# THE SUPREMUM OF AUTOCONVOLUTIONS, WITH APPLICATIONS TO ADDITIVE NUMBER THEORY

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ABSTRACT. We adapt a number-theoretic technique of Yu to prove a purely analytic theorem: if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is nonnegative and supported on an interval of length $I$, then the supremum of $f * f$ is at least $0.631\|f\|_1^2/I$. This improves the previous bound of $0.591389\|f\|_1^2/I$. Consequently, we improve the known bounds on several related number-theoretic problems. For a set $A \subseteq \{1, 2, \ldots, n\}$, let $g$ be the maximum multiplicity of any element of the multiset $\{a_1 + a_2 : a_i \in A\}$. Our main corollary is the inequality $gn > 0.631|A|^2$, which holds uniformly for all $g$, $n$, and $A$.

## 1. Introduction

One measure of the "flatness" of a nonnegative function $f$ is the ratio of its $L^\infty$ norm to its $L^1$ norm. If $f$ is supported on an interval of length $I$, then we trivially have $\|f\|_\infty \geq \|f\|_1/I$, and equality holds exactly if $f$ is the indicator function of the interval. However, it seems difficult for a convolution of nonnegative functions to be flat throughout its domain, and we seek an improved inequality that reflects this difficulty.

Define the autoconvolution of $f$ to be

$$f^{*2}(x) := f * f(x) := \int_{\mathbb{R}} f(y)f(x - y)\,dy,$$

and for integers $h > 2$ let $f^{*h} = f^{*(h-1)} * f$ be the $h$-fold autoconvolution. Since the $h$-fold autoconvolution is supported on an interval of length $hI$,

we have the trivial lower bound $\|f^{*h}\|_\infty \geq \|f^{*h}\|_1/hI = \|f\|_1^h/hI$. Our main theorem shows that the constant can be improved for even integers $h \geq 2$.

THEOREM 1.1. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is nonnegative and supported on an interval of length $I$, and $h$ is an even positive integer, then*

$$\|f^{*h}\|_\infty > \frac{1.262}{hI}\|f\|_1^h.$$

By replacing $f(x)$ with $c_1 f(c_2 x)$, we may assume without loss of generality that $I = 1/h$ and $\|f\|_1 = 1$, in which case $f^{*h}$ is supported on an interval of length 1 and the conclusion is simply

$$\|f^{*h}\|_\infty > 1.262.$$

In the case $h = 2$, this inequality improves the bound $\|f * f\|_\infty \geq 1.182778$ of the authors [5].

Before giving a one-paragraph summary of the proof of Theorem 1.1, we state some of its applications, which will be deduced from the theorem in Section 4. The main topic of [5] is to give upper and lower bounds on the constant appearing in Corollary 1.2, a result in continuous Ramsey theory concerning *centrally symmetric sets*, which are sets $S$ such that $S = c - S$ for some real number $c$. As a consequence of Theorem 1.1, we obtain the following improved lower bound.

COROLLARY 1.2. *Every Lebesgue measurable subset of $[0,1]$ with measure $\varepsilon$ contains a centrally symmetric subset with measure $0.631\varepsilon^2$.*

Although Theorem 1.1 is a purely analytic theorem, it has applications to discrete problems, specifically problems in additive number theory. If $A$ is a finite set, let $A + A$ be the multiset $\{a_1 + a_2 : a_i \in A\}$. We call the set $A$ a $B^*[g]$ *set* if no element of the multiset $A + A$ has multiplicity greater than $g$. When $g$ is even, it is common to call $A$ a $B_2[g/2]$ set. Theorem 1.1 yields the following upper bound on the size of $B^*[g]$.

COROLLARY 1.3. *If $A \subseteq \{1, 2, \ldots, n\}$ is a $B^*[g]$ set, then $|A| < 1.258883\sqrt{gn}$.*

In [2], by comparison, Cilleruelo and Vinuesa construct $B^*[g]$ sets with $|A| > (\frac{2}{\sqrt{\pi}} - \varepsilon)\sqrt{gn}$, provided that $g$ is sufficiently large in terms of $\varepsilon > 0$, and $n$ is sufficiently large in terms of $g$ and $\varepsilon$. The previous best upper bound is Yu's [8], who proved that for every fixed even integer $g \geq 2$,

$$\limsup_{n \to \infty} \left( \max_{\substack{A \subseteq \{1,2,\ldots,n\} \\ A \text{ is a } B^*[g] \text{ set}}} \frac{|A|}{\sqrt{gn}} \right) < 1.2649.$$

Our corollary improves on Yu's result in three ways. First, we have simplified and shortened the proof considerably. Second, we get a smaller constant. We do use a better kernel function—possibly even the optimal kernel—but that makes very little numerical difference. The bulk of the numerical improvement

comes from a lower bound for a diagonal quadratic form in the real parts of Fourier coefficients, inspired by a Fourier coefficient bound used in [4] and, in a different form, in [5]. The third improvement is that Corollary 1.3 is uniform—it applies to all $g$ and $n$. For example, the uniformity allows us to restate Corollary 1.3 as the following.

COROLLARY 1.4. *If* $A \subseteq \{1, 2, \ldots, n\}$ *has cardinality at least* $\varepsilon n$, *then there is an element* $s \in A + A$ *with multiplicity greater than* $0.631\varepsilon^2 n$.

These discrete results can also be phrased in terms of *Newman polynomials*, which are polynomials all of whose coefficients are 0 or 1. Inequalities for Newman polynomials that relate $p(1)$, $\deg(p)$, and $H(p^2)$ have recently received some attention ([3], [1]); here $H(q)$ is the *height* of the polynomial $q$, defined to be $H(q) := \max\{|q_i| : 0 \le i \le \deg(q)\}$ if $q(x) = \sum_{i=0}^{\deg(q)} q_i x^i$. Note that $p(1)$, for a Newman polynomial $p$, counts the number of coefficients of $p$ that equal 1. The main quantity of interest is the ratio

$$R(p) := \frac{H(p^2)(\deg(p) + 1)}{p(1)^2},$$

which we can bound from below for all polynomials with nonnegative real coefficients, not just Newman polynomials.

COROLLARY 1.5. *If* $p$ *is a polynomial with nonnegative real coefficients, then* $R(p) > 0.631$.

Berenhaut and Saidak [1] have constructed a sequence of polynomials that yields $R(p) \to 8/9 \approx 0.89$, and a construction of Dubickas [3] yields $R(p) \to 5/6 \approx 0.83$. The authors' work on $B^*[g]$ sets ([6], [5]) can be rephrased in terms of Newman polynomials; that work shows that if $\varepsilon > 0$, then there is a sequence of Newman polynomials with $p(1)/\deg(p) \to \varepsilon$ and $R(p) \to \pi/(1 + \sqrt{1 - \varepsilon})^2$. In particular, there is a sequence of Newman polynomials with $p(1)/\deg(p) \to 0$ and $R(p) \to \pi/4 \approx 0.7854$. We conjecture that $\pi/4$ is the best possible constant.

The proof of Theorem 1.1 proceeds by forming upper and lower bounds on the integral

$$\int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K(x) \, dx,$$

where $f \circ f$ is the autocorrelation of $f$ and $K$ is a kernel function to be chosen later. A trivial upper bound is used for $\int_{\mathbb{R}} f * f(x) K(x) \, dx$, while the upper bound on $\int_{\mathbb{R}} f \circ f(x) K(x) \, dx$ uses Parseval's identity to convert the integral into a sum over Fourier coefficients; the central coefficient is pulled out of the sum, and Cauchy–Schwarz is used to bound the remaining terms. The lower bound proceeds by using Parseval's identity, applied to Fourier coefficients with a period smaller than 1, to express the integral in terms of the Fourier coefficients of $f$ and $K$, and then bounding the resulting quadratic form in

the real parts of the Fourier coefficients. This strategy reflects Yu's approach from [8] in two ways: noting that the Fourier coefficients of $f * f + f \circ f$ have nonnegative real parts, and using Fourier coefficients to two different periods.

After defining some notation and special kernel functions in Section 2, we prove Theorem 1.1 in Section 3. The corollaries are deduced from the theorem in Section 4; we prove Corollaries 1.2 and 1.5 first and then derive the other two corollaries from Corollary 1.5. Finally, in Section 5 we indicate the extent to which Theorem 1.1 could be improved.

## 2. Notation and kernel functions

For any integrable function $g$, we use the notation

$$g * g(x) := \int_{\mathbb{R}} g(y)g(x-y)\,dy$$

for its autoconvolution and

$$g \circ g(x) := \int_{\mathbb{R}} g(y)g(x+y)\,dy$$

for its autocorrelation. We define its Fourier transform $\hat{g}$ by

$$\hat{g}(\xi) := \int_{\mathbb{R}} g(x)e^{-2\pi i \xi x}\,dx,$$

and we note that the Fourier transforms of its autoconvolution and autocorrelation satisfy

$$\widehat{g * g}(\xi) = \hat{g}(\xi)^2 \quad \text{and} \quad \widehat{g \circ g}(\xi) = |\hat{g}(\xi)|^2.$$

By a *probability density function*, or *pdf*, we mean a nonnegative function $g$ such that $\|g\|_1 = 1$. Note that for any pdf $g$, we have $\hat{g}(0) = 1$.

Often we will speak of the $j$th "Fourier coefficient" of a function $g$ defined on the real line; by this we mean the $j$th Fourier coefficient of the periodization $\sum_{k \in \mathbb{Z}} g(x+k)$ of $g$. The notation $\hat{g}(j)$ represents two equal quantities: the Fourier transform of $g$ evaluated at the real number $j$, and the $j$th Fourier coefficient of the periodization of $g$. In any case, the reader can generally replace occurrences of "Fourier coefficient" with "value of the Fourier transform at integers" without changing the meaning.

As mentioned in the Introduction, our method involves using Fourier coefficients corresponding to a period smaller than 1, which we can introduce as a differently normalized Fourier transform. We let $\delta$ be a parameter that lies between 0 and $\frac{1}{4}$ (we will at the end set $\delta = 13/100$), and for notational convenience we set $u := \delta + \frac{1}{2}$. We define the differently normalized Fourier transform $\tilde{g}$ of an integrable function $g$ by

$$\tilde{g}(\xi) := \frac{1}{u} \int_{\mathbb{R}} g(x)e^{-2\pi i x \xi/u}\,dx.$$

Note that in this notation,

$$\widetilde{g * g}(\xi) = u\tilde{g}(\xi)^2 \quad \text{and} \quad \widetilde{g \circ g}(\xi) = u|\tilde{g}(\xi)|^2,$$

while $\tilde{g}(0) = \frac{1}{u}$ for any pdf $g$.

We will invoke Parseval's identity, which is technically a statement involving Fourier coefficients rather than values of Fourier transforms. The following lemma establishes the form of Parseval's identity that will suffice for our purposes.

LEMMA 2.1. *For $i \in \{1, 2\}$, suppose that $g_i$ is a square-integrable function supported on $(-\alpha_i, \alpha_i)$. If $\alpha_1 + \alpha_2 \leq u$, then*

$$\int_{\mathbb{R}} g_1(x)\overline{g_2(x)}\, dx = u \sum_{r \in \mathbb{Z}} \tilde{g}_1(r)\overline{\tilde{g}_2(r)}.$$

*Proof.* Without loss of generality, we assume that $u \geq \alpha_1 \geq \alpha_2 > 0$, so that $\alpha_2 \leq u/2$. Define $G_i(x) = \sum_{k \in \mathbb{Z}} g_i(ku + x)$, so that each $G_i$ is a $u$-periodic function and for $r \in \mathbb{Z}$

$$\tilde{g}_i(r) = \frac{1}{u} \sum_{k \in \mathbb{Z}} \int_{ku-u/2}^{ku+u/2} g_i(x)e^{-2\pi i x r/u}\, dx$$

$$= \frac{1}{u} \sum_{k \in \mathbb{Z}} \int_{-u/2}^{u/2} g_i(ku + x)e^{-2\pi i(ku+x)r/u}\, dx$$

$$= \frac{1}{u} \int_{-u/2}^{u/2} \left( \sum_{k \in \mathbb{Z}} g_i(ku + x) \right) e^{-2\pi i x r/u}\, dx$$

$$= \frac{1}{u} \int_{-u/2}^{u/2} G_i(x)e^{-2\pi i x r/u}\, dx.$$

The fact that each $g_i$ is a square-integrable function with compact support implies that each $G_i$ is also square-integrable on any interval of length $u$. Parseval's identity is thus applicable, giving

$$\int_{-u/2}^{u/2} G_1(x)\overline{G_2(x)}\, dx = u \sum_{r \in \mathbb{Z}} \tilde{g}_1(r)\overline{\tilde{g}_2(r)}.$$

Notice that $G_2(x) = g_2(x)$ throughout the interval $(-\frac{u}{2}, \frac{u}{2})$, since $g_2$ is supported on $(-\alpha_2, \alpha_2)$. In particular,

$$\int_{-u/2}^{u/2} G_1(x)\overline{G_2(x)}\, dx = \int_{-\alpha_2}^{\alpha_2} G_1(x)\overline{g_2(x)}\, dx.$$

But notice that

$$G_1(x) = \sum_{k \in \mathbb{Z}} g_1(x + ku) = g_1(x) \quad \text{for } x \in (-\alpha_2, \alpha_2),$$

since the support of any term $g_1(x+ku)$ with $k \neq 0$ is contained in $(u - \alpha_1, \infty)$ or $(-\infty, -u + \alpha_1)$, both of which are disjoint from $(-\alpha_2, \alpha_2)$ due to the inequality $\alpha_1 + \alpha_2 \leq u$. Therefore,

$$\int_{-\alpha_2}^{\alpha_2} G_1(x)\overline{g_2(x)} \, dx = \int_{-\alpha_2}^{\alpha_2} g_1(x)\overline{g_2(x)} \, dx = \int_{\mathbb{R}} g_1(x)\overline{g_2(x)} \, dx,$$

which establishes the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We also use Parseval's identity with respect to the 1-periodic Fourier transform $\hat{g}$ in the following form.

LEMMA 2.2. *If $g_1$ and $g_2$ are square-integrable functions supported on $(-\frac{1}{2}, \frac{1}{2})$, then*

$$\int_{\mathbb{R}} g_1(x)\overline{g_2(x)} \, dx = \sum_{r \in \mathbb{Z}} \hat{g}_1(r)\overline{\hat{g}_2(r)}.$$

*In particular, $\|g_1\|_2^2 = \sum_{r \in \mathbb{Z}} |\hat{g}_1(r)|^2$.*

We now turn to describing two kernel functions that will appear in our proofs. We will need a first kernel function $K$ with the following four properties: $K$ is a pdf; $K$ is square-integrable; $K$ is supported on $(-\delta, \delta)$; and $\tilde{K}(j)$ is real and nonnegative for all integers $j$. We also need to be able to numerically compute $\|K\|_2^2$ and $\tilde{K}(j)$ for small $j$ to high accuracy; it will turn out that the primary consideration for $K$ is to have $\|K\|_2^2$ as small as possible.

We define our favored kernel $K_\text{ß}$ by setting

$$(2.1) \qquad K_\text{ß}(x) := \frac{1}{\delta} \text{ß} \circ \text{ß}\left(\frac{x}{\delta}\right), \quad \text{where } \text{ß}(x) := \begin{cases} \frac{2/\pi}{\sqrt{1-4x^2}}, & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $K_\text{ß}$ is nonnegative and supported on $(-\delta, \delta)$, and simple calculus verifies that $\|K_\text{ß}\|_1 = \|\text{ß}\|_1^2 = 1$. Since $K_\text{ß}$ is defined as an autocorrelation, its Fourier transform $\tilde{K}_\text{ß}(j)$ is automatically real and nonnegative; in particular,

$$(2.2) \qquad\qquad \tilde{K}_\text{ß}(j) = u|\tilde{\text{ß}}(\delta j)|^2 = \frac{1}{u}\left|J_0\left(\frac{\pi \delta j}{u}\right)\right|^2,$$

where $J_0(t)$ is the Bessel $J$-function of order 0. The fact that $|J_0(x)| < 1/\sqrt{x}$ for $x > 0$ implies that $\sum_{j \in \mathbb{Z}} \tilde{K}_\text{ß}(j)^2$ converges; it then follows from Lemma 2.1 that $K_\text{ß}$ is square-integrable. Moreover, we can use the formula (2.2) to accurately compute the $\tilde{K}_\text{ß}(j)$ numerically; we can also accurately compute $\|K_\text{ß}\|_2^2 = \frac{1}{\delta}\|\text{ß} \circ \text{ß}\|_2^2 < 0.5747/\delta$ using

$$\text{ß} \circ \text{ß}(x) = \begin{cases} \frac{2}{\pi^2|x|} E\left(1 - \frac{1}{x^2}\right), & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $E(x)$ is the complete elliptic integral of the first kind. We note that in [5] the authors showed that any autocorrelation (or autoconvolution) supported on $(-\delta, \delta)$ has two-norm-squared at least $0.5745/\delta$, and so our particular kernel $K_{\text{ß}}$ is at least nearly optimal.

We will also need a second kernel function $G$ that is $u$-periodic and at least 1 on the interval $[-\frac{1}{4}, \frac{1}{4}]$ and that has few nonzero $\sim$-Fourier coefficients, all of which we can compute explicitly. (With respect to this kernel function only, we speak literally of its Fourier coefficients, since it is a legitimate $u$-periodic function.) For this, we turn to Selberg's "magic functions":

LEMMA 2.3. *Let* $\frac{1}{2} < u < 1$ *be a real number and* $n > \frac{2u}{2u-1}$ *an integer, and define*

$$C_{u,n}(k) := \left(1 - \frac{k}{n}\right)\left(\cot\frac{\pi k}{n}\sin\frac{\pi k}{2u} + \cos\frac{\pi k}{2u}\right) + \frac{1}{\pi}\sin\frac{\pi k}{2u}$$

*and*

$$(2.3) \qquad G_{u,n}(x) := \frac{4u}{2un - 2u - n}\sum_{k=1}^{n-1} C_{u,n}(k)\cos\frac{2\pi kx}{u}.$$

*Then* $G_{u,n}(x)$ *is* $u$-*periodic, even, and square-integrable on* $[-\frac{u}{2}, \frac{u}{2}]$; *moreover,* $G_{u,n}(x) \geq 1$ *for* $-\frac{1}{4} \leq x \leq \frac{1}{4}$, *and*

$$\tilde{G}_{u,n}(k) = \begin{cases} \frac{2uC_{u,n}(|k|)}{2un-2u-n}, & \text{if } 1 \leq |k| < n, \\ 0, & \text{otherwise.} \end{cases}$$

Specifically, we will use $G_{63/100,22}$; we note for the record that

$$\min_{0 \leq x \leq 1/4} G_{63/100,22}(x) > 1.006.$$

*Proof of Lemma* 2.3. Up to changes of variables that are convenient to our present application, this lemma is directly from Montgomery's account [7]. Let $K$ be a positive integer. Define $e(u) := e^{2\pi iu}$ and $f(u) := -(1-u)\cot(\pi u) - 1/\pi$, and set [7, Chapter 1, equations (16) and (18)]

$$\Delta_K(x) := \sum_{k=-K}^{K}\left(1 - \frac{|k|}{K}\right)e(kx),$$

$$V_K(x) := \frac{1}{K+1}\sum_{k=1}^{K}f\left(\frac{k}{K+1}\right)\sin(2\pi kx).$$

Then define [7, Chapter 1, equation (20)]

$$B_K(x) := V_K(x) + \frac{1}{2(K+1)}\Delta_{K+1}(x)$$

and, for any real numbers $\alpha$ and $\beta$ such that $\alpha \le \beta \le \alpha + 1$, define [7, Chapter 1, equation $(21^+)$]

$$S_K^+(x) := \beta - \alpha + B_K(x - \beta) + B_K(\alpha - x).$$

It follows from Vaaler's lemma (see [7, page 6]) that $S_K^+(x) \ge \chi_{[\alpha,\beta]}(x)$; that is, $S_K^+(x) \ge 0$ for all $x$ and $S_K^+(x) \ge 1$ for $\alpha \le x \le \beta$. We are interested in the special case $\alpha = -\beta$.

Notice that for any positive integer $n$ and any real number $\beta$, we have

$$V_{n-1}(x - \beta) + V_{n-1}(-\beta - x)$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \left(\sin(2\pi k(x - \beta)) + \sin(2\pi k(-\beta - x))\right)$$

$$= \frac{2}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \left(-\sin(2\pi k\beta) \cos(2\pi kx)\right)$$

$$= \frac{2}{n} \sum_{k=1}^{n-1} \left(\left(1 - \frac{k}{n}\right) \cot \frac{\pi k}{n} + \frac{1}{\pi}\right) \sin(2\pi k\beta) \cos(2\pi kx).$$

Similarly, noting that

$$\Delta_n(x) = 1 + \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) (e(kx) + e(-kx)) = 1 + 2 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \cos(2\pi kx),$$

we have

$$\Delta_n(x - \beta) + \Delta_n(-\beta - x)$$

$$= 2 + 2 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \left(\cos(2\pi k(x - \beta)) + \cos(2\pi k(-\beta - x))\right)$$

$$= 2 + 4 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) (\cos(2\pi k\beta) \cos(2\pi kx)).$$

Therefore, $B_{n-1}(x - \beta) + B_{n-1}(-\beta - x)$ is equal to

$$V_{n-1}(x - \beta) + V_{n-1}(-\beta - x) + \frac{1}{2n}\left(\Delta_n(x - \beta) + \Delta_n(-\beta - x)\right),$$

which we expand into the form

$$\frac{1}{n} + \frac{2}{n} \sum_{k=1}^{n-1} \left(\left(1 - \frac{k}{n}\right) \left(\cot \frac{\pi k}{n} \sin(2\pi k\beta) + \cos(2\pi k\beta)\right) + \frac{\sin(2\pi k\beta)}{\pi}\right)$$
$$\times \cos(2\pi kx).$$

If we now set $\alpha := -\frac{1}{4u}$ and $\beta := \frac{1}{4u}$ for a real number $\frac{1}{2} < u < 1$, then the function

$$S_{n-1}^+(x) = \frac{1}{2u} + \frac{1}{n}$$
$$+ \frac{2}{n} \sum_{k=1}^{n-1} \left( \left(1 - \frac{k}{n}\right) \left( \cot \frac{\pi k}{n} \sin \frac{\pi k}{2u} + \cos \frac{\pi k}{2u} \right) + \frac{1}{\pi} \sin \frac{\pi k}{2u} \right)$$
$$\times \cos(2\pi k x)$$

satisfies $S_{n-1}^+(x) \geq 1$ for $|x| \leq \frac{1}{4u}$.

We now stipulate that $n > 2u/(2u - 1)$, so that $2un - 2u - n > 0$. Because

$$G_{u,n}(x) = \frac{2un}{2un - 2u - n} \left( S_{n-1}^+ \left( \frac{x}{u} \right) - \left( \frac{1}{2u} + \frac{1}{n} \right) \right),$$

the inequality $S_{n-1}^+(x) \geq 1$ for $|x| \leq \frac{1}{4u}$ implies that

$$G_{u,n}(x) \geq \frac{2un}{2un - 2u - n} \left( 1 - \frac{2u + n}{2un} \right) = 1 \quad \text{for} \quad \left| \frac{x}{u} \right| \leq \frac{1}{4u};$$

that is, $G_{u,n}(x) \geq 1$ for $|x| \leq \frac{1}{4}$ as claimed. The other properties of $G_{u,n}(x)$ follow directly from its definition (2.3). $\square$

These kernel functions $K_\beta$ and $G_{u,n}$ are quite good, but they have been chosen primarily for their computational convenience rather than for their optimality. For instance, there is no particular reason why an optimal $K$ would *have* to be an autocorrelation, and Selberg's functions enjoy many additional properties that are not relevant for our purposes.

## 3. Proof of Theorem 1.1

We first note that it suffices to prove Theorem 1.1 in the case $h = 2$: if $h \geq 4$ is an even integer, then the function $f^{*h/2}$ is supported on an interval of length $hI/2$, and we can apply the theorem for twofold autoconvolutions to $f^{*h/2}$, obtaining the required lower bound for the $L^\infty$ norm of $(f^{*h/2})^{*2} = f^{*h}$. Therefore, we may assume that $h = 2$ from now on; as noted after the statement of the theorem, we may also assume that $f$ is a pdf supported on an interval of length $\frac{1}{2}$. In fact, we may assume that $f$ is supported on $(-\frac{1}{4}, \frac{1}{4})$ by replacing $f(x)$ with $f(x - x_0)$ if necessary. For such a function $f$, we need to prove that $\|f * f\|_\infty > 1.262$.

As described earlier, the proof of Theorem 1.1 proceeds by forming upper and lower bounds on the integral

$$\int_{\mathbb{R}} \left( f * f(x) + f \circ f(x) \right) K(x) \, dx.$$

The simple Lemmas 3.1 and 3.2 provide the required upper bound, using standard inequalities and 1-periodic Fourier analysis (so the coefficients $\hat{f}(j)$

appear, for example). The lower bound is provided by the more complicated Lemmas 3.3 and 3.4; the second kernel function $G$ makes its appearance in Lemma 3.4, as does $u$-periodic Fourier analysis (including the coefficients $\tilde{f}(j)$, for example). All four lemmas are stated with general kernels $K$ and $G$ and unspecified parameter $u$, so that other choices can be easily examined for possible improvements to the final outcome. Once these four lemmas are established, the proof of Theorem 1.1 can be completed; it is here that we use the specific kernels $K_\beta$ and $G_{u,n}$ (with $n = 22$) and the specific value $u = 63/100$ corresponding to $\delta = 13/100$.

LEMMA 3.1. *For any pdf $K$, we have $\int_{\mathbb{R}} (f * f(x)) K(x)\, dx \leq \|f * f\|_\infty$.*

*Proof.* Hölder's inequality immediately gives

$$\int_{\mathbb{R}} \big( f * f(x) \big) K(x)\, dx \leq \|f * f\|_\infty \|K\|_1,$$

and $\|K\|_1 = 1$ by assumption. □

LEMMA 3.2. *Let $f$ be a square-integrable pdf that is supported on $(-\frac{1}{4}, \frac{1}{4})$, and let $K$ be a square-integrable pdf that is supported on $(-\frac{1}{2}, \frac{1}{2})$. Then*

$$\int_{\mathbb{R}} \big( f \circ f(x) \big) K(x)\, dx \leq 1 + \sqrt{\|f * f\|_\infty - 1}\, \sqrt{\|K\|_2^2 - 1}.$$

*Proof.* Note that $f \circ f$ is supported on $(-\frac{1}{2}, \frac{1}{2})$; also, $f \circ f$ is square-integrable, since $f$ is a square-integrable function with compact support. Therefore, we may apply Lemma 2.2:

$$\int_{\mathbb{R}} \big( f \circ f(x) \big) K(x)\, dx = \int_{\mathbb{R}} \big( f \circ f(x) \big) \overline{K(x)}\, dx = \sum_{r \in \mathbb{Z}} \widehat{f \circ f}(r) \overline{\hat{K}(r)}$$

$$= \sum_{r \in \mathbb{Z}} |\hat{f}(r)|^2 \overline{\hat{K}(r)} = 1 + \sum_{r \neq 0} |\hat{f}(r)|^2 \overline{\hat{K}(r)},$$

since $\hat{f}(0) = \hat{K}(0) = 1$. The Cauchy–Schwarz inequality now yields

$$\int_{\mathbb{R}} \big( f \circ f(x) \big) K(x)\, dx \leq 1 + \left( \sum_{r \neq 0} |\hat{f}(r)|^4 \right)^{1/2} \left( \sum_{r \neq 0} |\hat{K}(r)|^2 \right)^{1/2}$$

$$= 1 + \left( \sum_{r \in \mathbb{Z}} |\hat{f}(r)|^4 - 1 \right)^{1/2} \left( \sum_{r \in \mathbb{Z}} |\hat{K}(r)|^2 - 1 \right)^{1/2}$$

$$= 1 + \left( \sum_{r \in \mathbb{Z}} |\widehat{f * f}(r)|^2 - 1 \right)^{1/2} \left( \sum_{r \in \mathbb{Z}} |\hat{K}(r)|^2 - 1 \right)^{1/2}.$$

Since $f * f$ is also square-integrable and supported on $(-\frac{1}{2}, \frac{1}{2})$, two applications of Lemma 2.2 now yield

$$\int_{\mathbb{R}} \big(f \circ f(x)\big) K(x)\, dx \le 1 + (\|f * f\|_2^2 - 1)^{1/2}(\|K\|_2^2 - 1)^{1/2}$$

$$\le 1 + (\|f * f\|_\infty - 1)^{1/2}(\|K\|_2^2 - 1)^{1/2}$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Recall that $0 < \delta < \frac{1}{4}$ is a parameter and $u = \delta + \frac{1}{2}$, and that $\tilde{g}$ refers to the Fourier transform normalized using the parameter $u$.

LEMMA 3.3. *Let $f$ be a square-integrable pdf that is supported on $(-\frac{1}{4}, \frac{1}{4})$, and let $K$ be a pdf that is supported on $(-\delta, \delta)$. Then*

$$\int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K(x)\, dx = \frac{2}{u} + 2u^2 \sum_{j \ne 0} (\Re \tilde{f}(j))^2 \Re \tilde{K}(j).$$

*Proof.* The function $f * f + f \circ f$ is square-integrable and supported on $(-\frac{1}{2}, \frac{1}{2})$. Since the inequality $\frac{1}{2} + \delta \le u$ is satisfied, we may apply Lemma 2.1 to obtain

$$\int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K(x)\, dx = \int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) \overline{K(x)}\, dx$$

$$= u \sum_{j \in \mathbb{Z}} \big(\widetilde{f * f}(j) + \widetilde{f \circ f}(j)\big) \overline{\tilde{K}(j)}$$

$$= u^2 \sum_{j \in \mathbb{Z}} \big(\tilde{f}(j)^2 + |\tilde{f}(j)|^2\big) \tilde{K}(j).$$

As the left-hand side is real, we may take real parts term by term on the right-hand side:

$$\int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K(x)\, dx = u^2 \sum_{j \in \mathbb{Z}} \Re\big(\tilde{f}(j)^2 + |\tilde{f}(j)|^2\big) \Re \tilde{K}(j)$$

$$= 2u^2 \sum_{j \in \mathbb{Z}} (\Re \tilde{f}(j))^2 \Re \tilde{K}(j),$$

using the fact that $\Re(z^2 + |z|^2) = 2(\Re z)^2$ for any complex number $z$. Continuing,

$$(3.1) \qquad \int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K(x)\, dx$$

$$= 2u^2 \left( (\Re \tilde{f}(0))^2 \Re \tilde{K}(0) + \sum_{j \ne 0} (\Re \tilde{f}(j))^2 \Re \tilde{K}(j) \right)$$

$$= \frac{2}{u} + 2u^2 \sum_{j \ne 0} (\Re \tilde{f}(j))^2 \Re \tilde{K}(j),$$

since $\tilde{f}(0) = \tilde{K}(0) = \frac{1}{u}$. This establishes the lemma.                     $\square$

We comment that Lemma 3.3 implies

$$\int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K_{\mathfrak{B}}(x)\,dx \geq \frac{2}{u},$$

since $\tilde{K}_{\mathfrak{B}}(j) \geq 0$ for all $j$. Combining this lower bound with the upper bounds of Lemmas 3.1 and 3.2 applied with $K = K_{\mathfrak{B}}$ gives

$$\frac{2}{u} \leq \int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K_{\mathfrak{B}}(x)\,dx$$
$$\leq \|f * f\|_\infty + 1 + \sqrt{\|f * f\|_\infty - 1}\sqrt{0.5747/\delta - 1};$$

setting $\delta := 0.1184$ and solving for $\|f * f\|_\infty$ yields $\|f * f\|_\infty \geq 1.25087$, already an improvement over earlier work.

We should be able to do better, of course, than simply throwing away all the $j \neq 0$ terms on the right-hand side of Lemma 3.3. To do so and thus achieve the statement $\|f * f\|_\infty \geq 1.262$ of Theorem 1.1, we utilize our second kernel function $G$ and the following lemma. Let $\operatorname{Spec}(G) := \{j \in \mathbb{Z} : \tilde{G}(j) \neq 0\}$ denote the support of the $\sim$-Fourier series for the function $G$.

LEMMA 3.4. *Let $f$ be a square-integrable pdf supported on $(-\frac{1}{4}, \frac{1}{4})$. Let $G$ be an even, real-valued, square-integrable function that is $u$-periodic, takes positive values on $(-\frac{1}{4}, \frac{1}{4})$, and satisfies $\tilde{G}(0) = 0$. Let $K$ be a function supported on $(-\delta, \delta)$, with $\tilde{K}(j) \geq 0$ for all integers $j$ and $\tilde{K}(j) > 0$ for $j \in \operatorname{Spec}(G)$. Then*

$$u^2 \sum_{j \neq 0} (\Re\tilde{f}(j))^2 \Re\tilde{K}(j) \geq \Big(\min_{0 \leq x \leq 1/4} G(x)\Big)^2 \cdot \Big(\sum_{j \in \operatorname{Spec}(G)} \frac{\tilde{G}(j)^2}{\tilde{K}(j)}\Big)^{-1}.$$

*Proof.* We observe that

$$\min_{0 \leq x \leq 1/4} G(x) = \Big(\min_{-1/4 \leq x \leq 1/4} G(x)\Big) \int_{-1/4}^{1/4} f(x)\,dx$$
$$\leq \int_{-1/4}^{1/4} f(x)G(x)\,dx = \int_{\mathbb{R}} f(x)\overline{G(x)}\,dx,$$

since $f$ is supported on $(-\frac{1}{4}, \frac{1}{4})$. Lemma 2.1 then gives

$$\min_{0 \leq x \leq 1/4} G(x) \leq u \sum_{j=-\infty}^{\infty} \tilde{f}(j)\overline{\tilde{G}(j)}.$$

Taking real parts of both sides, and noting that $\tilde{G}(j)$ is real under the hypotheses on $G$, yields

$$\min_{0 \leq x \leq 1/4} G(x) \leq u \sum_{j=-\infty}^{\infty} \Re\tilde{f}(j) \cdot \tilde{G}(j) = u \sum_{j \in \operatorname{Spec}(G)} \Re\tilde{f}(j) \cdot \tilde{G}(j).$$

We now have, using Cauchy–Schwarz in the middle inequality,

$$\left( \min_{0 \leq x \leq 1/4} G(x) \right)^2$$

$$\leq u^2 \left( \sum_{j \in \mathrm{Spec}(G)} \Re \tilde{f}(j) \cdot \tilde{G}(j) \right)^2$$

$$= u^2 \left( \sum_{j \in \mathrm{Spec}(G)} \Re \tilde{f}(j) \sqrt{\tilde{K}(j)} \cdot \frac{\tilde{G}(j)}{\sqrt{\tilde{K}(j)}} \right)^2$$

$$\leq u^2 \left( \sum_{j \in \mathrm{Spec}(G)} (\Re \tilde{f}(j))^2 \tilde{K}(j) \right) \left( \sum_{j \in \mathrm{Spec}(G)} \frac{\tilde{G}(j)^2}{\tilde{K}(j)} \right)$$

$$\leq \left( u^2 \sum_{j \neq 0} (\Re \tilde{f}(j))^2 \tilde{K}(j) \right) \left( \sum_{j \in \mathrm{Spec}(G)} \frac{\tilde{G}(j)^2}{\tilde{K}(j)} \right),$$

where we have used the hypothesis that $\tilde{K}(j) \geq 0$ for all integers $j$ and $\tilde{K}(j) > 0$ for $j \in \mathrm{Spec}(G)$ in the last inequalities (as well as $\tilde{G}(0) = 0$, so that $0 \notin \mathrm{Spec}(G)$). $\qquad\square$

*Proof of Theorem* 1.1. We assume, as we have observed we may, that $h = 2$ and that $f$ is a pdf supported on $(-\frac{1}{4}, \frac{1}{4})$. We would like to apply Lemmas 3.1–3.4 with the choices $\delta := 13/100$, $u := \delta + 1/2 = 63/100$, $K := K_\beta$ as defined in equation (2.1), and $G := G_{u,22}$ as defined in equation (2.3). We already saw in Section 2 that $K_\beta$ is a square-integrable pdf supported on $(-\delta, \delta)$ whose $\sim$-Fourier coefficients are nonnegative; a calculation shows that $\tilde{K}(j) > 0$ for $1 \leq |j| \leq 21$. We also know from Lemma 2.3 that $G_{u,22}$ is an even, real-valued, square-integrable $u$-periodic function that takes positive values on $(-\frac{1}{4}, \frac{1}{4})$; moreover, $\tilde{G}_{u,22}(0) = 0$ and $\mathrm{Spec}(G_{u,22}) = \{j : 1 \leq |j| \leq 21\}$. Therefore, all the hypotheses of Lemmas 3.1–3.4 are satisfied.

By Lemmas 3.1 and 3.2, we have

$$(3.2) \qquad \int_{\mathbb{R}} \big( f * f(x) + f \circ f(x) \big) K_\beta(x) \, dx$$

$$\leq \|f * f\|_\infty + 1 + \sqrt{\|f * f\|_\infty - 1} \sqrt{\|K_\beta\|_2^2 - 1}$$

$$\leq \|f * f\|_\infty + 1 + 2L \sqrt{\|f * f\|_\infty - 1},$$

where

$$L := 0.9248 > \frac{1}{2} \sqrt{0.5747/\delta - 1} > \frac{1}{2} \sqrt{\|K_\beta\|_2^2 - 1}.$$

(The constant in equation (3.2) has been called $2L$ rather than $L$ simply for later convenience.) On the other hand, by Lemmas 3.3 and 3.4, we

have

$$\int_{\mathbb{R}} \big(f * f(x) + f \circ f(x)\big) K_{\text{ß}}(x)\, dx$$

$$\geq \frac{2}{u} + 2\Big(\min_{0 \leq x \leq 1/4} G_{u,22}(x)\Big)^2 \cdot \Big(\sum_{j \in \text{Spec}(G_{u,22})} \frac{\tilde{G}_{u,22}(j)^2}{\tilde{K}_{\text{ß}}(j)}\Big)^{-1} \geq R,$$

where

$$R := 3.20874 < \frac{2}{u} + 2(1.006)^2 \Big(2\sum_{j=1}^{21} \frac{(2uC_{u,22}(j)/(44u - 2u - 22))^2}{J_0(\pi\delta j/u)^2/u}\Big)^{-1}$$

$$= \frac{2}{u} + \frac{(1.006)^2(22u - u - 11)^2}{u^3}\Big(\sum_{j=1}^{21} \frac{C_{u,22}(j)^2}{J_0(\pi\delta j/u)^2}\Big)^{-1}.$$

Combining these inequalities yields

$$(3.3) \qquad \|f * f\|_\infty + 1 + 2L\sqrt{\|f * f\|_\infty - 1} \geq R.$$

Using the notation $Q = \sqrt{\|f * f\|_\infty - 1}$, equation (3.3) becomes $(Q^2 + 1) + 1 + 2LQ \geq R$, or equivalently $Q^2 + 2LQ + (2 - R) \geq 0$. The discriminant $4L^2 - 4(2 - R)$ is nonnegative, and so $Q$ must be at least as large as the positive root of $x^2 + 2Lx + (2 - R)$, which is $(-2L + \sqrt{4L^2 - 4(2 - R)})/2$. This establishes that

$$\sqrt{\|f * f\|_\infty - 1} \geq \sqrt{L^2 + R - 2} - L,$$

and it follows from the fact that $\sqrt{L^2 + R - 2} - L$ is nonnegative that

$$\|f * f\|_\infty \geq \big(\sqrt{L^2 + R - 2} - L\big)^2 + 1 > 1.262,$$

as claimed. $\qquad\qquad\square$

## 4. Deriving the corollaries

We now derive Corollaries 1.2 and 1.5 from Theorem 1.1, then derive Corollaries 1.3 and 1.4 from Corollary 1.5. We let

$$\chi_S(x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S \end{cases}$$

denote the indicator function of a set $S$, and we let $\mu$ denote Lebesgue measure on $\mathbb{R}$.

*Proof of Corollary* 1.2. Let $B$ be a set with measure $\varepsilon$ supported in $[0, 1]$, so that $\|\chi_B\|_1 = \varepsilon$. Theorem 1.1 applied with $h = 2$ tells us that

$$\|\chi_B * \chi_B\|_\infty > \frac{1.262}{2 \cdot 1}\|\chi_B\|_1^2 = 0.631\varepsilon^2.$$

On the other hand,

$$\chi_B * \chi_B(x) = \int_{\mathbb{R}} \chi_B(y)\chi_B(x-y)\,dy = \mu(C_x),$$

where $C_x = B \cap (x - B)$ is the largest centrally symmetric subset of $B$ with center $x/2$. Therefore, $\|\chi_B * \chi_B\|_\infty$ is the measure of the largest centrally symmetric subset of $B$, which establishes the corollary.  □

*Proof of Corollary* 1.5. Suppose that $p(x) = \sum_{i=0}^{\deg(p)} p_i x^i$ with all of the $p_i$ nonnegative, and set

$$g(x) := \sum_{i=0}^{\deg(p)} p_i \chi_{(i-1/2,i+1/2)}(x).$$

Clearly, $g$ is nonnegative and supported on the interval $(-\frac{1}{2}, \deg(p) + \frac{1}{2})$, and $\|g\|_1 = \sum_{i=0}^{\deg(p)} p_i = p(1)$. Theorem 1.1 applied with $h = 2$ gives

$$(4.1) \qquad \|g * g\|_\infty > \frac{1.262}{2I}\|g\|_1^2 = \frac{0.631}{\deg(p)+1}p(1)^2.$$

However, note that

$$g * g(x) = \sum_{i=0}^{\deg(p)} \sum_{j=0}^{\deg(p)} p_i p_j \chi_{(i-1/2,i+1/2)} * \chi_{(j-1/2,j+1/2)}(x)$$

$$= \sum_{i=0}^{\deg(p)} \sum_{j=0}^{\deg(p)} p_i p_j \Lambda(x - i - j),$$

where $\Lambda(x) := \max\{0, 1 - |x|\}$ is a tent function with corners only at integers. In particular, $g * g$ is a piecewise linear function with corners only at integers, and so $\|g * g\|_\infty = \max\{g * g(k) : k \in \mathbb{Z}\}$. Moreover, for $k$ an integer

$$g * g(k) = \sum_{i=0}^{\deg(p)} \sum_{j=0}^{\deg(p)} p_i p_j \Lambda(k - i - j) = \sum_{\substack{i=0 \\ i+j=k}}^{\deg(p)} \sum_{j=0}^{\deg(p)} p_i p_j$$

(since $\Lambda(0) = 1$ while $\Lambda(n) = 0$ for every nonzero integer $n$), which is exactly the coefficient of $x^k$ in $p(x)^2$. Therefore, $\|g * g\|_\infty = H(p^2)$, and so the inequality (4.1) is equivalent to the assertion of the corollary.  □

*Proof of Corollaries* 1.3 *and* 1.4. Fix $A \subseteq \{0, 1, \ldots, n - 1\}$, and let $g$ be the maximum multiplicity of an element of the multiset $A + A$, so that $A$ is a $B^*[g]$ set. Define $p(x) = \sum_{a \in A} x^a$, and observe that $p(1) = |A|$. Note that the coefficient of $x^k$ in $p(x)^2$ is exactly the multiplicity of $k$ as an element of

$A + A$, so that $H(p^2) = g$. Corollary 1.5 then yields

$$0.631 < R(p) = \frac{H(p^2)(\deg(p) + 1)}{p(1)^2} \leq \frac{gn}{|A|^2},$$

since $\deg(p) \leq n - 1$. Solving this inequality for $|A|$ establishes Corollary 1.3; on the other hand, setting $|A| = \varepsilon n$ and solving the inequality for $g$ establishes Corollary 1.4.                                                                   □

## 5. Open problems

We conclude by mentioning a few open problems associated to twofold autoconvolutions. The function

$$h(x) = \begin{cases} 1/\sqrt{2x}, & \text{if } 0 < x < 1/2, \\ 0, & \text{otherwise} \end{cases}$$

is a pdf whose autoconvolution is

$$h * h(x) = \begin{cases} \pi/2, & \text{if } 0 < x \leq 1/2, \\ \pi/2 - 2\arctan\sqrt{2x - 1}, & \text{if } 1/2 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\|h * h\|_1 = \|h\|_1^2 = 1$, while $\|h * h\|_\infty = \pi/2$ and $\|h * h\|_2^2 = \log 4$ (here log is the natural logarithm). We believe that this function is extremal in two ways. First, this function demonstrates that the constant 1.262 in Theorem 1.1 cannot be increased beyond $\pi/2 \approx 1.5708$, and we believe that this constant represents the truth:

CONJECTURE 5.1. *If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is nonnegative and supported on an interval of length $I$, then*

$$\|g * g\|_\infty \geq \frac{\pi/2}{2I}\|g\|_1^2.$$

Dubickas [3] gives a sequence of functions supported on $[0, 1]$, which take only the values $\pm 1$ on that interval, satisfying $\|f * f\|_\infty \to 0$. That example shows that the hypothesis of nonnegativity in Conjecture 5.1 is necessary.

Second, Hölder's inequality gives the upper bound $\|g\|_2^2 \leq \|g\|_\infty\|g\|_1$, which can be an equality if $g$ is a multiple of an indicator function. However, when we apply this inequality in the last step of the proof of Lemma 3.2, we apply it to the function $f * f$, which seems far from being constant on its support. We conjecture that an improvement is possible here, with the function $h$ again providing the best possible constant. This conjecture would imply that the constant 1.262 in Theorem 1.1 could be improved to 1.3674.

CONJECTURE 5.2. *If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is nonnegative, then*

$$\|g * g\|_2^2 \leq \frac{\log 16}{\pi}\|g * g\|_\infty\|g * g\|_1.$$

The analogous discrete inequality cannot be improved in this manner. Set $\sum_i q_i x^i = (\sum_i^{\deg(p)} p_i x^i)^2$, where $p_i \geq 0$; the inequality that we refer to is $\sum q_i^2 \leq \max q_i \cdot \sum q_i$. To see that this cannot be improved, set $p_i = 1$ if $i = 2^k$ with $0 \leq k < N$, and set $p_i = 0$ otherwise. In this case, $\sum q_i^2 = 2N^2 - N$, while $\max q_i = 2$ and $\sum q_i = N^2$, and the inequality is seen to be asymptotically sharp.

Finally, consider a pdf $g$ that is supported on an interval of length 1. Theorem 1.1 indicates that $\|g^{*h}\|_\infty > 1.262/h$ for even integers $h$, while the central limit theorem implies that much more is true: for large integers $h$ there is a constant $c > 0$ such that $\|g^{*h}\|_\infty \geq c/\sqrt{h}$. It seems likely that $1/\sqrt{h}$ is the true rate of decay as $h \to \infty$, at least under some assumption on $g$ such as piecewise continuity, but we have not succeeded in proving this.

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