

## POINT SIMPLICIALITY IN CHOQUET REPRESENTATION THEORY

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ABSTRACT. Let  $\mathcal{H}$  be a function space on a compact space  $K$ . If  $\mathcal{H}$  is not simplicial, we can ask *at which points* of  $K$  there exist unique maximal representing measures. We shall call the set of such points the *set of simpliciality*. The aim of this paper is to examine topological, algebraic and measure-theoretic properties of the set of simpliciality. We shall also define and investigate sets of points enjoying other simplicial-like properties.

### 1. Introduction

The conception of the infinite-dimensional simplex in locally convex spaces was introduced by Choquet, see [3]. Later, it was generalized by means of measure theory for general (nonconvex) compact spaces as the simplicial function space.

Several authors (e.g., Chu [4], Köhn [6], Lima [7]) have studied simpliciality restricted on faces generated by a given point. In Köhn's paper [6], there is an implicit definition of point simpliciality and some equivalent conditions for it. We should also mention an abstract framework due to Simons [14].

In this paper, we define a point of simpliciality, this enables us to consider simpliciality as a point phenomenon. Then we define the set of simpliciality as the set of all points of simpliciality.

Moreover, we use a more general setup (the framework of function spaces), than that used in previous work (cited above) concerned with simpliciality of faces. That was limited to compact convex subsets of locally convex spaces.

The main results of this paper (Theorems 4.1, 4.5, 5.6, 6.2, 6.4) describe properties of the set of simpliciality (and Bauer simpliciality). We also define "generalized" simpliciality in the set of Radon probability measures. This

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provides a characterization of measures supported by the set of simpliciality (Theorem 5.1), which enables us to show that the set of simpliciality is measure extremal (Theorem 5.6).

We obtain an analogy to the following well-known fact [2].

**Fact 1.** Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the Choquet boundary  $\text{Ch}_{\mathcal{H}}(K)$  is a  $G_{\delta}$ -set, and a measure  $\mu \in \mathcal{M}^1(K)$  is maximal if and only if  $\mu(\text{Ch}_{\mathcal{H}}(K)) = 1$ .

The promised analogy, which should be compared to the above fact, is contained in Theorem 4.5 and in the equivalence (2)  $\Leftrightarrow$  (4) in Theorem 5.1. Let us mention the statement here.

**Fact 2.** Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the set of simpliciality  $\text{Sim}_{\mathcal{H}}(K)$  is a  $G_{\delta}$ -set, and, for a measure  $\mu \in \mathcal{M}^1(K)$ , there exists a unique maximal measure  $\nu \in \mathcal{M}^1(K)$ ,  $\mu \preceq \nu$  if and only if  $\mu(\text{Sim}_{\mathcal{H}}(K)) = 1$ .

We refer the reader to the next section for notation and definitions not explained here.

## 2. Preliminaries

At the beginning, we introduce some notation and basic facts concerning Choquet's theory, for details, see e.g., [1], [2], [8], or [12]. All topological spaces in this paper are supposed to be Hausdorff. Let  $K$  be a compact space. The symbol  $C(K)$  stands for the Banach space of all real continuous functions on  $K$  equipped with the sup-norm. A subspace  $\mathcal{H}$  of  $C(K)$  is called a *function space* on  $K$  provided it separates points of  $K$  and contains all constant functions. Notice that the function space  $\mathcal{H}$  does not have to be closed. Let us denote the set of all Radon measures, positive Radon measures and probability Radon measures  $\mathcal{M}(K)$ ,  $\mathcal{M}^+(K)$ , and  $\mathcal{M}^1(K)$ , respectively. These sets of measures are equipped with the weak\* topology. We say that a measure  $\mu \in \mathcal{M}^1(K)$  *represents* a point  $x \in K$  if  $f(x) = \mu f$  for all  $f \in \mathcal{H}$ . If a measure  $\mu$  represents a point  $x \in K$ , we also say that  $x$  is the *barycenter* of  $\mu$ , and we denote  $x = r_{\mu}$ . Since  $\mathcal{H}$  separates points of  $K$ , the barycenter of  $\mu$ , if it exists, is determined uniquely. The set of all measures representing a point  $x \in K$  will be denoted by  $\mathcal{M}_x(\mathcal{H})$ . Further, define an equivalence on  $\mathcal{M}^1(K)$  by

$$\mu \sim \nu \quad \text{if } \mu - \nu \in \mathcal{H}^{\perp},$$

where  $\mathcal{H}^{\perp}$  stands for the annihilator of  $\mathcal{H}$  defined as  $\mathcal{H}^{\perp} = \{\mu \in \mathcal{M}(K) : \mu f = 0 \text{ for all } f \in \mathcal{H}\}$ .

We shall denote the set of all measures  $\nu \in \mathcal{M}^1(K)$  which are equivalent to a given measure  $\mu \in \mathcal{M}^1(K)$  as  $\mathcal{M}_{\mu}(\mathcal{H})$ . Clearly,  $\mathcal{M}_{\varepsilon_x}(\mathcal{H}) = \mathcal{M}_x(\mathcal{H})$ , where  $\varepsilon_x$  denotes the Dirac measure at a point  $x \in K$ . The symbol  $\text{Ch}_{\mathcal{H}}(K)$  stands

for the *Choquet boundary*, which is, by definition, the set of all  $x \in K$  having only one representing measure  $\varepsilon_x$ .

We present two examples of function spaces.

**Example 2.1.** Let  $X$  be a compact convex subset of a locally convex space. The set of all continuous affine functions  $A^c(X)$  is a function space on  $X$ . The Choquet boundary corresponds with the set of extreme points of  $K$ . We will refer to this setting as to the “convex case,” and the symbol  $X$  will always stand for a compact convex subset of a locally convex space. As we will see later (definition of the state space), every compact space (with a function space defined on it) can be considered to be embedded into a certain compact convex set (which depends on the function space). In this sense, the “convex case” is the most important example.

**Example 2.2.** Let  $U$  be a bounded open subset of the Euclidean space  $\mathbb{R}^m$ . Then  $H(U)$ , the family of all continuous functions on  $\overline{U}$  which are harmonic on  $U$ , is a function space on the compact set  $\overline{U}$ . The Choquet boundary of  $H(U)$  corresponds with the set  $U_{\text{reg}}$  of regular points of  $U$  (see [11, Theorem, p. 625]).

We define the *state space* of a function space  $\mathcal{H}$  as

$$S(\mathcal{H}) = \{\varphi \in \mathcal{H}^* : 0 \leq \varphi, \|\varphi\| = 1\}.$$

It is well known that  $\mathcal{H}^*$  is isometrically isomorphic to the quotient space

$$\mathcal{M}(K)/\mathcal{H}^\perp$$

and that

$$S(\mathcal{H}) = \pi(\mathcal{M}^1(K)).$$

Here,  $\pi$  stands for the quotient mapping from  $\mathcal{M}(K)$  to  $\mathcal{H}^*$ . Furthermore, define homeomorphic embedding  $\phi : K \rightarrow S(\mathcal{H}) : x \mapsto \phi_x$  by  $\phi_x = \pi(\varepsilon_x)$ .

A Borel bounded function  $f$  on  $K$  is said to be  $\mathcal{H}$ -affine if  $f(x) = \mu f$ , for all  $x \in K$  and  $\mu \in \mathcal{M}_x(\mathcal{H})$ . Let us denote the set of all  $\mathcal{H}$ -affine functions on  $K$  as  $\mathcal{A}(\mathcal{H})$  and the set of all continuous  $\mathcal{H}$ -affine functions on  $K$  as  $\mathcal{A}^c(\mathcal{H})$ . It is a closed subspace of  $C(K)$  and it contains  $\mathcal{H}$ . In the “convex case,” one has  $\mathcal{A}^c(\mathcal{H}) = \mathcal{H} = A^c(X)$ , hence  $\mathcal{A}^c(\mathcal{H})$  coincides with the set of all continuous affine functions on  $X$ .

A Borel bounded function  $f$  on  $K$  is said to be  $\mathcal{H}$ -convex if  $f(x) \leq \mu f$ , for all  $x \in K$  and  $\mu \in \mathcal{M}_x(\mathcal{H})$ . Denote the set of all  $\mathcal{H}$ -convex functions on  $K$  as  $\mathcal{K}(\mathcal{H})$  and the set of all continuous  $\mathcal{H}$ -convex functions on  $K$  as  $\mathcal{K}^c(\mathcal{H})$ . The cone of all continuous  $\mathcal{H}$ -convex functions induces so-called *Choquet ordering*  $\preceq$  on  $\mathcal{M}^+(K)$  by

$$\mu \preceq \nu \quad \text{if } \mu f \leq \nu f \text{ for all } f \in \mathcal{K}^c(\mathcal{H}).$$

**LEMMA 2.3.** *Let  $f$  be a semicontinuous  $\mathcal{H}$ -convex function and  $\mu, \nu \in \mathcal{M}^+(K)$ . If  $\mu \preceq \nu$ , then  $\mu f \leq \nu f$ .*

*Proof.* Can be found in [9, Lemma 2.7] □

For each measure  $\mu \in \mathcal{M}^1(K)$ , there exists a maximal (in the Choquet ordering) measure  $\nu \in \mathcal{M}^1(K)$  such that  $\mu \preceq \nu$ . The set of all maximal measures representing a point  $x \in K$  will be denoted  $\mathcal{M}_x^{\max}(\mathcal{H})$ . If  $K$  is metrizable, then the Choquet boundary is a Borel measurable set and a measure  $\mu$  is maximal if and only if  $\mu(\text{Ch}_{\mathcal{H}}(K)) = 1$ .

A function space  $\mathcal{H}$  on a compact space  $K$  is called *simplicial* if, for every  $x \in K$ , there exists a unique maximal measure  $\mu \in \mathcal{M}_x(\mathcal{H})$ . Moreover, if  $\text{Ch}_{\mathcal{H}}(K)$  is closed, then  $\mathcal{H}$  is called a *Bauer simplicial space*. We shortly say that a compact convex set  $X$  is a *Choquet simplex*, if  $A^c(X)$  is simplicial.

For a bounded function  $f$  on  $K$ , its upper envelope  $f^*$  is defined as

$$f^* = \inf\{h : h \geq f, h \in \mathcal{H}\},$$

and its lower envelope  $f_*$  as

$$f_* = \sup\{h : h \leq f, h \in \mathcal{H}\}.$$

An upper envelope is upper semicontinuous and  $-f^* \in \mathcal{K}(\mathcal{H})$ .

The following lemma is called the Mokobodzki maximality test ([9, Theorem 2.8]).

**LEMMA 2.4.** *A measure  $\mu \in \mathcal{M}^1(K)$  is maximal if and only if  $\mu f = \mu f^*$  for all  $f \in \mathcal{K}^c(\mathcal{H})$ .*

Let  $X$  be a compact convex subset of a locally convex space, we say that a subset  $F \subset X$  is extremal if for any  $x, y \in X, t \in (0, 1)$  is  $x, y \in F$ , provided  $tx + (1 - t)y \in F$ . If  $F$  is extremal and convex, we say that  $F$  is a *face*. Let  $x \in X$ , and define the *smallest face*  $\text{face}(x)$  containing  $x$  as the intersection of all faces containing  $x$ .

The following proposition is due to Köhn [6, Proposition 2].

**PROPOSITION 2.5.** *Let  $X$  be a compact convex subset of a locally convex space,  $x \in X$ . Then there exists a unique maximal measure representing the point  $x$  if and only if, for every  $y \in \text{face}(x)$ , there exists a unique maximal measure representing the point  $y$ .*

A Borel set  $B \subset K$  is called *measure convex* if every measure  $\mu \in \mathcal{M}^1(K)$  such that  $\mu(B) = 1$  has its barycenter in  $B$ , provided it has the barycenter. A Borel set  $B \subset K$  is called *measure extremal* if for each  $x \in B$  and for each  $\mu \in \mathcal{M}_x(\mathcal{H})$ , it is  $\mu(B) = 1$ . Define the following functionals

$$Q^\mu f = \inf\{\mu h : h \geq f, h \in \mathcal{H}\} \quad \text{and} \quad P^\mu f = \mu f^*,$$

where  $f$  is a bounded Borel function and  $\mu \in \mathcal{M}^1(K)$ . If  $\mu = \varepsilon_x$  for some  $x \in K$ , then

$$Q^\mu f = P^\mu f = f^*(x).$$

Similarly, define functionals  $Q_\mu f = \sup\{\mu h : h \leq f, h \in \mathcal{H}\}$ , and  $P_\mu f = \mu f_*$ .

LEMMA 2.6. For each  $f \in C(K)$  and  $\mu \in \mathcal{M}^1(K)$ , we have

$$Q^\mu f = \sup\{\nu f : \nu \in \mathcal{M}_\mu(\mathcal{H})\}$$

and the supremum is attained.

*Proof.* See [9, Proposition 2.3]. □

COROLLARY 2.7 (Bauer). For each  $f \in C(K)$  and  $x \in K$ , we have

$$f^*(x) = \sup\{\nu f : \nu \in \mathcal{M}_x(\mathcal{H})\}$$

and the supremum is attained.

LEMMA 2.8. For each  $f \in C(K)$  and each  $\mu \in \mathcal{M}^1(K)$ , we have

$$P^\mu f = \sup\{\nu f : \nu \in \mathcal{M}^1(K), \mu \preceq \nu\}$$

and the supremum is attained.

*Proof.* This lemma can be easily proved replacing  $Q^\mu$  by  $P^\mu$  in the proof of Proposition 2.3 in [9]. □

LEMMA 2.9. If a sequence  $(f_n) \subset C(K)$  converges uniformly on  $K$  to a function  $f \in C(K)$ , then the sequence  $(f_n^*)$  converges to  $f^*$  uniformly on  $K$ .

*Proof.* Follows from the inequality  $|f_n^* - f_m^*| \leq \|f_n - f_m\|$ . □

### 3. Examples

Let us start with the pivotal definition of this paper and present some examples.

DEFINITION 3.1. Let  $\mathcal{H}$  be a function space on a compact space  $K$ . We say that  $x \in K$  is a point of simpliciality if there exists only one maximal measure representing the point  $x$ . We denote the set of all points of simpliciality by  $\text{Sim}_{\mathcal{H}}(K)$ , and call it the set of simpliciality. The complement  $K \setminus \text{Sim}_{\mathcal{H}}(K)$  is called the set of nonsimpliciality.

**Remark 3.2.** Clearly,  $\text{Ch}_{\mathcal{H}}(K) \subset \text{Sim}_{\mathcal{H}}(K)$ , in particular, the set of simpliciality is nonempty; and  $\text{Sim}_{\mathcal{H}}(K) = K$  if and only if  $\mathcal{H}$  is simplicial.

**Remark 3.3.** In the “convex case,” for a compact convex set  $X$ , denote the set of simpliciality  $\text{Sim}(X)$ , that is  $\text{Sim}(X)$  stands for  $\text{Sim}_{A^c(X)}(X)$ .

**Example 3.4.** Consider a square in  $\mathbb{R}^2$ . It is a compact convex set which is not a simplex. The set of simpliciality consists of its edges.

**Example 3.5.** Let us introduce “McDonald’s nonsimplex” (Example 1.9 in [10]). Choose  $\mu \in \mathcal{M}([0, 1])$  such that the positive and negative variations  $\mu^+, \mu^-$  are in  $\mathcal{M}^1([0, 1])$  and  $\text{spt}(\mu^+) = \text{spt}(\mu^-) = [0, 1]$ . Define

$$\mathcal{H} = \text{Ker } \mu = \{f \in C([0, 1]) : \mu f = 0\}.$$

Obviously,

$$\mathcal{H}^\perp = \{\nu \in \mathcal{M}([0, 1]) : \nu = \alpha\mu \text{ for some } \alpha \in \mathbb{R}\}.$$

We claim that  $\mathcal{H}$  is a function space on  $[0, 1]$ . Indeed, it contains constant functions since  $\mu([0, 1]) = 0$ , and it separates points of  $[0, 1]$ : choose  $x, y \in [0, 1]$  and suppose  $f(x) = f(y)$  for every  $f \in \mathcal{H}$ . Then  $\varepsilon_x - \varepsilon_y \in \mathcal{H}^\perp$ , hence

$$\varepsilon_x - \varepsilon_y = \alpha\mu = \alpha\mu^+ - \alpha\mu^-,$$

for some  $\alpha \in \mathbb{R}$ . Since  $\text{spt}(\mu^+) = \text{spt}(\mu^-) = [0, 1]$ , one gets  $\alpha = 0$ , and thus  $x = y$ . Further, we will show that  $\text{Ch}_{\mathcal{H}}([0, 1]) = [0, 1]$ . Choose  $x \in [0, 1]$  and  $\nu \in \mathcal{M}_x(\mathcal{H})$ . Then  $\nu - \varepsilon_x \in \mathcal{H}^\perp$ , and thus  $\nu - \varepsilon_x = \alpha\mu$ , for some  $\alpha \in \mathbb{R}$ . Similarly, as above, we have  $\nu = \varepsilon_x$ , and thus  $\text{Ch}_{\mathcal{H}}([0, 1]) = [0, 1]$ . McDonald's nonsimplex is defined as the state space  $S(\mathcal{H})$  of  $\mathcal{H}$ . Let us show that it is not a simplex. Since  $s := \pi(\mu^+) = \pi(\mu^-) \in S(\mathcal{H})$  and

$$r_{\phi\mu^+} = \pi(\mu^+) = \pi(\mu^-) = r_{\phi\mu^-},$$

we see that the point  $s$  has two different representing measures  $\phi\mu^+, \phi\mu^-$  supported by  $\text{Ch}_{\mathcal{H}}([0, 1])$ . Therefore  $\phi\mu^+$  and  $\phi\mu^-$  are maximal, and so  $s$  is not a point of simpliciality.

Now, we want to find the set of simpliciality  $\text{Sim}(S(\mathcal{H}))$ . Suppose that  $\Lambda_1$  and  $\Lambda_2$  are maximal probability measures on  $S(\mathcal{H})$  representing a point  $x \in S(\mathcal{H})$ . Then there exist maximal measures  $\lambda_1, \lambda_2 \in \mathcal{M}^1([0, 1])$  such that  $\Lambda_1 = \phi\lambda_1$  and  $\Lambda_2 = \phi\lambda_2$ . Then  $\lambda_1 - \lambda_2 \in \mathcal{H}^\perp$ , and thus

$$\lambda_1 - \lambda_2 = \alpha\mu = \alpha\mu^+ - \alpha\mu^-,$$

for some  $\alpha \in \mathbb{R}$ . Without loss of generality, assume that  $\alpha \geq 0$ . Since  $\lambda_1 \geq \alpha\mu^+$ , we get  $\alpha \leq 1$ . Hence,

$$\lambda_1 = \alpha\mu^+ + (1 - \alpha)\gamma$$

and

$$\lambda_2 = \alpha\mu^- + (1 - \alpha)\gamma,$$

where  $\gamma$  is a measure from  $\mathcal{M}^1([0, 1])$ . If  $\alpha > 0$ , then we have two different maximal measures  $\Lambda_1, \Lambda_2$  representing the point

$$x = \alpha\phi\mu^+ + (1 - \alpha)\phi\gamma = \alpha\phi\mu^- + (1 - \alpha)\phi\gamma$$

and thus  $x \notin \text{Sim}(S(\mathcal{H}))$ . We conclude that

$$S(\mathcal{H}) \setminus \text{Sim}(S(\mathcal{H})) = \{\alpha\phi\mu^+ + (1 - \alpha)\phi\gamma : \alpha \in (0, 1], \gamma \in \mathcal{M}^1([0, 1])\}.$$

**Example 3.6.** The last example deals with convex functions on  $[0, 1]$ . A similar set of functions was investigated from the point of view of the noncompact Choquet theory by Rakestraw [13]. Let us define the following set of functions

$$Z = \{f : [0, 1] \rightarrow [0, \infty), f \text{ convex}, f(0) + f(1) = 1\}.$$

This set is convex and compact in  $\{f : [0, 1] \rightarrow \mathbb{R}, \text{ bounded, continuous on } (0, 1)\}$  with respect to the topology of pointwise convergence.

The set of extreme points is

$$\text{ext}(Z) = \{g_y^1, g_y^2 : y \in [0, 1]\},$$

where

$$g_y^1(x) = \begin{cases} 0, & 0 \leq x \leq y < 1, \\ \frac{x-y}{1-y}, & 0 \leq y < x \leq 1, \end{cases}$$

for  $y \in [0, 1)$ ,

$$g_y^2(x) = \begin{cases} 1 - \frac{x}{y}, & 0 \leq x < y \leq 1, \\ 0, & 0 < y \leq x \leq 1, \end{cases}$$

for  $y \in (0, 1]$ , and

$$g_1^1(x) = \chi_{\{1\}}(x), \quad g_0^2(x) = \chi_{\{0\}}(x).$$

A function  $f \in Z$  belongs to the set of simpliciality if and only if it satisfies at least one of these conditions:

- $f$  is affine on  $(0, 1)$ ,
- $\inf_{x \in [0, 1]} f(x) = 0$ .

Verification of these facts is rather technical but elementary.

In the following three sections, we will present the main results concerning the set of simpliciality and the set of Bauer simpliciality. The letter “ $K$ ” stands for a compact space, whereas we use the letter “ $X$ ” instead, for a convex compact subset of a locally convex space.

#### 4. Properties of the set of simpliciality

In the previous section, we introduced examples of nonsimplicial function spaces for which we were able to find the sets of simpliciality. Now let us investigate some general properties of the set of simpliciality.

**THEOREM 4.1.** *Let  $X$  be a compact convex subset of a locally convex space. Then the set of simpliciality  $\text{Sim}(X)$  is extremal and, consequently, the set of nonsimpliciality  $X \setminus \text{Sim}(X)$  is convex.*

*Proof.* According to Proposition 2.5, if the set  $\text{Sim}_{\mathcal{H}}(K)$  contains a point  $x \in X$ , it also contains  $\text{face}(x)$ . Hence,

$$\text{Sim}(X) = \bigcup_{x \in \text{Sim}(X)} \{x\} \subset \bigcup_{x \in \text{Sim}(X)} \text{face}(x) \subset \text{Sim}(X).$$

Then

$$\text{Sim}(X) = \bigcup_{x \in \text{Sim}(X)} \text{face}(x).$$

It is straightforward to verify that a union of faces is an extremal set. This finishes the proof. Another possibility is to prove first that  $X \setminus \text{Sim}(X)$  is convex. □

**Remark 4.2.** In the “convex case,” the set  $\text{Sim}(X)$  is the complementary set to  $X \setminus \text{Sim}(X)$  and we have a disjoint union  $X \setminus \text{Sim}(X) \cup (X \setminus \text{Sim}(X))' = X$ . Recall that, for a subset  $A$  of a compact convex set  $X$ , the *complementary set*  $A'$  is defined as the union of all faces of  $X$  disjoint with  $A$ .

**Remark 4.3.** In general, if a compact space  $K$  does not have an algebraic structure, we can ask whether the set  $\text{Sim}_{\mathcal{H}}(K)$  is measure extremal, or equivalently, whether the set  $K \setminus \text{Sim}_{\mathcal{H}}(K)$  is measure convex. But we do not know yet that these sets are Borel measurable. Recall that in the “convex case” a measure convex set is convex, but a convex set is not necessarily measure convex, and similarly, a measure extremal set is extremal, but an extremal set is not necessarily measure extremal. For counterexamples, see [5], and [9, Examples 4.3].

Now, let us present some characterizations of point simpliciality for function spaces which will be useful further on. “Global version” of Proposition 4.4 for the “convex case” can be found in [12, Theorem, p. 56].

**PROPOSITION 4.4.** *Let  $\mathcal{H}$  be a function space on a compact space  $K$  and  $M$  a dense (with respect to the norm topology) subset of  $\mathcal{K}^c(\mathcal{H})$ . Let  $x \in K$ . The following assertions are equivalent:*

- (1)  $x \in \text{Sim}_{\mathcal{H}}(K)$ ,
- (2)  $f^*(x) = \mu f^*$  for all  $f \in M$  and  $\mu \in \mathcal{M}_x(\mathcal{H})$ ,
- (3)  $f^*(x) = \mu f$  for all  $f \in M$  and  $\mu \in \mathcal{M}_x^{\max}(\mathcal{H})$ ,
- (4)  $(f + g)^*(x) = f^*(x) + g^*(x)$  for all  $f, g \in M$ .

*Proof.* If  $M = \mathcal{K}^c(\mathcal{H})$ , the proof is similar to the proof of the “global version” in the “convex case,” [12, Theorem, p. 56]. For an arbitrary dense  $M \subset \mathcal{K}^c(\mathcal{H})$ , the proof follows from Lemma 2.9 and the Lebesgue Dominated theorem. □

**THEOREM 4.5.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the set of simpliciality  $\text{Sim}_{\mathcal{H}}(K)$  is a  $G_{\delta}$ -set.*

*Proof.* Since  $K$  is metrizable, the space  $C(K)$  is separable, and thus  $\mathcal{K}^c(\mathcal{H})$  is such. Choose a dense countable set  $M \subset \mathcal{K}^c(\mathcal{H})$ . According to (4), in Proposition 4.4, we have

$$\text{Sim}_{\mathcal{H}}(K) = \{x \in K : f^*(x) + g^*(x) = (f + g)^*(x) \text{ for all } f, g \in M\}.$$

Hence,

$$\begin{aligned} \text{Sim}_{\mathcal{H}}(K) &= \bigcap_{k \in \mathbb{N}} \bigcap_{f, g \in M} \bigcap_{(h \in \mathcal{H}, h \geq f + g)} \left\{ x \in K : f^*(x) + g^*(x) - h(x) < \frac{1}{k} \right\} \\ &= \bigcap_{k \in \mathbb{N}} \bigcap_{f, g \in M} \bigcap_{(h \in \mathcal{N}, h \geq f + g)} \left\{ x \in K : f^*(x) + g^*(x) - h(x) < \frac{1}{k} \right\}, \end{aligned}$$



where  $N$  is a dense countable subset of  $\mathcal{H}$ . The function  $f^* + g^* - h$  is upper semicontinuous, hence the set  $\{x \in K : f^*(x) + g^*(x) - h(x) < \frac{1}{k}\}$  is open for each  $k \in \mathbb{N}$ . We conclude that  $\text{Sim}_{\mathcal{H}}(K)$  is a  $G_{\delta}$ -set.  $\square$

**Remark 4.6.** The set of simpliciality can be closed in the compact space  $K$  as we saw in Example 3.4. But in the “convex case,” it cannot be open in  $X$ . Moreover, its interior in  $X$  is empty (of course, provided  $\text{Sim}(X) \neq X$ ). Indeed, if there exists a point  $x$  in interior of  $\text{Sim}(X)$ , then for arbitrary point  $y \in K \setminus \text{Sim}(X)$  one can find  $z \in \text{Sim}(X)$  on the line segment, say  $z = \lambda x + (1 - \lambda)y$ , for some  $\lambda \in (0, 1)$ . Let  $\mu_x \in \mathcal{M}_x^{\max}(A^c(X))$  and  $\mu_y^1, \mu_y^2 \in \mathcal{M}_y^{\max}(A^c(X))$ . Then

$$\lambda\mu_x + (1 - \lambda)\mu_y^1$$

and

$$\lambda\mu_x + (1 - \lambda)\mu_y^2$$

are two different maximal measures representing the point  $z$ , which is a contradiction to  $z \in \text{Sim}(X)$ .

Generally, in a “nonconvex case,” the set of simpliciality can be open. To show this, consider the set  $K = \{[0, 0], [1, 1], [1, -1], [-1, -1], [-1, 1]\} \subset \mathbb{R}^2$  equipped with the relative topology from  $\mathbb{R}^2$ . Then  $K$  is a compact set and restrictions of affine functions form a function space  $\mathcal{H}$ . Clearly,  $\text{Sim}_{\mathcal{H}}(K) = \{[1, 1], [1, -1], [-1, -1], [-1, 1]\}$ , which is an open set in  $K$ .

**COROLLARY 4.7.** *Let  $X$  be a compact convex subset of a locally convex space. The set  $X \setminus \text{Sim}(X)$  of nonsimpliciality is dense in  $X$ , provided it is not empty.*

*Proof.* Follows immediately from Remark 4.6.  $\square$

### 5. Generalized simpliciality

Let  $K$  be a compact space. We know that a point  $x \in K$  is a point of simpliciality if there exists only one maximal measure  $\nu \in \mathcal{M}^1(K)$  representing  $x$ . That is, if there exists a unique maximal measure  $\nu \in \mathcal{M}^1(K)$ , such that  $\varepsilon_x \preceq \nu$ . For every  $x \in K$  and  $\nu \in \mathcal{M}^1(K)$ , the following equivalence holds

$$\varepsilon_x \preceq \nu \quad \text{if and only if} \quad \varepsilon_x \sim \nu.$$

In general, for measures  $\mu, \nu \in \mathcal{M}^1(K)$ , we have only implication

$$\text{if } \mu \preceq \nu, \text{ then } \mu \sim \nu.$$

We can ask when, for a given measure  $\mu \in \mathcal{M}^1(K)$ , there exists a unique maximal measure  $\nu \in \mathcal{M}^1(K)$  such that  $\mu \preceq \nu$ , and when there exists a unique maximal measure  $\nu \in \mathcal{M}^1(K)$  such that  $\mu \sim \nu$ . We will see that such measure  $\mu$  can be characterized by the functionals  $P^\mu$  and  $Q^\mu$ , respectively, in a similar way like a point of simpliciality in Proposition 4.4(4).

We shall say that a measure  $\mu \in \mathcal{M}^1(K)$  belongs to the set PS if

$$P^\mu f + P^\mu g = P^\mu(f + g)$$

for all  $f, g \in \mathcal{K}^c(\mathcal{H})$ . Similarly, we shall say that a measure  $\mu \in \mathcal{M}^1(K)$  belongs to the set QS if

$$Q^\mu f + Q^\mu g = Q^\mu(f + g)$$

for all  $f, g \in \mathcal{K}^c(\mathcal{H})$ .

**THEOREM 5.1.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$  and  $\mu \in \mathcal{M}^1(K)$ . The following assertions are equivalent:*

- (1)  $\mu \in \text{PS}$ ,
- (2) there exists only one maximal measure  $\nu \in \mathcal{M}^1(K)$  such that  $\mu \preceq \nu$ ,
- (3) for every maximal measure  $\nu \in \mathcal{M}^1(K)$ ,  $\mu \preceq \nu$  and every  $f \in \mathcal{K}^c(\mathcal{H})$ , we have  $\nu f = P^\mu f$ ,
- (4)  $\mu$  is supported by the set  $\text{Sim}_{\mathcal{H}}(K)$ , that is  $\mu(\text{Sim}_{\mathcal{H}}(K)) = 1$ .

*Proof.* (1)  $\Rightarrow$  (2) Suppose that

$$P^\mu f + P^\mu g = P^\mu(f + g),$$

for all  $f, g \in \mathcal{K}^c(\mathcal{H})$ . Let us define a linear functional  $\varphi$  on  $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$  as

$$\varphi(f - g) = P^\mu f - P^\mu g \quad \text{for } f, g \in \mathcal{K}^c(\mathcal{H}).$$

It is straightforward to verify that the definition does not depend on the choice of  $f, g \in \mathcal{K}^c(\mathcal{H})$ . Hence the functional  $\varphi$  is well defined. Further, it is bounded and  $\|\varphi\| = 1$ . Indeed,

$$(1) \quad \varphi(f - g) = P^\mu f - P^\mu g \leq P^\mu(f - g) \leq \|f - g\|$$

implies after changing  $f$  and  $g$  that

$$|\varphi(f - g)| \leq \|f - g\|,$$

and hence  $\|\varphi\| \leq 1$ . Further,  $\varphi(1) = 1$  implies  $\|\varphi\| = 1$ . The first inequality in (1) follows from

$$P^\mu f = P^\mu[(f - g) + g] \leq P^\mu(f - g) + P^\mu g \quad \text{for all } f, g \in \mathcal{K}^c(\mathcal{H}).$$

Since  $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$  is a dense subspace of  $C(K)$ , there exists a uniquely determined linear extension  $\nu$  of  $\varphi$  to whole  $C(K)$ , such that  $\|\nu\| = \|\varphi\|$ . Using Riesz's representation theorem, one can assume that  $\nu \in \mathcal{M}^1(K)$ . Take a function  $f \in \mathcal{K}^c(\mathcal{H})$ , then  $\nu f = P^\mu f \geq \mu f$ . That is  $\mu \preceq \nu$ . According to Lemma 2.8, for any measure  $\lambda \in \mathcal{M}^1(K)$  such that  $\mu \preceq \lambda$  we have  $\lambda \preceq \nu$ . We conclude that  $\nu$  is the unique maximal measure such that  $\mu \preceq \nu$ .

(2)  $\Rightarrow$  (3) Follows immediately from Lemma 2.8.

(3)  $\Rightarrow$  (1) Let  $f, g \in \mathcal{K}^c(\mathcal{H})$  and  $\mu \in \mathcal{M}^1(K)$ . Take a maximal measure  $\nu \in \mathcal{M}^1(K)$ ,  $\mu \preceq \nu$ . Then  $P^\mu(f + g) = \nu(f + g) = \nu f + \nu g = P^\mu f + P^\mu g$ .

(1)  $\Rightarrow$  (4) Suppose that for  $f, g \in \mathcal{K}^c(\mathcal{H})$  is

$$P^\mu f + P^\mu g = P^\mu(f + g).$$

Hence,

$$\mu f^* + \mu g^* = \mu(f + g)^*,$$

and

$$\mu(f^* + g^* - (f + g)^*) = 0.$$

Since the function  $f^* + g^* - (f + g)^*$  is nonnegative, we have  $f^* + g^* = (f + g)^*$   $\mu$ -almost everywhere. Using characterization (4) in Proposition 4.4, we get  $\mu(\text{Sim}_{\mathcal{H}}(K)) = 1$ .

Implication (4)  $\Rightarrow$  (1) can be proven by following the previous lines backwards. □

**Remark 5.2.** The equivalences (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) in the previous Theorem 5.1 were proven in [6, Proposition 1] for a compact convex set in a locally convex space. The equivalence (2)  $\Leftrightarrow$  (4) is analogous to this well-known statement: a probability Radon measure on a metrizable compact space is maximal if and only if it is supported by the Choquet boundary [2].

**THEOREM 5.3.** *The following assertions are equivalent for a measure  $\mu \in \mathcal{M}^1(K)$ :*

- (1)  $\mu \in \text{QS}$ ,
- (2) *there exists only one maximal measure  $\nu \in \mathcal{M}^1(K)$  such that  $\mu \sim \nu$ ,*
- (3) *for every maximal measure  $\nu \in \mathcal{M}_{\mu}(\mathcal{H})$  and every  $f \in \mathcal{K}^c(\mathcal{H})$ , we have  $\nu f = \text{Q}^{\mu} f$ .*

*Proof.* Analogous to the proof of equivalences (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) in Theorem 5.1. □

**Remark 5.4.** The following statements are easy to verify.

- $\{\varepsilon_x : x \in \text{Sim}_{\mathcal{H}}(K)\} \subset \text{QS} \subset \text{PS}$ .

In the “convex case,” we have:

- $\text{QS} = \{\mu \in \mathcal{M}^1(X) : r_{\mu} \in \text{Sim}(X)\}$ ,
- $X$  is a simplex if and only if, for every  $\mu \in \mathcal{M}^1(X)$ , there exists a unique maximal measure  $\nu \in \mathcal{M}^1(X)$ , such that  $\mu \preceq \nu$ , and this is the case if and only if, for every  $\mu \in \mathcal{M}^1(X)$ , there exists a unique maximal measure  $\nu \in \mathcal{M}^1(X)$  such that  $\mu \sim \nu$ .

**Remark 5.5.** According to the Bauer characterization, we know that  $x \in \text{Ch}_{\mathcal{H}}(K)$  if and only if  $f^*(x) = f_*(x)$ , for all  $f \in C(K)$ , cf. Corollary 2.7. In the same way, one can define the “generalized boundaries”

$$\partial_P = \{\mu \in \mathcal{M}^1(K) : P^{\mu} f = P_{\mu} f \text{ for all } f \in C(K)\}$$

and

$$\partial_Q = \{\mu \in \mathcal{M}^1(K) : Q^{\mu} f = Q_{\mu} f \text{ for all } f \in C(K)\}.$$

Let  $\mathcal{H}$  be a function space on a compact space  $K$ . The Mokobodzki test immediately yields that

$$\mu \in \partial_P \quad \text{if and only if} \quad \mu \text{ is maximal,}$$

and

$$\mu \in \partial_Q \quad \text{if and only if} \quad \mathcal{M}_\mu(\mathcal{H}) = \{\mu\}.$$

Clearly,  $\partial_Q \subset \partial_P \subset \text{PS}$ .

**THEOREM 5.6.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the set  $\text{Sim}_{\mathcal{H}}(K)$  of simpliciality is measure extremal and the set  $K \setminus \text{Sim}_{\mathcal{H}}(K)$  of nonsimpliciality is measure convex.*

*Proof.* From Theorem 4.5, we know that the set  $\text{Sim}_{\mathcal{H}}(K)$  is Borel measurable. Choose  $x \in \text{Sim}_{\mathcal{H}}(K)$  and  $\mu \in \mathcal{M}_x(\mathcal{H})$ . Then there is a unique maximal measure  $\nu \in \mathcal{M}^1(K)$  such that  $\mu \preceq \nu$ . According to Theorem 5.1, we have  $\mu(\text{Sim}_{\mathcal{H}}(K)) = 1$ . It means that  $\text{Sim}_{\mathcal{H}}(K)$  is measure extremal. Now choose a measure  $\lambda \in \mathcal{M}^1(K)$  having its barycenter  $r_\lambda$  in  $K$  and suppose  $\lambda(K \setminus \text{Sim}_{\mathcal{H}}(K)) = 1$ . If  $r_\lambda \in \text{Sim}_{\mathcal{H}}(K)$ , we get a contradiction to measure extremality of  $\text{Sim}_{\mathcal{H}}(K)$ . We conclude that  $K \setminus \text{Sim}_{\mathcal{H}}(K)$  is measure convex.  $\square$

**Remark 5.7.** We can reformulate the previous result as follows. If  $x \in K$  is a point of simpliciality, then  $\mu$ -almost all points in  $K$  are points of simpliciality, provided  $\mu \in \mathcal{M}_x(\mathcal{H})$ .

## 6. The set of Bauer simpliciality

To define *point Bauer simpliciality*, one needs some characterization of Bauer simpliciality which enables us to localize the conception of closed Choquet boundary of a function space. We used the following Proposition 6.1. For its proof in the “convex case,” see [7, Theorem 7].

**PROPOSITION 6.1.** *Let  $\mathcal{H}$  be a function space on a compact space  $K$ . Then  $\mathcal{H}$  is a Bauer simplicial space if and only if it is simplicial and a CE-space.*

Recall that a function space is called a *CE-space*, if the upper envelopes of continuous functions are continuous.

Let  $\mathcal{H}$  be a function space on a compact space  $K$ . We say that an  $x \in K$  is a *point of continuity of envelopes* (or a *CE-point*) if the upper envelopes  $f^*$  are continuous at the point  $x$  for all  $f \in C(K)$ . Denote the set of all such points of  $K$  by  $\text{CE}_{\mathcal{H}}(K)$ .

**THEOREM 6.2.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . The Choquet boundary  $\text{Ch}_{\mathcal{H}}(K)$  is contained in the set  $\text{CE}_{\mathcal{H}}(K)$ , in particular,  $\text{CE}_{\mathcal{H}}(K)$  is nonempty.*

*Proof.* Consider  $x \in \text{Ch}_{\mathcal{H}}(K)$  and a sequence  $x_n \in K$  such that  $x_n \rightarrow x$ . We want to show  $f^*(x_n) \rightarrow f^*(x)$  for an arbitrary  $f \in C(K)$ . By Corollary 2.7, there exists  $\mu_n \in \mathcal{M}_{x_n}(\mathcal{H})$ , for every  $n \in \mathbb{N}$ , such that  $f^*(x_n) = \mu_n f$ . We claim that the sequence  $(\mu_n)$  converges to  $\varepsilon_x$ . If it be to the contrary, there exists a neighborhood  $U$  of  $\varepsilon_x$  such that  $\mu_n \notin U$ , for infinitely many  $n \in \mathbb{N}$ . By compactness of  $\mathcal{M}^1(K)$ , we can find a subsequence  $(\mu_{n_k}), \mu_{n_k} \notin U$  and a measure  $\mu \in \mathcal{M}^1(K)$  such that  $\mu_{n_k} \rightarrow \mu$ . Especially, for  $h \in \mathcal{H}$ , we have  $\mu_{n_k} h \rightarrow \mu h$ . Since  $\mu_{n_k} h = h(x_{n_k})$ , for every  $k \in \mathbb{N}$ , and  $h(x_{n_k}) \rightarrow h(x)$ , we get  $\mu h = h(x)$ . Hence,  $\mu \in \mathcal{M}_x(\mathcal{H})$ , which, together with  $x \in \text{Ch}_{\mathcal{H}}(K)$ , yields  $\mu = \varepsilon_x$ . Contradiction.

Thus, we get  $f^*(x_n) = \mu_n f \rightarrow f(x)$ . Since  $x \in \text{Ch}_{\mathcal{H}}(K)$ , we have  $f^*(x) = f(x)$ , which finishes the proof.  $\square$

Now, being inspired by Proposition 6.1, we define the *set of Bauer simpliciality*,  $\text{BSim}_{\mathcal{H}}(K)$ , as

$$\text{BSim}_{\mathcal{H}}(K) = \text{Sim}_{\mathcal{H}}(K) \cap \text{CE}_{\mathcal{H}}(K).$$

**Remark 6.3.** Remark 3.2 and Proposition 6.2 yield

$$\text{Ch}_{\mathcal{H}}(K) \subset \text{BSim}_{\mathcal{H}}(K).$$

In particular,  $\text{BSim}_{\mathcal{H}}(K)$  is nonempty. By Proposition 6.1, a function space  $\mathcal{H}$  is Bauer simplicial if and only if  $\text{BSim}_{\mathcal{H}}(K) = K$ .

**THEOREM 6.4.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the set  $\text{CE}_{\mathcal{H}}(K)$  is a dense  $G_{\delta}$ -set subset of  $K$ .*

*Proof.* Since  $K$  is metrizable,  $C(K)$  is separable. Let  $M$  be a countable dense subset of  $C(K)$ . Using Lemma 2.9, it is easy to verify that  $x \in \text{CE}_{\mathcal{H}}(K)$  if (and only if)  $f^*$  is continuous at the point  $x$  for every  $f \in M$ . Consider a function  $g \in M$ . As  $g^*$  is upper semicontinuous, the set of points of continuity of  $g^*$  is a dense  $G_{\delta}$ -set. Since  $M$  is countable, the set  $\text{CE}_{\mathcal{H}}(K)$  is also a  $G_{\delta}$ -set and the Baire Category Theorem yields that  $\text{CE}_{\mathcal{H}}(K)$  is dense in  $K$ .  $\square$

**COROLLARY 6.5.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the set  $\text{CE}_{\mathcal{H}}(K)$  is residual, that is, its complement is of the first category.*

**COROLLARY 6.6.** *Let  $\mathcal{H}$  be a function space on a metrizable compact space  $K$ . Then the set  $\text{BSim}_{\mathcal{H}}(K)$  is a  $G_{\delta}$ -set.*

*Proof.* Follows immediately from Theorems 4.5 and 6.4.  $\square$

**Question 6.7.** The Example 3.5 can be generalized in the following way. Let

$$M \subset \{\mu \in \mathcal{M}([0, 1]) : \mu^+, \mu^- \in \mathcal{M}^1([0, 1]), \text{spt}(\mu^+) = \text{spt}(\mu^-) = [0, 1]\}$$

and define the function space  $\mathcal{H}$  on the compact space  $[0, 1]$  as follows

$$\mathcal{H} = M^{\perp} = \{f \in C([0, 1]) : \mu f = 0 \text{ for all } \mu \in M\}.$$

In this general setting, we were not able to find the set of simpliciality  $\text{Sim}(S(\mathcal{H}))$ .

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