + \cdots + t^{d_i - 1}) = \sum_{i=0}^{\Delta} h_i t^i.$$

(Set $h_i = 0$ for $i < 0$ and $i > \Delta$.) Then

(1) *$h_{\Delta - i} = h_i$ for all i.*
(2) *μ is the smallest nonnegative integer such that $h_\mu \geq h_{\mu+1}$.*
(3) *$h_i = h_\mu$ for $\mu \leq i \leq \Delta - \mu$.*

Proof. (1) follows immediately from the observation that

$$t^\Delta H\left(\frac{1}{t}\right) = \prod_{i=1}^{n+1} t^{d_i - 1}\left(1 + \frac{1}{t} + \cdots + \frac{1}{t^{d_i - 1}}\right) = H(t).$$

We prove (2) and (3) together by induction. If $n = 0$, then $H(t) = 1 + \cdots + t^{d_1 - 1}$, $\Delta = d_1 - 1$ and $\mu = \delta = 0$. In this case, (2) and (3) are trivially satisfied.

If $n > 0$, then set $\tilde{\delta} = \sum_{i=1}^{n-1}(d_i - 1)$ and $\tilde{\mu} = \min\{\tilde{\delta}, \lfloor \frac{\delta}{2} \rfloor\}$. Let $\tilde{h}_i$ be the coefficient of t^i in the polynomial $\tilde{H}(t) = \prod_{i=1}^{n}(1 + t + \cdots + t^{d_i - 1})$ with the understanding that $\tilde{h}_i = 0$ for $i < 0$ and $i > \delta$. By induction, the $\tilde{h}_i$ are strictly increasing for $0 \le i \le \tilde{\mu}$, are constant for $\tilde{\mu} \le i \le \delta - \tilde{\mu}$ and are strictly decreasing for $\delta - \tilde{\mu} \le i \le \delta$.

Since $H(t) = (1 + t + \cdots + t^{d_{n+1}-1})\tilde{H}(t)$, we have that $h_i = \tilde{h}_{i-d_{n+1}+1} + \cdots + \tilde{h}_i$ for every i. In order to prove (2), we want to find out for how long the h_i are strictly increasing. To do so, we consider

$$h_i - h_{i-1} = \tilde{h}_i - \tilde{h}_{i-d_{n+1}} = \tilde{h}_{\delta-i} - \tilde{h}_{i-d_{n+1}}.$$

For this quantity to be positive, we first need that $0 \le i \le \delta$, since otherwise $\tilde{h}_i = 0$ and $\tilde{h}_{i-d_{n+1}} \ge 0$. For i in this interval, and since, of course, $i - d_{n+1} < i = \delta - (\delta - i)$, we have $\tilde{h}_{\delta-i} - \tilde{h}_{i-d_{n+1}} > 0$ if and only if $i - d_{n+1} < \min\{\tilde{\mu}, \delta - i\}$. In the interesting case, when $i \ge d_{n+1}$, we have $\min\{\tilde{\mu}, \delta - i\} = \min\{\lfloor \frac{\delta}{2} \rfloor, \delta - i\}$. Thus, $h_i - h_{i-1} > 0$ if and only if $0 \le i \le \delta$ and $i - d_{n+1} < \min\{\lfloor \frac{\delta}{2} \rfloor, \delta - i\}$.

Note that if $i \le \frac{\delta}{2}$, then the latter inequality is trivial. If $\frac{\delta}{2} < i \le \delta$, then $i - d_{n+1} < \min\{\lfloor \frac{\delta}{2} \rfloor, \delta - i\}$ is the same as $i - d_{n+1} < \delta - i$, which is the same as $2i < \delta + d_{n+1}$, which is the same as $2i \le \delta + d_{n+1} - 1 = \Delta$, which is the same as $i \le \lfloor \frac{\Delta}{2} \rfloor$. Thus, $h_i > h_{i-1}$ if and only if $0 \le i \le \min\{\delta, \lfloor \frac{\Delta}{2} \rfloor\}$. So, the smallest nonnegative i such that $h_{i+1} \le h_i$ is $\mu = \min\{\delta, \lfloor \frac{\Delta}{2} \rfloor\}$ as required.

To prove (3), first consider the case in which $\mu = \lfloor \frac{\Delta}{2} \rfloor$. Then by (1), $h_{\Delta-\mu} = h_\mu$, but there are no other values of i strictly between μ and $\Delta - \mu$. If $\mu = \delta < \lfloor \frac{\Delta}{2} \rfloor$, then $d_{n+1} > \delta$. In this case, h_i attains its maximum value as $\sum_{j=0}^{\delta} \tilde{h}_j$. This is attained when $\delta \le i \le d_{n+1} - 1$. But, $\delta = \mu$ and $d_{n+1} - 1 = \Delta - \mu$ as required. □

Corollary 3.11. *Let I be an ideal in $S = k[x_1, \ldots, x_n]$ generated by a semi-regular sequence $f_1, \ldots, f_{n+1}$ of degrees $d_1 \le \cdots \le d_{n+1}$ where $d_{n+1} \le \sum_{i=1}^{n}(d_i - 1)$. Let ρ be the Castelnuovo–Mumford regularity of S/I. Then the coefficient of $t^{\rho+1}$ in*

$$\frac{\prod_{i=1}^{n+1}(1 - t^{d_i})}{(1-t)^n}$$

is 0 if and only if $\Delta = \sum_{i=1}^{n+1}(d_i - 1)$ is odd. When Δ is odd the graded Betti numbers of I are completely determined by the degree sequence $d_1 \le \cdots \le d_{n+1}$.

Note that if the hypothesis that $d_{n+1} \le \sum_{i=1}^{n}(d_i - 1)$ fails, then f_{n+1} is in the ideal generated by the regular sequence $f_1, \ldots, f_n$, and thus the first n polynomials generate I. So, this hypothesis does not exclude any interesting cases.

Proof. Note that

$$\frac{\prod_{i=1}^{n+1}(1 - t^{d_i})}{(1-t)^n} = (1-t)\prod_{i=1}^{n+1}(1 + t + \cdots + t^{d_i - 1}).$$

In the language of Proposition 3.10 for the polynomial $H(t) = \prod_{i=1}^{n+1}(1 + t + \cdots + t^{d_i-1})$, we have that $\rho = \mu$. Since $d_{n+1} \le \sum_{i=1}^{n}(d_i - 1)$, we have that

$\lfloor\frac{\Delta}{2}\rfloor \le \delta$, so that $\rho = \lfloor\frac{\Delta}{2}\rfloor$. If Δ is even, the coefficient of $t^{\rho+1}$ in $H(t)$ is less than the coefficient of t^{ρ}. If Δ is odd, then the two coefficients are equal. For

$$\frac{\prod_{i=1}^{n+1}(1-t^{d_i})}{(1-t)^n}$$

this means that the coefficient of $t^{\rho+1}$ is 0, if Δ is odd and negative if Δ is even.

The last statement follows from the discussion following Corollary 3.11. □

EXAMPLE 3.12. For a final example, we demonstrate that the open set of generic forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ for which the generic ideal has minimal graded Betti numbers can be strictly smaller then the open set of semi-regular sequences of those degrees. Let $S = k[x_1, \ldots, x_4]$ where k has characteristic zero, and let I be the ideal generated by 5 generic forms of degree 3. Let J be the ideal generated by the forms x_1^3, x_2^3, x_3^3 x_4^3, $(x_1 + x_2 + x_3 + x_4)^3$. By Stanley [7], we know that both I and J are generated by semi-regular sequences, so they both have the same Hilbert series

$$\begin{aligned} H(t) &= |1 + 4t + 10t^2 + 15t^3 + 15t^4 + 6t^5 - 6t^6 + \cdots| \\ &= 1 + 4t + 10t^2 + 15t^3 + 15t^4 + 6t^5. \end{aligned}$$

The Betti diagrams of S/I and S/J, however, are not equal. (Note that although this is an almost complete intersection, $\Delta = \sum_{i=1}^{5}(d-1) = 10$ is even.) The Betti diagram for S/I turns out to be:

total:	1	5	16	18	6
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	5	.	.	.
3:	.	.	.	.	.
4:	.	.	16	9	.
5:	.	.	.	9	6.

In this case, most of the information in the Betti diagram of S/I is beyond the purview of Theorem 3.6. We know, however, what the entries in the first two columns should be, while $\beta_{2,6}(S/I)$ and $\beta_{4,9}(S/I)$ can be read off the much used equation

$$H_{S/I}(t) = \frac{\sum_{d=0}^{\infty}\sum_{i=0}^{n}(-1)^i\beta_{i,d}(S/I)t^d}{(1-t)^n} = \frac{P_{S/I}(-t,t)}{(1-t)^n}.$$

The values of $\beta_{2,7}(S/I)$, $\beta_{3,7}(S/I)$, $\beta_{3,8}(S/I)$, and $\beta_{4,8}(S/I)$ cannot be determined by the methods in this paper (note that in the calculation above, the coefficient of $t^{\rho+1} = t^6$ is nonzero). Migliore and Miró-Roig, however, are able to compute the entire resolution for almost complete intersections in dimension $n = 4$ when $\sum_{i=1}^{5} d_i$ is odd and $d_2 + d_3 + d_4 < d_1 + d_5 + 4$ (Proposition 3.15 [5]). Thus, we calculate the remaining Betti numbers using their results.

The Betti diagram of S/J (easily produced using Macaulay 2) is

total:	1	5	17	20	7
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	5	.	.	.
3:	.	.	.	.	.
4:	.	.	16	10	1
5:	.	.	1	10	6

and it is obviously strictly larger then the diagram of S/I.

Finally, we observe that neither Betti diagram agrees with $(1+st^2)^5$ in t degree $4=\rho-1$.

4. Converse of the main theorem

In this section, we prove a converse to Theorem 3.6. In particular, let $I \subset S$ such that S/I is Artinian. Write ρ to be the Castelnuovo–Mumford regularity of S/I. If the number of second syzygies in degrees $\leq \rho-1$ agrees with the number of expected Koszul relations, then I can be generated by a semi-regular sequence.

First, we need a few lemmas.

Lemma 4.1. *The ideal $\mathfrak{m}^s$ can be generated by a semi-regular sequence of monomials for all $s \in \mathbb{N}$.*

Proof. Recall from the Introduction that if S/I is Artinian of Castelnuovo–Mumford regularity ρ, then any f of degree ρ that is not in I is semi-regular modulo I. We use this principal to construct an ascending chain of ideals generated by semi-regular sequences in $\mathfrak{m}^s$. Let I_0 be the ideal generated by the regular sequence $x_1^s, \ldots, x_n^s$. For $j \geq 0$, say that we have constructed a semi-regular sequence of monomials of length $n+j$ generating an ideal $I_j \subseteq \mathfrak{m}^s$. If $I_j = \mathfrak{m}^s$, then we are done. Otherwise, choose any monomial x^μ not in I_j and of degree equal to the Castelnuovo–Mumford regularity of S/I_j. Note that this degree is at least s. Append x^μ to our sequence, and let the new sequence generate I_{j+1}, a strictly larger ideal contained in $\mathfrak{m}^s$. Since S is Noetherian, the chain of ideals must terminate. By construction, it can only terminate at $\mathfrak{m}^s$. □

Remark 4.2. Note that the semi-regular sequence constructed in Lemma 4.1 is far from minimal if s and n are at least 2. However, Fröberg's Conjecture implies that a generic sequence of forms of degree s and of length $\dim S_s$ is a semi-regular sequence that minimally generates $\mathfrak{m}^s$. In fact, that $\mathfrak{m}^s$ can be minimally generated by a semi-regular sequence is equivalent to Fröberg's Conjecture in the special case that all forms in the sequence have degree s.

LEMMA 4.3. *Let $M \subseteq N$ be finitely generated graded S-modules such that minimal generating sets for M and for N have the same number of elements for each degree. Then $M = N$.*

Proof. If the conclusion is false, then let d be the smallest degree in which $M_d \neq N_d$. Without loss of generality, we may replace M and N by their submodules, generated by their components of degrees less than or equal to d. Then N is minimally generated by the minimal generators of M together with a nonempty set of elements of N_d forming a cobasis for the subspace M_d. This contradicts that all minimal generating sets for N have the same cardinality. $\square$

THEOREM 4.4. *Let S/I be Artinian with Castelnuovo–Mumford regularity ρ, and suppose that for $j \leq \rho$ we have that $\beta_{2,j}(S/I)$ is the coefficient of s^2t^j in*

$$\prod_{e=1}^{\rho+1}(1+st^e)^{\beta_{1,e}(S/I)}.$$

Then I can be generated by a semi-regular sequence.

REMARK 4.5. In general, I may fail to be *minimally* generated by a semi-regular sequence: in the construction that we give in the proof below, the semi-regular sequence *ends* with a minimal generating set. Consider for example the ideal

$$I = (x_1^2, x_1x_2x_3, x_1x_2^3, x_1x_3^3, x_2^4, x_2^3x_3, x_2^2x_3^2, x_2x_3^3, x_3^4) \subset k[x_,x_2,x_3] = S.$$

The Hilbert series of S/I is $1 + 3t + 5t^2 + 6t^3$ and its Betti diagram is

total:	1	9	14	6
0:	1	.	.	.
1:	.	1	.	.
2:	.	1	1	.
3:	.	7	13	6

so that the hypothesis of the theorem is satisfied. But, we know that I cannot be minimally generated by a semi-regular sequence, for if so, then it cannot have a non-Koszul second syzygy in degree 4 (this is a due to a result of Migliore and Miró-Roig [4, remark 3.16]). Since I clearly has a non-Koszul syzygy in that degree, we conclude that it cannot be minimally generated by a semi-regular sequence.

Proof. Let $f_1, \ldots, f_r$ be a minimal set of generators of I of degrees $d_1, \ldots, d_r$. We first show that all syzygies of $f_1, \ldots, f_r$ of degree less than or equal to ρ are generated by Koszul syzygies.

Let $F_\bullet \to S/I$ be a minimal free resolution of S/I and let $K_\bullet$ be the Koszul complex of $f_1, \ldots, f_r$. If the map from F_1 to F_0 is chosen so that the basis elements of F_1 go to the generators $f_1, \ldots, f_r$, then we have a commutative diagram

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d] & K_0 \arrow[d] \\ F_1 \arrow[r] & F_0 \end{tikzcd}$$

in which the vertical arrows are isomorphisms. Let Γ be the kernel of the top row and and let Ω be the kernel of the bottom row. Then $\Gamma \cong \Omega$ and $\Gamma_\rho \cong \Omega_\rho$ under the isomorphism above. Furthermore, Ω_ρ is minimally generated by the image of $(F_2)_\rho$. Note that the hypothesis on the graded Betti numbers implies that $(F_2)_\rho$ and $(K_2)_\rho$ have the same number of basis elements of each degree.

We claim that the image of $(K_2)_\rho$ in K_1 is minimally generated by the images of the basis elements of $(K_2)_\rho$. If this is not the case, then there is some basis element κ_σ whose image $f_{\sigma_2}\kappa_{\sigma_1} - f_{\sigma_1}\kappa_{\sigma_2}$ is equal to the image of some S-linear combination of the other basis elements $\sum_{\omega \neq \sigma} g_\omega \kappa_\omega$. Comparing the coefficients of κ_{σ_1} in the images of these two elements, we find that $f_{\sigma_2} = \sum_{j \neq \sigma_1, \sigma_2} \pm g_{\{\sigma_1, j\}} f_j$. Thus, f_{σ_2} is in the ideal generated by the other f_j, contradicting that $f_1, \ldots, f_r$ is a minimal set of generators for I.

The image of $(K_2)_\rho$ in K_1 is contained in Γ_ρ, so that the image of $(K_2)_\rho$ in F_1 is contained in Ω_ρ. It follows immediately from the last paragraph that the image of $(K_2)_\rho$ in Ω_ρ is minimally generated by the images of the basis elements. It then follows from Lemma 4.3 that $(K_2)_\rho$ surjects on to Ω_ρ, which is to say that all of the syzygies of $f_1, \ldots, f_r$ of degree less than or equal to ρ are Koszul syzygies.

Notice that $\mathfrak{m}^{\rho+1} \subseteq I$. Let $g_1, \ldots, g_N$ be a semi-regular sequence generating $\mathfrak{m}^{\rho+1}$ (such a sequence exists by Lemma 4.1). We claim that $g_1, \ldots, g_N$, $f_1, \ldots, f_r$ is a semi-regular sequence. To prove this, it is enough to show that the map

$$\big(S/\big(\mathfrak{m}^{\rho+1} + (f_1, \ldots, f_{i-1})\big)\big)_{a-d_i} \xrightarrow{f_i} \big(S/\big(\mathfrak{m}^{\rho+1} + (f_1, \ldots, f_{i-1})\big)\big)_a$$

has maximal rank for all a.

If $a < d_i$ or $a \geq \rho + 1$, then multiplication by f_i has maximal rank because either the domain or the range is zero-dimensional. If $d_i \leq a \leq \rho$, then we will show that the map is injective, hence, also maximum rank. If $h \in S_{a-d_i}$ is such that $hf_i \in (\mathfrak{m}^{\rho+1} + (f_1, \ldots, f_{i-1}))$, then $hf_i \in (f_1, \ldots, f_{i-1})$ gives a syzygy of degree $\leq a$ in $f_1, \ldots, f_i$ (note that $\deg(h) > 0$ since f_i is a minimal generator of I). Since the only second syzygies of degree $\leq a$ are Koszul (because $a \leq \rho$), we conclude that $h \in (f_1, \ldots, f_{i-1}) \subseteq ((f_1, \ldots, f_{i-1}) + \mathfrak{m}^{\rho+1})$ as required. $\square$

EXAMPLE 4.6. In Theorem 4.4, the Artinian hypothesis is very important; one can easily find counterexamples if it is omitted. For instance, let $S = k[x_1, x_2]$ and suppose that $I = (x_1^2, x_1x_2)$. Then the regularity of S/I is 1, and, for $j \leq 1$, the coefficients of s^2t^j in $P_{S/I}(s,t) = 1 + 2st^2 + s^2t^3$ and $(1+st^2)^2 = 1 + 2st^2 + s^2t^4$ agree; they are all zero. However, there cannot be any semi-regular sequence generating I, because $I \subset (x_1)$. That is for any generating set $f_1, \ldots, f_r$, $x_1|f_1$ and $x_1|f_2$, so f_1, f_2 fails to be semi-regular.

Corollary 4.7. *Suppose that S/I is Artinian with Castelnuovo–Mumford regularity ρ where I is minimally generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$, and that $\beta_{2,j}(S/I)$ agrees with the coefficients of s^2t^j in*

$$\prod_{e=1}^{\rho+1} (1 + st^e)^{\beta_{1,e}(S/I)},$$

for $j \leq \rho$. Let $F_\bullet$ be a minimal resolution of S/I, and $K_\bullet$ be the Koszul complex on $f_1, \ldots, f_r$. Then $F_\bullet^{(\rho-2)} \cong K_\bullet^{(\rho-2)}$.

Proof. By Theorem 4.4, we can find a semi-regular sequence, $g_1, \ldots, g_N$, $f_1, \ldots, f_r$ generating I such that $g_1, \ldots, g_N$ generate $\mathfrak{m}^{\rho+1}$. We can assume that degree$(g_i) \geq \rho + 1$ for all $i = 1, \ldots, N$. By Theorem 3.6, it follows that

$$F_\bullet^{(\rho-2)} \cong K(g_1, \ldots, g_N, f_1, \ldots, f_r)_\bullet^{(\rho-2)}.$$

But because $g_1, \ldots, g_N$ all have degree $\geq \rho + 1$, it is clear that

$$K(g_1, \ldots, g_N, f_1, \ldots, f_r)_\bullet^{(\rho-2)} \cong K_\bullet^{(\rho-2)}.$$

□

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