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COHEN–MACAULAYNESS WITH RESPECT TO SERRE CLASSES

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Abstract. Let R be a commutative Noetherian ring. The notion of regular sequences with respect to a Serre class of R-modules is introduced and some of their essential properties are given. Then in the local case, we explore a theory of Cohen–Macaulayness with respect to Serre classes.

1. Introduction

The concept of almost vanishing has been studied by a number of authors, see [GR] and [RSS]. They introduced along the way the notion of almost zero modules in two different ways over not necessarily Noetherian rings. We do not require to give the definition of the almost zero modules, but we list here two basic properties of them:

(i) for any exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

the module M is almost zero if and only if each of M' and M'' is almost zero, and

(ii) if $\{M_i\}$ is a directed system consisting of almost zero modules, then its direct limit $\varinjlim_i M_i$ is almost zero.

A subclass of the class of all modules is called a Serre class, if it is closed under taking submodules, quotients and extensions. A Serre class which is closed under taking the direct limit of any direct system of its objects is called a torsion theory, see [St]. So, the class of almost zero modules is a torsion theory.

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In view of [RSS, Definition 1.2], a sequence $\underline{x} := x_1, \dots, x_r$ of elements of a certain ring A is called almost regular sequence, if the A-module

$$\big((x_1, \dots, x_{i-1})A :_A x_i\big)/(x_1, \dots, x_{i-1})A$$

is almost zero for each $i = 1, \dots, r$. If every system of parameters for A is an almost regular sequence, A is said to be almost Cohen–Macaulay. We refer the reader to [RSS, Proposition 1.3] for a connection between this definition and the monomial conjecture. These observations motivate us to introduce a new generalization of the notions of regular sequences and Cohen–Macaulay modules by using Serre classes. In fact, we do this by replacing the class of almost zero modules with an arbitrary Serre class of modules. It is worth pointing out that some of the existing generalizations of Cohen–Macaulayness can be viewed as special cases of our definition. More precisely, consider the following two examples.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. Consider the torsion theory $\mathcal{T}_0 := \{N \in R\text{-}\mathbf{Mod} \mid \operatorname{Supp}_R N \subseteq \operatorname{Supp}_R(R/\mathfrak{m})\}$, where R-**Mod** is the category of R-modules and R-homomorphisms. In the literature, the concept of M-sequence with respect to $\mathcal{T}_0$ is called M-filter regular sequence, see [CST] and [SV]. Cuong et al. introduced the notion of f-modules in [CST]. They studied modules called f-modules satisfying the condition that every system of parameters is a filter regular sequence, see [CST]. Now, consider the torsion theory $\mathcal{T}_1 := \{N \in R\text{-}\mathbf{Mod} \mid \dim N \leq 1\}$. The concept of M-sequence with respect to $\mathcal{T}_1$ is called generalized filter regular sequence on M. This notion, as a generalization of the notion of filter regular sequences, first appeared in [N]. Following [N], the concept of generalized f-modules was introduced in [NM]. An R-module M is called a generalized f-module, if every system of parameters for M is a generalized filter regular sequence. Thus, our theory will include the notions of f-modules and generalized f-modules.

Throughout this paper, R is a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and M an R-module. Always, "$\mathcal{S}$" stands for a Serre class.

In Section 2, we introduce the notion of weak M-sequences with respect to a Serre class $\mathcal{S}$. We define the $\mathcal{S} - \text{c. grade}_R(\mathfrak{a}, M)$ as the supremum of the lengths of weak M-sequences with respect to $\mathcal{S}$ in $\mathfrak{a}$. After summarizing some results, we characterize $\mathcal{S} - \text{c. grade}_R(\mathfrak{a}, M)$ via some homological tools such as Ext-modules, Koszul complexes and local cohomology modules. This provides a common language for expressing some results concerning several types of sequences that have appeared in different papers.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. We say that M is $\mathcal{S}$-Cohen–Macaulay, if every system of parameters for M is a weak M-sequence with respect to $\mathcal{S}$. In Section 3, we exhibit some of the basic properties of $\mathcal{S}$-Cohen–Macaulay modules. Some connections between the notions of $\mathcal{S}$-Cohen–Macaulayness and Cohen–Macaulayness are given.

We show that most of the remarkable properties of Cohen–Macaulay modules remain valid for $\mathcal{S}$-Cohen–Macaulay modules. Especially, they behave well with respect to flat extensions, annihilators of local cohomology modules, non-Cohen–Macaulay locus and quotient by weak sequences with respect to the Serre class $\mathcal{S}$.

2. Grade of ideals with respect to Serre classes

Let $\mathcal{S}$ be a subcategory of the category of R-modules and R-homomorphisms. Then $\mathcal{S}$ is said to be a *Serre class* (or Serre subcategory), if for any exact sequence of R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

the R-module M belongs to $\mathcal{S}$ if and only if each of L and N belongs to $\mathcal{S}$. The key to the work in this paper is given by the following easy lemma.

Lemma 2.1. *Let M be a finitely generated R-module and K an R-module. Then the following hold*:

(i) *$M \in \mathcal{S}$ if and only if $R/\mathfrak{p} \in \mathcal{S}$ for all $\mathfrak{p} \in \operatorname{Supp} M$. In particular, for any two finitely generated R-modules N, L with $\operatorname{Supp}_R N = \operatorname{Supp}_R L$, we have $N \in \mathcal{S}$ if and only if $L \in \mathcal{S}$.*

(ii) *If S is closed under taking direct sums, then $K \in \mathcal{S}$ if and only if $R/\mathfrak{p} \in \mathcal{S}$, for all $\mathfrak{p} \in \operatorname{Supp} K$. In particular for any two R-modules N, L with $\operatorname{Supp}_R N = \operatorname{Supp}_R L$, it follows that $N \in \mathcal{S}$ if and only if $L \in \mathcal{S}$.*

Proof. (i) First, assume that $M \in \mathcal{S}$. Let $\mathfrak{p} \in \operatorname{Supp}_R M$. So there exists $m \in M$ such that $(0 :_R m) \subseteq \mathfrak{p}$. Since $R/(0 :_R m) \cong Rm \subseteq M$, it turns out that $R/(0 :_R m) \in \mathcal{S}$. From the natural epimorphism $R/(0 :_R m) \longrightarrow R/\mathfrak{p}$, we get $R/\mathfrak{p} \in \mathcal{S}$.

Now, we prove the converse. There is a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

of submodules of M such that for each j, $M_j/M_{j-1} \cong R/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Supp} M$. By using short exact sequences, the situation can be reduced to the trivial case $\ell = 1$.

(ii) Since $\mathcal{S}$ is closed under taking direct limits, we can assume that K is a finitely generated R-module. The remainder of the proof is similar to (i). □

Definition 2.2. Let M be an R-module. A sequence $\underline{x} := x_1, \ldots, x_r$ of elements of R is called a weak M-sequence with respect to $\mathcal{S}$ if for each $i = 1, \ldots, r$ the R-module $((x_1, \ldots, x_{i-1})M :_M x_i)/(x_1, \ldots, x_{i-1})M$ belongs to $\mathcal{S}$. If in addition $M/\underline{x}M \notin \mathcal{S}$, we say that $\underline{x}$ is an M-sequence with respect to $\mathcal{S}$.

Notation. For an R-module L, we denote $\{\mathfrak{p} \in \operatorname{Supp}_R L | R/\mathfrak{p} \notin \mathcal{S}\}$ by $\mathcal{S} - \operatorname{Supp}_R L$ and $\{\mathfrak{p} \in \operatorname{Ass}_R L | R/\mathfrak{p} \notin \mathcal{S}\}$ by $\mathcal{S} - \operatorname{Ass}_R L$.

In order to exploit Lemma 2.1 and Definition 2.2, we give the relation between regular sequences with respect to $\mathcal{S}$ and ordinary regular sequences.

LEMMA 2.3. *Let M be a finitely generated R-module and $\underline{x} := x_1, \ldots, x_r$ a sequence of elements of R. Then the following conditions are equivalent*:

(i) $x_i \notin \bigcup_{\mathfrak{p} \in \mathcal{S} - \operatorname{Ass}_R M/(x_1,\ldots,x_{i-1})M} \mathfrak{p}$ *for all* $i = 1, \ldots, r$.
(ii) *The sequence $x_1, \ldots, x_r$ is a weak M-sequence with respect to $\mathcal{S}$.*
(iii) *For any $\mathfrak{p} \in \mathcal{S} - \operatorname{Supp}_R(M)$, the elements $x_1/1, \ldots, x_r/1$ of the local ring $R_\mathfrak{p}$ form a weak $M_\mathfrak{p}$-sequence.*
(iv) *The sequence $x_1^{n_1}, \ldots, x_r^{n_r}$ is a weak M-sequence with respect to $\mathcal{S}$ for all positive integers $n_1, \ldots, n_r$.*

Proof. (i) $\Rightarrow$ (ii) We can and do assume that $r = 1$. In view of Lemma 2.1, it is enough to show that $\{R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Supp}_R(0 :_M x_1)\} \subseteq \mathcal{S}$. To establish this, suppose on the contrary that, there is $\mathfrak{p} \in \operatorname{Supp}_R(0 :_M x_1)$ such that $R/\mathfrak{p} \notin \mathcal{S}$. So there exists $\mathfrak{q} \in \operatorname{Ass}_R(0 :_M x_1)$, which is contained in $\mathfrak{p}$. The natural epimorphism $R/\mathfrak{q} \longrightarrow R/\mathfrak{p}$ and the condition $R/\mathfrak{p} \notin \mathcal{S}$ imply that $R/\mathfrak{q} \notin \mathcal{S}$. Since $\mathfrak{q} \in \mathcal{S} - \operatorname{Ass}_R(0 :_M x_1)$ and $x_1 \in \mathfrak{q}$, we get a contradiction.

(ii) $\Rightarrow$ (iii) This implication is an immediate consequence of Lemma 2.1.

(iii) $\Rightarrow$ (i) Assume that $\mathfrak{p} \in \mathcal{S} - \operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M)$ for some $i = 1, \ldots, r$. Then $\mathfrak{p} \in \mathcal{S} - \operatorname{Supp}_R(M)$ and

$$\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}/(x_1/1, \ldots, x_{i-1}/1)M_\mathfrak{p}).$$

Hence, by our assumptions, we have $x_i/1 \notin \mathfrak{p}R_\mathfrak{p}$. Consequently, $x_i \notin \mathfrak{p}$.

(iii) $\Leftrightarrow$ (iv) is clear. $\square$

Now, we establish a preliminary lemma.

LEMMA 2.4. *Let M be a finitely generated R-module and let $\underline{x} := x_1, \ldots, x_r$ be a weak M-sequence with respect to $\mathcal{S}$ in $\mathfrak{a}$. Then the following holds*:

(i) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) \in \mathcal{S}$ *for all* $0 \le i \le r - 1$.
(ii) $\operatorname{Ext}^r_R(R/\mathfrak{a}, M) \notin \mathcal{S}$ *if and only if* $\operatorname{Hom}_R(R/\mathfrak{a}, M/\underline{x}M) \notin \mathcal{S}$.

Proof. (i) Let $0 \le i \le r - 1$ and $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Ext}^i_R(R/\mathfrak{a}, M))$. Assume that $R/\mathfrak{p} \notin \mathcal{S}$. We have $\mathfrak{p} \in \operatorname{Supp}_R(M)$. By the implication (ii) $\Rightarrow$ (iii) of Lemma 2.3, the elements $x_1/1, \ldots, x_r/1$ of the local ring $R_\mathfrak{p}$ form a weak $M_\mathfrak{p}$-sequence. Therefore,

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, M)_\mathfrak{p} \cong \operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}_\mathfrak{p}, M_\mathfrak{p}) = 0,$$

which is a contradiction. So $R/\mathfrak{p} \in \mathcal{S}$. Now, Lemma 2.1 implies that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) \in \mathcal{S}$.

(ii) Let $\mathfrak{p} \in \mathcal{S} - \operatorname{Supp}_R(M)$. By Lemma 2.3, the elements $x_1/1, \ldots, x_r/1$ of the local ring $R_\mathfrak{p}$ form a weak $M_\mathfrak{p}$-sequence. Therefore, in view of [BH, Lemma 1.2.4], we have

$$\operatorname{Ext}^r_{R_\mathfrak{p}}\left(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, M_\mathfrak{p}\right) \cong \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, M_\mathfrak{p}/\underline{x}M_\mathfrak{p}).$$

This shows that $\mathcal{S}-\operatorname{Supp}_R(\operatorname{Ext}^r_R(R/\mathfrak{a},M))=\mathcal{S}-\operatorname{Supp}_R(\operatorname{Hom}_R(R/\mathfrak{a},M/\underline{x}M))$. Hence, the desired result follows from Lemma 2.1. □

Let $\mathfrak{a}$ be an ideal of the ring R. Suppose $\underline{y}:=y_1,\dots,y_r$ is a system of generators of $\mathfrak{a}$. We denote the Koszul complex of $\underline{y}$ by $\mathbb{K}_\bullet(\underline{y})$. For an R-module M, the Koszul complex with coefficients in M, is defined by $\mathbb{K}^\bullet(\underline{y},M):=\operatorname{Hom}_R(\mathbb{K}_\bullet(\underline{y}),M)$.

We need the following lemma in Definition 2.6 below. The symbol $\mathbb{N}_0$ will denote the set of nonnegative integers.

LEMMA 2.5. *Let $\mathfrak{a}$ be an ideal of the ring R and M an R-module. Suppose that $\underline{y}:=y_1,\dots,y_r$ is a system of generators of $\mathfrak{a}$. The number* $\inf\{i\in\mathbb{N}_0|H^i(\mathbb{K}^\bullet(\underline{y},M))\notin\mathcal{S}\}$ *does not depend on the choice of the generating sets of $\mathfrak{a}$.*

Proof. Let $\underline{x}:=x_1,\dots,x_s$ be another generating set for $\mathfrak{a}$. Set $\underline{x}':=x_1,\dots,x_s,y_1,\dots,y_r$. In view of [BH, Proposition 1.6.21], $H_\bullet(\underline{x}',M)\cong H_\bullet(\underline{x},M)\otimes_R\bigwedge R^r$. Therefore $H_i(\underline{x}',M)\in\mathcal{S}$ if and only if $H_i(\underline{x},M)\in\mathcal{S}$, since $\bigwedge R^r$ is a non-zero finitely generated free R-module. Set $n=r+s$. Thus, the symmetry of Koszul cohomology and Koszul homology implies that $H^{n-i}(\mathbb{K}^\bullet(\underline{x}',M))\in\mathcal{S}$ if and only if $H^{s-i}(\mathbb{K}^\bullet(\underline{x},M))\in\mathcal{S}$, see [BH, Proposition 1.6.10(d)]. Set $\underline{y}':=y_1,\dots,y_r,x_1,\dots,x_s$. By the same reason, we have $H^{n-i}(\mathbb{K}^\bullet(\underline{y}',M))\in\mathcal{S}$ if and only $H^{r-i}(\mathbb{K}^\bullet(\underline{y},M))\in\mathcal{S}$. The claim becomes clear from the fact that Koszul complex $\mathbb{K}^\bullet(\underline{y}',M)$ is invariant (up to isomorphism) under permutation of $\underline{y}'$. □

DEFINITION 2.6. Let M be an R-module and $\mathfrak{a}$ an ideal of R. Let $\underline{x}:=x_1,\dots,x_s$ be a generating set for $\mathfrak{a}$. The notions of local cohomology grade, Ext grade, Koszul grade and classical grade of $\mathfrak{a}$ on M with respect to $\mathcal{S}$, are defined, respectively as follows:

(i) $\mathcal{S}-\text{H. grade}_R(\mathfrak{a},M):=\inf\{i\in\mathbb{N}_0|H^i_{\mathfrak{a}}(M)\notin\mathcal{S}\}$,
(ii) $\mathcal{S}-\text{E. grade}_R(\mathfrak{a},M):=\inf\{i\in\mathbb{N}_0|\operatorname{Ext}^i_R(R/\mathfrak{a},N)\notin\mathcal{S}\}$,
(iii) $\mathcal{S}-\text{K. grade}_R(\mathfrak{a},M):=\inf\{i\in\mathbb{N}_0|H^i(\mathbb{K}^\bullet(\underline{x},M))\notin\mathcal{S}\}$,
(iv) $\mathcal{S}-\text{c. grade}_R(\mathfrak{a},M):=\sup\{\ell\in\mathbb{N}_0|y_1,\dots,y_\ell$ is a weak M-sequence in $\mathfrak{a}$ with respect to $\mathcal{S}\}$.

Here inf and sup are formed in $\mathbb{Z}\cup\{\pm\infty\}$ with the convention that $\inf\emptyset=+\infty$ and $\sup\emptyset=-\infty$.

In the case $\mathcal{S}=\{0\}$ for simplicity, we use the notation $\text{K. grade}_R(\mathfrak{a},M)$, $\text{E. grade}_R(\mathfrak{a},M)$ and $\text{H. grade}_R(\mathfrak{a},M)$, instead of $\mathcal{S}-\text{K. grade}_R(\mathfrak{a},M)$, $\mathcal{S}-\text{E. grade}_R(\mathfrak{a},M)$ and $\mathcal{S}-\text{H. grade}_R(\mathfrak{a},M)$.

PROPOSITION 2.7. *Let M be a finitely generated R-module. Then the following holds*:

(i) $\mathcal{S}-\text{E. grade}_R(\mathfrak{a},M)=\inf\{\text{E. grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})|\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)\}$.

(ii) $\mathcal{S}-\mathrm{K.\ grade}_R(\mathfrak{a},M)=\inf\{\mathrm{K.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})|\mathfrak{p}\in\mathcal{S}-\mathrm{Supp}_R(M)\}$.

(iii) *If $\mathcal{S}$ is closed under taking direct sums, then*

$$\mathcal{S}-\mathrm{H.\ grade}_R(\mathfrak{a},M)=\inf\{\mathrm{H.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})|\mathfrak{p}\in\mathcal{S}-\mathrm{Supp}_R(M)\}.$$

Proof. (i) First, assume that $s:=\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)<\infty$. So $\mathrm{Ext}^s_R(R/\mathfrak{a},M)\notin\mathcal{S}$ and $\mathrm{Ext}^i_R(\frac{R}{\mathfrak{a}},M)\in\mathcal{S}$ for all $0\leq i<s$. By using Lemma 2.1(i), $\mathrm{Ext}^i_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}},M_\mathfrak{p})=0$ for all $\mathfrak{p}\in\mathcal{S}-\mathrm{Supp}_R(M)$ and all $0\leq i<s$. Therefore, $s\leq\mathrm{E.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})$ for all $\mathfrak{p}\in\mathcal{S}-\mathrm{Supp}_R(M)$. Again by Lemma 2.1(i), $R/\mathfrak{q}\notin\mathcal{S}$ for some $\mathfrak{q}\in\mathrm{Supp}_R(\mathrm{Ext}^s_R(\frac{R}{\mathfrak{a}},M))$. The fact $\mathrm{Ext}^s_{R_\mathfrak{q}}(\frac{R_\mathfrak{q}}{\mathfrak{a}R_\mathfrak{q}},M_\mathfrak{q})\neq 0$ implies that $\mathrm{E.\ grade}_{R_\mathfrak{q}}(\mathfrak{a}R_\mathfrak{q},M_\mathfrak{q})\leq s$. Now, we can assume that $\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)=\infty$. In the case $\mathcal{S}-\mathrm{Supp}_R(M)=\emptyset$, we have nothing to prove. Hence, we can assume that $\mathcal{S}-\mathrm{Supp}_R(M)\neq\emptyset$. Let $\mathfrak{p}\in\mathcal{S}-\mathrm{Supp}_R(M)$. Since $\mathrm{Ext}^i_R(R/\mathfrak{a},M)\in\mathcal{S}$ for all $i\geq 0$, so $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})=0$ for all $i\geq 0$. Consequently, $\mathrm{E.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})=\infty$.

By the same argument as (i), we can prove (ii) and (iii). Only note that in the case (iii) we use Lemma 2.1(ii) instead of Lemma 2.1(i), since $H^i_\mathfrak{a}(M)$ is not necessarily a finitely generated R-module. $\square$

The following is the main result of this section.

THEOREM 2.8. *Let M be a finitely generated R-module. Then the following hold*:

(i) *Let $\underline{x}:=x_1,\dots,x_r$ be a maximal weak M-sequence with respect to $\mathcal{S}$ in $\mathfrak{a}$. Then $r=\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)$.*

(ii) $\mathcal{S}-\mathrm{c.\ grade}_R(\mathfrak{a},M)=\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)=\mathcal{S}-\mathrm{K.\ grade}_R(\mathfrak{a},M)$.

(iii) *If $\mathcal{S}$ is closed under taking direct sums, then $\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)=\mathcal{S}-\mathrm{H.\ grade}_R(\mathfrak{a},M)$.*

(iv) *If $M/\mathfrak{a}M\notin\mathcal{S}$, then $\mathcal{S}-\mathrm{K.\ grade}_R(\mathfrak{a},M)<\infty$ and all maximal M-sequences with respect to $\mathcal{S}$ in $\mathfrak{a}$ have a common length.*

Proof. (i) Since M is a finitely generated, it follows from the maximality of $\underline{x}$ and Lemma 2.3 that $\mathfrak{a}\subseteq\mathfrak{p}$ for some $\mathfrak{p}\in\mathcal{S}-\mathrm{Ass}_R(M/\underline{x}M)$. Hence

$$\mathfrak{p}\in\mathrm{Ass}_R(M/\underline{x}M)\cap\mathrm{Supp}_R(R/\mathfrak{a})=\mathrm{Ass}_R\big(\mathrm{Hom}_R(R/\mathfrak{a},M/\underline{x}M)\big),$$

see [BH, Exercise 1.2.27]. Lemma 2.1 implies that $\mathrm{Hom}_R(R/\mathfrak{a},M/\underline{x}M)\notin\mathcal{S}$. Now, the conclusion follows from Lemma 2.4.

(ii) The inequality $\mathcal{S}-\mathrm{c.\ grade}_R(\mathfrak{a},M)\leq\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)$, becomes clear by Lemma 2.4(i). Therefore, without loss of generality, we can assume that $r:=\mathcal{S}-\mathrm{c.\ grade}_R(\mathfrak{a},M)<\infty$. So, the inequality from the other side follows from (i).

In order to prove the second equality, recall that for all $\mathfrak{p}\in\mathcal{S}-\mathrm{Supp}_R(M)$ we have

$$\mathrm{E.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p})=\mathrm{K.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p},M_\mathfrak{p}),$$

see [Str, Theorem 6.1.6]. In view of Proposition 2.7(i) and (ii), the assertion follows.

(iii) This follows from Proposition 2.7(iii) and [Str, Proposition 5.3.15], which state that

$$\mathrm{E.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p}, M_\mathfrak{p}) = \mathrm{H.\ grade}_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p}, M_\mathfrak{p}).$$

(iv) Assume that $\mathfrak{a}$ can be generated by n elements $\underline{y} := y_1, \dots, y_r$. We have

$$H^n(K(\underline{y}, M)) \cong M/\mathfrak{a}M \notin \mathcal{S}$$

and consequently $\mathcal{S} - \mathrm{K.\ grade}_R(\mathfrak{a}, M) < \infty$. So, in view of (i) and (ii), all maximal M-sequences with respect to $\mathcal{S}$ in $\mathfrak{a}$ have the same length. □

Example 2.9. (i) In Theorem 2.8(iii), the assumption "$\mathcal{S}$ is closed under taking direct sums" is really needed. To see this, let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$ and let $\mathcal{S}$ be the Serre class of all finitely generated R-modules. It is easy to see that $\mathcal{S} - \mathrm{E.\ grade}_R(\mathfrak{m}, R) = \infty \neq d = \mathcal{S} - \mathrm{H.\ grade}_R(\mathfrak{m}, R)$.

(ii) In Theorem 2.8(ii), the finitely generated assumption on M is necessary. To see this, let $R = K[[x, y]]$ and set $M := \bigoplus_{0 \neq r \in (X,Y)} R/rR$. By [Str, p. 91], we have $\mathrm{E.\ grade}_R(\mathfrak{m}, M) = 1$ and $\mathrm{c.\ grade}_R(\mathfrak{m}, M) = 0$.

We denote the category of R-modules and R-homomorphisms, by R-**Mod**. For an R-module L, set $\mathcal{T}(L) := \{N \in R\text{-}\mathbf{Mod} | \operatorname{Supp}_R N \subseteq \operatorname{Supp}_R L\}$. It is easy to see that $\mathcal{T}(L)$ is a Serre class, which is closed under taking direct sums. Such Serre classes are called torsion theories. In the following proposition, we give a characterization of torsion theories over Noetherian rings.

Proposition 2.10. *Let $\mathcal{T}$ be a torsion theory. Then $\mathcal{T} = \mathcal{T}(L)$ for some R-module L.*

Proof. Set $L := \bigoplus_{\mathfrak{p} \in \Sigma} R/\mathfrak{p}$, where $\Sigma = \{\mathfrak{p} \in \operatorname{Spec} R | R/\mathfrak{p} \in \mathcal{T}\}$ is a subset of $\operatorname{Spec} R$. Let N be an R-module. It follows from Lemma 2.1(ii) that $N \in \mathcal{T}$ if and only if $\operatorname{Supp}_R N \subseteq \Sigma$. The claim follows from the fact that $\operatorname{Supp}_R N \subseteq \Sigma$ if and only if $N \in \mathcal{T}(L)$. □

Let $(R, \mathfrak{m})$ be a local ring and set $\mathcal{T}(R/\mathfrak{m}) := \{N \in R\text{-}\mathbf{Mod} | \operatorname{Supp}_R N \subseteq \operatorname{Supp}_R(R/\mathfrak{m})\}$. In the literature, the concept of M-sequence with respect to $\mathcal{T}(R/\mathfrak{m})$ is called M-filter regular sequence and the maximal length of such sequences in $\mathfrak{a}$ is denoted by $f - \operatorname{depth}_R(\mathfrak{a}, M)$, see [CST] and [Mel2]. The following corollary can be found in [LT, Proposition 3.2], [LT, Theorem 3.9], [LT, Theorem 3.10] and [Mel1, Theorem 5.5].

Corollary 2.11. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module such that* $\operatorname{Supp}_R(M/\mathfrak{a}M) \nsubseteq \{\mathfrak{m}\}$. *Then*

$$f - \operatorname{depth}_R(\mathfrak{a}, M) = \inf\{i : H^i_\mathfrak{a}(M) \textit{ is not Artinian}\}$$

$$= \inf\{i : H^i(K^\bullet(\mathfrak{a}, M)) \textit{ is not Artinian}\}$$
$$= \inf\{i : \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \textit{ is not Artinian}\}.$$

Proof. Note that $\inf\{i : H^i_{\mathfrak{a}}(M)$ is not Artinian$\} = \inf\{i : \operatorname{Supp}_R H^i_{\mathfrak{a}}(M) \nsubseteq \{\mathfrak{m}\}\}$, see [Ma, Lemma 2.4]. Also, we know that for any finitely generated R-module N, $\operatorname{Supp}_R N \subseteq \{\mathfrak{m}\}$ if and only if N is an Artinian R-module. The desired result follows from Theorem 2.8. □

By [N, Definition 2.1], a sequence $\underline{x} := x_1, \ldots, x_r$ of elements of $\mathfrak{a}$ is a generalized regular sequence of M, if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M)$ satisfying $\dim(R/\mathfrak{p}) > 1$. She called, the length of a maximal generalized regular sequence of M in $\mathfrak{a}$, generalized depth of M in $\mathfrak{a}$ and denoted it by $g - \operatorname{depth}_R(\mathfrak{a}, M)$, see [N, Definition 4.2]. Set $X_1 := \{\mathfrak{p} \in \operatorname{Spec} R : \dim(R/\mathfrak{p}) \leq 1\}$ and $M_1 := \bigoplus_{\mathfrak{p} \in X_1} R/\mathfrak{p}$. Consider the $\mathcal{T} := \mathcal{T}(M_1) = \{N \in R\text{-}\mathbf{Mod} | \operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M_1\} = \{N \in R\text{-}\mathbf{Mod} | \dim N \leq 1\}$. It is easy to see that $\underline{x}$ is a generalized regular sequence of M if and only if $\underline{x}$ is an M-sequence with respect to $\mathcal{T}$. In the local case, the first and the second equality in the following corollary is in [N, Proposition 4.4] and [N, Proposition 4.5].

Corollary 2.12. *Let M be a finitely generated R-module such that $\dim(M/\mathfrak{a}M) \geq 2$. Then*

$$\begin{aligned} g - \operatorname{depth}_R(\mathfrak{a}, M) &= \inf\{i : \dim H^i_{\mathfrak{a}}(M) > 1\} \\ &= \inf\{i : \dim \operatorname{Ext}^i_R(R/\mathfrak{a}, M) > 1\} \\ &= \inf\{i : \dim H^i(K^\bullet(\mathfrak{a}, M)) > 1\}. \end{aligned}$$

Let j be an integer such that $0 \leq j < \dim R$ and set $X_j = \{\mathfrak{p} \in \operatorname{Spec} R : \dim(R/\mathfrak{p}) \leq j\}$. Consider the R-module $M_j := \bigoplus_{\mathfrak{p} \in X_j} R/\mathfrak{p}$ and set

$$\begin{aligned} \mathcal{T}_j := \mathcal{T}(M_j) &= \{N \in R\text{-}\mathbf{Mod} | \operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M_j\} \\ &= \{N \in R\text{-}\mathbf{Mod} | \dim N \leq j\}. \end{aligned}$$

In the local case, the first equality in the following corollary first appeared in [Q, p. 9] and [BN, Lemma 2.4]. The second equality is in [CHK, Lemma 2.3].

Corollary 2.13. *Let M be a finitely generated R-module such that $\dim(M/\mathfrak{a}M) \geq j + 1$. Then*

$$\begin{aligned} \mathcal{T}_j - \operatorname{depth}_R(\mathfrak{a}, M) &= \inf\{i : \dim H^i_{\mathfrak{a}}(M) > j\} \\ &= \inf\{i : \dim \operatorname{Ext}^i_R(R/\mathfrak{a}, M) > j\} \\ &= \inf\{i : \dim H^i(K(\mathfrak{a}, M)) > j\}. \end{aligned}$$

Let $\mathfrak{b}$ be an ideal of R. Recall from [A, Definition 1.3] that a sequence $a_1, \ldots, a_r$ is a $\mathfrak{b}$-filter regular M-sequence if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R M/(x_1, \ldots, x_{i-1})M \setminus \operatorname{V}(\mathfrak{b})$. Consider the torsion theory $\mathcal{T}_{\mathfrak{b}} := \mathcal{T}(R/\mathfrak{b}) = \{N \in R\text{-}\mathbf{Mod} |$

$\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R(R/\mathfrak{b})\}$. It is easy to see that $\underline{x}$ is a $\mathfrak{b}$-filter regular M-sequence if and only if $\underline{x}$ is an M-sequence with respect to $\mathcal{T}_{\mathfrak{b}}$. The first equality in the following corollary was first appeared in [A, Theorem 1.7].

Corollary 2.14. *Let M be a finitely generated R-module such that $\mathrm{Supp}_R(M/\mathfrak{a}M) \nsubseteq \mathrm{V}(\mathfrak{b})$. Then*

$$\begin{aligned}\mathcal{T}_{\mathfrak{b}} - \mathrm{grade}(\mathfrak{a}, M) &= \inf\{i : \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \notin \mathcal{T}_{\mathfrak{b}}\}\\ &= \inf\{i : H^i(K^\bullet(\mathfrak{a}, M)) \notin \mathcal{T}_{\mathfrak{b}}\}\\ &= \inf\{i : H^i_{\mathfrak{a}}(M) \notin \mathcal{T}_{\mathfrak{b}}\}.\end{aligned}$$

Remark 2.15. After the submission of this paper, the paper [AM] was published. The notion of regular sequences with respect to Serre classes of R-modules can be found in [AM] as Definition 2.6. The authors considered the Serre classes of R-modules that satisfy a condition $\mathcal{C}_{\mathfrak{a}}$, see [AM, Definition 2.1]. By a completely different method, [AM, Theorem 2.18] independently provides another proof of Theorem 2.8 for Serre classes of R-modules that satisfy the condition $\mathcal{C}_{\mathfrak{a}}$. Also, it provides another proof of Proposition 2.7(i) for Serre classes of R-modules that are closed under taking the injective envelope.

3. Cohen–Macaulayness with respect to Serre classes

In this section, we introduce the concept of $\mathcal{S}$-Cohen–Macaulay modules. Let M be an R-module and $\mathfrak{q}$ a prime ideal of R. By $\mathrm{ht}_M(\mathfrak{q})$, we mean the Krull dimension of the $R_{\mathfrak{q}}$-module $M_{\mathfrak{q}}$. First of all, consider the following definition.

Definition 3.1. Let M be an R-module and $\mathfrak{a}$ an ideal of R. The height of $\mathfrak{a}$ on M and the Krull dimension of M with respect to $\mathcal{S}$ are defined as follows:

(i) $\mathcal{S} - \mathrm{ht}_M(\mathfrak{a}) := \inf\{\mathrm{ht}_M(\mathfrak{q}) | \mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M) \cap \mathrm{V}(\mathfrak{a})\}$,
(ii) $\mathcal{S} - \dim(M) := \sup\{\mathrm{ht}_M(\mathfrak{q}) | \mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M)\}$.

In the following lemma, we investigate the relation between $\mathcal{S} - \mathrm{E.\ grade}_R(\mathfrak{a}, M)$ and $\mathcal{S} - \mathrm{ht}_M(\mathfrak{a})$.

Lemma 3.2. *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Then $\mathcal{S} - \mathrm{E.\ grade}_R(\mathfrak{a}, M) \leq \mathcal{S} - \mathrm{ht}_M(\mathfrak{a})$.*

Proof. The result follows from the following inequality and equalities.

$$\begin{aligned}\mathcal{S} - \mathrm{E.\ grade}_R(\mathfrak{a}, M) &= \inf\{\mathrm{E.\ grade}_{R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) | \mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M)\}\\ &= \inf\{\mathrm{E.\ grade}_{R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) | \mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M/\mathfrak{a}M)\}\\ &\leq \inf\{\mathrm{ht}_{M_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}) | \mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M/\mathfrak{a}M)\}\\ &= \inf\{\mathrm{ht}_{M_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}) | \mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M/\mathfrak{a}M)\}\\ &= \mathcal{S} - \mathrm{ht}_M(\mathfrak{a}),\end{aligned}$$

where the first equality follows from the Proposition 2.7. □

For a subset X of $\operatorname{Spec} R$, we denote the set of minimal members of X with respect to inclusion, by $\min(X)$. We say that an ideal $\mathfrak{a}$ of R is unmixed on M with respect to $\mathcal{S}$, if $\mathcal{S}-\operatorname{Ass}_R(M/\mathfrak{a}M)=\{\mathfrak{p}\in\min(\operatorname{Supp}_R(M/\mathfrak{a}M)) : R/\mathfrak{p}\notin\mathcal{S}\}$.

PROPOSITION 3.3. *Let M be a finitely generated R-module. Then the following are equivalent*:

(i) $\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{a},M)=\mathcal{S}-\mathrm{ht}_M(\mathfrak{a})$ *for all ideals $\mathfrak{a}$ of R.*
(ii) $\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{p},M)=\mathcal{S}-\mathrm{ht}_M(\mathfrak{p})=\mathrm{ht}_M(\mathfrak{p})$ *for all* $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$.
(iii) *For any $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$, $M_\mathfrak{p}$ is Cohen–Macaulay.*
(iv) *Any ideal $\mathfrak{a}$ which is generated by $\mathrm{ht}_M(\mathfrak{a})$ elements is unmixed on M with respect to $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii) Note that $\mathcal{S}-\mathrm{ht}_M(\mathfrak{p})=\mathrm{ht}_M(\mathfrak{p})$ for each $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$. So this implication is clear.

(ii) $\Rightarrow$ (iii) Let $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$. Then the claim follows from the following inequalities:

$$\begin{aligned}\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{p},M)&\le E-\mathrm{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p},M_\mathfrak{p})\\&\le \mathrm{ht}_{M_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p})\\&=\mathcal{S}-\mathrm{ht}_M(\mathfrak{p})\\&=\mathcal{S}-\mathrm{E.\ grade}_R(\mathfrak{p},M),\end{aligned}$$

where the first inequality follows from the Proposition 2.7.

(iii) $\Rightarrow$ (iv) Let $\mathfrak{a}$ be an ideal, which can be generated by $\mathrm{ht}_M(\mathfrak{a})$ elements, $\{a_1,\dots,a_n\}$. Let $\mathfrak{p}\in\mathcal{S}-\operatorname{Ass}_R(M/\mathfrak{a}M)$. Then $M_\mathfrak{p}$ is Cohen–Macaulay and $\mathrm{ht}_{M_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p})=n$. Therefore, the elements $a_1/1,\dots,a_n/1$ of the local ring $R_\mathfrak{p}$ form an $M_\mathfrak{p}$-sequence. So $M_\mathfrak{p}/\mathfrak{a}M_\mathfrak{p}$ is Cohen–Macaulay. Since $\mathfrak{p}R_\mathfrak{p}\in\operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{a}M_\mathfrak{p})$, we get that $\mathfrak{p}\in\min(\operatorname{Supp}_R(M/\mathfrak{a}M))$.

(iv) $\Rightarrow$ (iii) Let $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$ be such that $\mathrm{ht}_M(\mathfrak{p})=n$. From this, one can find the elements $x_1,\dots,x_n$ of the ideal $\mathfrak{p}$ such that $\mathrm{ht}_M(x_1,\dots,x_i)=i$ for all $1\le i\le n$. By our assumption, $x_i\notin\bigcup_{\mathfrak{q}\in\mathcal{S}-\operatorname{Ass}_R\frac{M}{(x_1,\dots,x_{i-1})M}}\mathfrak{q}$ for all $1\le i\le n$. In view of Lemma 2.3, the elements $x_1/1,\dots,x_n/1$ of the local ring $R_\mathfrak{p}$ become an $M_\mathfrak{p}$-sequence. Therefore, $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})\ge\dim M_\mathfrak{p}$, which is what we wanted.

(iii) $\Rightarrow$ (i) The proof is like the proof of Lemma 3.2. □

DEFINITION 3.4. Let $(R,\mathfrak{m})$ be a local ring and M a finitely generated R-module. Then M is called an $\mathcal{S}$-*Cohen–Macaulay* R-module if any system of parameters of M form a weak M-sequence with respect to $\mathcal{S}$. The ring R is called $\mathcal{S}$-Cohen–Macaulay if R is $\mathcal{S}$-Cohen–Macaulay over itself.

In the sequel, we need the following theorem, which provides some equivalent conditions to Definition 3.4.

THEOREM 3.5. *Let* $(R,\mathfrak{m})$ *be a local ring and* M *a finitely generated* R*-module. Then the following are equivalent*:

(i) M *is an* $\mathcal{S}$*-Cohen–Macaulay* R*-module.*

(ii) *For any* $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$, $M_{\mathfrak{p}}$ *is Cohen–Macaulay and*

$$\operatorname{ht}_M(\mathfrak{p})+\dim(R/\mathfrak{p})=\dim M.$$

(iii) *For any* $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$, $\operatorname{E.\,grade}_{\mathcal{S}}(\mathfrak{p},M)=\mathcal{S}-\operatorname{ht}_M(\mathfrak{p})=\operatorname{ht}_M(\mathfrak{p})$ *and*

$$\operatorname{ht}_M(\mathfrak{p})+\dim(R/\mathfrak{p})=\dim M.$$

(iv) $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=\dim M-\dim(R/\mathfrak{p})$ *for any* $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$.

(v) *If* $x_1,\ldots,x_d$ *is a system of parameters for* M, *then*

$$\dim(R/\mathfrak{p})=\dim M-i$$

for all $\mathfrak{p}\in\mathcal{S}-\operatorname{Ass}_R(M/(x_1,\ldots,x_i)M)$ *and all* $0\le i\le d:=\dim M$.

(vi) $\operatorname{ht}(\mathfrak{p}/\mathfrak{q})+\operatorname{ht}_M(\mathfrak{q})=\operatorname{ht}_M(\mathfrak{p})$ *for all* $\mathfrak{q}\subseteq\mathfrak{p}$ *of* $\mathcal{S}-\operatorname{Supp}_R(M)\cup\{\mathfrak{m}\}$, $M_{\mathfrak{p}}$ *is Cohen Macaulay for all* $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$ *and* $\dim(R/\mathfrak{p})=\dim M$ *for all* $\mathfrak{p}\in\min(\mathcal{S}-\operatorname{Supp}_R(M))$.

Proof. (ii) $\Leftrightarrow$ (iv) is easy and (ii) $\Leftrightarrow$ (iii) follows from Proposition 3.3.

(i) $\Rightarrow$ (iii) Let $\mathfrak{p}\in\mathcal{S}-\operatorname{Supp}_R(M)$ and suppose that $\dim(R/\mathfrak{p})=\dim M-i$. Then there exists a subsystem $x_1,\ldots,x_i$ of a system of parameters for M contained in $\mathfrak{p}$, see [BH, Proposition A.4]. In view of our assumption, the sequence $x_1,\ldots,x_i$ is a weak M-sequence with respect to $\mathcal{S}$. So by Lemma 3.2 and Theorem 2.8(ii),

$$\mathcal{S}-\operatorname{E.\,grade}_R(\mathfrak{p},M)\le\operatorname{ht}_M(\mathfrak{p})\le i\le\mathcal{S}-\operatorname{E.\,grade}_R(\mathfrak{p},M).$$

Therefore, $\mathcal{S}-\operatorname{E.\,grade}_R(\mathfrak{p},M)=\operatorname{ht}_M(\mathfrak{p})=i$.

(iii) $\Rightarrow$ (v) First, we claim that if $\underline{x}:=x_1,\ldots,x_\ell$ is a subset of a system of parameters for M, then $\underline{x}$ form a weak M-sequence with respect to $\mathcal{S}$. We prove this claim by induction on ℓ. For the case $\ell=0$, we have nothing to prove. Now suppose inductively, $\ell>0$ and the result has been proved for all subset of system of parameters of length less than ℓ. Then, by the inductive hypothesis $x_1,\ldots,x_{\ell-1}$ is a weak M-sequence with respect to $\mathcal{S}$. Let $\mathfrak{p}\in\mathcal{S}-\operatorname{Ass}_R(M/(x_1,\ldots,x_{\ell-1})M)$. In view of Lemma 2.3, we have that $x_1,\ldots,x_{\ell-1}$ is a maximal weak M-sequence with respect to $\mathcal{S}$ in $\mathfrak{p}$. By the assumption and Theorem 2.8(i), we have $\dim(R/\mathfrak{p})=\dim M-\ell+1$. Since $\underline{x}$ is a subset of a system of parameters for M, one can deduce that $x_\ell\notin\mathfrak{p}$. Now, the implication (i) $\Rightarrow$ (ii) of Lemma 2.3 shows that $\underline{x}$ is a weak M-sequence with respect to $\mathcal{S}$, as claimed.

Let $\underline{x}:=x_1,\ldots,x_i$ be a subset of a system of parameters for M and let $\mathfrak{p}\in\mathcal{S}-\operatorname{Ass}_R(\frac{M}{(x_1,\ldots,x_i)M})$. So $x_1,\ldots,x_i$ is a maximal weak M-sequence with respect to $\mathcal{S}$ in $\mathfrak{p}$ and consequently $i=\mathcal{S}-\operatorname{E.\,grade}_R(\mathfrak{p},M)$. Now, the assertion becomes clear.

(v) $\Rightarrow$ (i) Let $\underline{x} := x_1, \ldots, x_d$ be a system of parameters for M. In view of our assumption, we have that $x_i \notin \bigcup_{\mathfrak{p} \in \mathcal{S}-\mathrm{Ass}_R M/(x_1,\ldots,x_{i-1})M} \mathfrak{p}$ for all $i = 1, \ldots, d$. Therefore, this implication follows from Lemma 2.3 and Definition 3.4.

(ii) $\Rightarrow$ (vi) Assume that $\mathfrak{p} \in \min(\mathcal{S} - \mathrm{Supp}_R(M))$. Let $\mathfrak{q} \in \mathrm{Supp}\, M$ be such that $\mathfrak{q} \subseteq \mathfrak{p}$. The epimorphism $R/\mathfrak{q} \longrightarrow R/\mathfrak{p}$ shows that $R/\mathfrak{q} \notin \mathcal{S}$ and consequently $\mathfrak{p} = \mathfrak{q}$. Hence, $\mathrm{ht}_M(\mathfrak{p}) = 0$. Therefore, by assumption, we get that $\dim R/\mathfrak{p} = \dim M$.

Let $\mathfrak{p}, \mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M) \cup \{\mathfrak{m}\}$ be such that $\mathfrak{q} \subseteq \mathfrak{p}$. In order to show that $\mathrm{ht}_M(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}/\mathfrak{q}) + \mathrm{ht}_M(\mathfrak{q})$, we can assume that $\mathfrak{p} \neq \mathfrak{q}$. To do this, first we assume that $\mathfrak{p} \neq \mathfrak{m}$. So $\mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M)$. Since $M_\mathfrak{p}$ is Cohen–Macaulay, we have:

$$\mathrm{ht}_M(\mathfrak{p}) = \dim(M_\mathfrak{p}) = \dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) + \mathrm{ht}_{M_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}/\mathfrak{q}) + \mathrm{ht}_M(\mathfrak{q}).$$

Now, we consider the case $\mathfrak{p} = \mathfrak{m}$. Note that $\mathfrak{p} \neq \mathfrak{q}$. Therefore, $\mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M)$. So

$$\mathrm{ht}_M(\mathfrak{q}) + \mathrm{ht}(\mathfrak{m}/\mathfrak{q}) = \dim(M_\mathfrak{q}) + \dim(R/\mathfrak{q}) = \dim M = \mathrm{ht}_M(\mathfrak{m}).$$

It remains to show the implication (vi) $\Rightarrow$ (ii). This is clear. $\square$

For an R-module L, we denote $\{\mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R L \,|\, \dim L = \dim(R/\mathfrak{p})\}$ by $\mathcal{S} - \mathrm{Assh}_R L$. Now, we are ready to present some of the basic properties of $\mathcal{S}$-Cohen–Macaulay modules.

PROPOSITION 3.6. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated $\mathcal{S}$-Cohen–Macaulay and x a weak M-sequence in $\mathfrak{m}$ with respect to $\mathcal{S}$. If $\mathcal{S} - \mathrm{Assh}_R(M/xM) \neq \emptyset$, then M/xM is an $\mathcal{S}$-Cohen–Macaulay R-module.*

Proof. First, we show that $\dim(M/xM) = \dim M - 1$. It is known that $\dim M \leq \dim(M/xM) + 1$. Let $\mathfrak{q} \in \mathcal{S} - \mathrm{Assh}_R(M/xM)$. Then by Theorem 2.8(i), Lemma 3.2 and Theorem 3.5(ii) we have:

$$\begin{aligned}
1 &\leq \mathcal{S} - \mathrm{E.\ grade}_R(\mathfrak{q}, M) \\
&\leq \mathcal{S} - \mathrm{ht}_M(\mathfrak{q}) \\
&= \mathrm{ht}_M(\mathfrak{q}) \\
&= \dim M - \dim(R/\mathfrak{q}) \\
&= \dim M - \dim(M/xM) \leq 1,
\end{aligned}$$

which implies that $\dim(M/xM) = \dim M - 1$.

Let $\mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M/xM)$. So $x \in \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M)$. By Theorem 3.5(ii) $M_\mathfrak{p}$ is Cohen–Macaulay and $\mathrm{ht}_M(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim M$. Also by Lemma 2.3 the element $x/1$ of the local ring $R_\mathfrak{p}$ becomes an $M_\mathfrak{p}$-sequence. This implies that $M_\mathfrak{p}/xM_\mathfrak{p}$ is Cohen–Macaulay and $\mathrm{ht}_{\frac{M}{xM}}(\mathfrak{p}) = \mathrm{ht}_M(\mathfrak{p}) - 1$. Therefore, $\mathrm{ht}_{\frac{M}{xM}}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(M/xM)$. Again by Theorem 3.5(ii), M/xM is $\mathcal{S}$-Cohen–Macaulay. $\square$

COROLLARY 3.7. *Let* $(R,\mathfrak{m})$ *be a local ring, x a weak M-sequence in* $\mathfrak{m}$ *with respect to $\mathcal{S}$ and M a finitely generated R-module. Then the following hold*:

(i) *If* $\frac{M}{xM}$ *is equidimensional and M is $\mathcal{S}$-Cohen–Macaulay, then* $\frac{M}{xM}$ *is $\mathcal{S}$-Cohen–Macaulay.*

(ii) *If M is an f-module* (*generalized f-module*), *then* $\frac{M}{xM}$ *is an f-module* (*generalized f-module*).

Proof. (i) Without loss of generality, we can assume that $M/xM \notin \mathcal{S}$. It follows from Lemma 2.1 that there exists $\mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M/xM)$. Let $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\mathfrak{p} \in \min(\mathrm{Supp}_R(M/xM))$. The natural epimorphism $R/\mathfrak{p} \longrightarrow R/\mathfrak{q}$ shows that $R/\mathfrak{p} \notin \mathcal{S}$. On the other hand $\frac{M}{xM}$ is an equidimensional R-module. Therefore, $\mathcal{S} - \mathrm{Assh}_R(M/xM) \neq \emptyset$.

(ii) Set $\mathcal{S} := \{N \in R\text{-}\mathbf{Mod} \mid \mathrm{Supp}_R N \subseteq \{\mathfrak{m}\}\}$ ($\mathcal{S} := \{N \in R\text{-}\mathbf{Mod} \mid \dim N \leq 1\}$). Then the notions of $\mathcal{S}$-Cohen–Macaulay and f-module (generalized f-module) are equivalent. Now, we prove the claim. We can assume that $M/xM \notin \mathcal{S}$. Let $\mathfrak{p} \in \mathrm{Assh}_R(M/xM)$. Therefore, $\dim R/\mathfrak{p} > 0$ ($\dim R/\mathfrak{p} > 1$). Thus, in both cases, we have $\mathcal{S} - \mathrm{Assh}_R(M/xM) \neq \emptyset$. □

REMARK 3.8. Let $(R,\mathfrak{m})$ be a local ring and M a finitely generated R-module. Assume that x is an M-regular element. This is well known that M is Cohen–Macaulay if and only if M/xM is Cohen–Macaulay. Having this fact and Corollary 3.7 in mind, one might ask whether the converse of Corollary 3.7(i) is true. This is not necessarily true. To see this, consider the following example. Let $(R,\mathfrak{m})$ be a 2-dimensional local domain such that its completion $\widehat{R}$ has an associated prime $\mathfrak{p}$ with $\dim \widehat{R}/\mathfrak{p} = 1$. Such rings were constructed by Nagata [Na, p. 203, Example 2]. Theorem 3.5(iv) implies that $\widehat{R}$ is not an f-module. Since $\mathrm{depth}\, \widehat{R} = \mathrm{depth}\, R > 0$, there exists an element $x \in \widehat{R}$, which is $\widehat{R}$-regular. So $\dim \widehat{R}/x\widehat{R} = 1$. On the other hand, any 1-dimensional local ring is an f-module. Therefore $\widehat{R}/x\widehat{R}$ is an f-module.

Let $f : R \longrightarrow A$ be a flat homomorphism of rings and $\mathcal{S}$ a Serre class of A-modules. Set $\mathcal{S}^c = \{M \in R\text{-}\mathbf{Mod} \mid M \otimes_R A \in \mathcal{S}\}$. It is routine to show that $\mathcal{S}^c$ is a Serre class of R-modules.

THEOREM 3.9. *Let* $f : (R,\mathfrak{m}) \longrightarrow (A,\mathfrak{n})$ *be a flat local homomorphism of Noetherian local rings. Let $\mathcal{S}$ be a Serre class of A-modules and M a finitely generated R-module. Then $M \otimes_R A$ is $\mathcal{S}$-Cohen–Macaulay if and only if the following three conditions are satisfied*:

(i) *M is $\mathcal{S}^c$-Cohen–Macaulay.*

(ii) $\frac{A_\mathfrak{q}}{f^{-1}(\mathfrak{q})A_\mathfrak{q}}$ *is Cohen–Macaulay for all* $\mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_A(M \otimes_R A)$.

(iii) $\mathrm{ht}(\frac{\mathfrak{q}}{f^{-1}(\mathfrak{q})A}) + \dim(\frac{A}{\mathfrak{q}}) = \dim(\frac{A}{f^{-1}(\mathfrak{q})A})$ *for all* $\mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_A(M \otimes_R A)$.

Proof. We first bring the following easy results (a) and (b). (a) Let $\mathfrak{p} \in \mathcal{S}^c - \mathrm{Supp}_R(M)$. The condition $R/\mathfrak{p} \notin \mathcal{S}^c$ implies that $A/\mathfrak{p}A \notin \mathcal{S}$. In view of

Lemma 2.1, one can find $\mathfrak{q}' \in \mathcal{S} - \operatorname{Supp}_A(A/\mathfrak{p}A)$. Let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{p}A$ such that $\mathfrak{q} \subseteq \mathfrak{q}'$. Hence, $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Also the natural epimorphism $A/\mathfrak{q} \longrightarrow A/\mathfrak{q}'$ shows that $A/\mathfrak{q} \notin \mathcal{S}$. Therefore, $\mathfrak{q} \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$.

(b) Let the situation and notation be as in (a). Then we have

$$\begin{aligned}
\operatorname{ht}_M(\mathfrak{p}) + \dim(R/\mathfrak{p}) &= \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}) + \dim M_\mathfrak{p} + \dim(R/\mathfrak{p}) \\
&= \dim(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} A_\mathfrak{q}) + \dim(R/\mathfrak{p}) \\
&= \operatorname{ht}_{M \otimes_R A}(\mathfrak{q}) + \dim(R/\mathfrak{p}) \\
&= \operatorname{ht}_{M \otimes_R A}(\mathfrak{q}) + \dim(A/\mathfrak{p}A) - \dim(A/\mathfrak{m}A) \\
&\geq \operatorname{ht}_{M \otimes_R A}(\mathfrak{q}) + \dim(A/\mathfrak{q}) - \dim(A/\mathfrak{m}A),
\end{aligned}$$

where the second equality follows from [BH, Theorem A.11(b)], for the natural flat homomorphism $R_\mathfrak{p} \longrightarrow A_\mathfrak{q}$, and the forth equality follows from [BH, Theorem A.11(a)], for the natural flat homomorphism $R/\mathfrak{p} \longrightarrow A/\mathfrak{p}A$.

Now, we are ready to prove our claims.

Assume that $M \otimes_R A$ is $\mathcal{S}$-Cohen–Macaulay. Let $\mathfrak{p} \in \mathcal{S}^c - \operatorname{Supp}_R(M)$. Then by (a) there exists $\mathfrak{q} \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$ such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$ and $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{p}A$. By our assumption and Theorem 3.5 $(M \otimes_R A)_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} A_\mathfrak{q}$ is Cohen–Macaulay. So $M_\mathfrak{p}$ and $A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}$ are Cohen–Macaulay.

On the other hand from (b) and Theorem 3.5(ii), we have

$$\begin{aligned}
\operatorname{ht}_M(\mathfrak{p}) + \dim(R/\mathfrak{p}) &\geq \operatorname{ht}_{M \otimes_R A}(\mathfrak{q}) + \dim(A/\mathfrak{q}) - \dim(A/\mathfrak{m}A) \\
&= \dim(M \otimes_R A) - \dim(A/\mathfrak{m}A) \\
&= \dim M.
\end{aligned}$$

Consequently, $\operatorname{ht}_M(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim M$. Again, by Theorem 3.5(ii), M becomes $\mathcal{S}^c$-Cohen–Macaulay. This completes the proof of (i).

With the notation as (ii), the desired result of (ii) follows from the Cohen–Macaulayness of $(M \otimes_R A)_\mathfrak{q} \cong M_{f^{-1}(\mathfrak{q})} \otimes_{R_\mathfrak{p}} A_\mathfrak{q}$.

In order to prove (iii), first, we claim that:

(c) Let $\mathfrak{p} \in \mathcal{S}^c - \operatorname{Supp}_R(M)$. If Q is a minimal prime ideal of $\mathfrak{p}A$ and $A/Q \notin \mathcal{S}$, then $\dim(A/Q) = \dim(A/\mathfrak{p}A)$.

To establish the claim, consider the following:

$$\begin{aligned}
\dim(A/\mathfrak{p}A) &\geq \dim(A/Q) \\
&= \operatorname{ht}_{M \otimes_R A}(\mathfrak{n}) - \operatorname{ht}_{M \otimes_R A}(Q) \\
&= \operatorname{ht}_{M \otimes_R A}(\mathfrak{n}) - \operatorname{ht}_M(\mathfrak{p}) \\
&= \dim M + \dim(A/\mathfrak{m}A) - \operatorname{ht}_M(\mathfrak{p}) \\
&= \dim(R/\mathfrak{p}) + \dim(A/\mathfrak{m}A) \\
&= \dim(A/\mathfrak{p}A),
\end{aligned}$$

where the first equality follows from Theorem 3.5(ii) and the second equality follows from [BH, Theorem A.11(b)] for the natural flat homomorphism $R_\mathfrak{p} \longrightarrow A_Q$. Note that we have $Q \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$ and $Q \cap R = \mathfrak{p}$.

Let $\mathfrak{q} \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$ and $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Then $A/\mathfrak{p}A \notin \mathcal{S}$. So by (a) we can find $\mathfrak{q}_1 \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$ such that $\mathfrak{q}_1$ is a minimal prime ideal of $\mathfrak{p}A$ and $\mathfrak{q}_1 \subseteq \mathfrak{q}$. Keep in mind that $M \otimes_R A$ is $\mathcal{S}$-Cohen–Macaulay. Hence, Theorem 3.5(vi) implies that

$$\begin{aligned}
&\dim(A/\mathfrak{q}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}A)\\
&\quad \geq \operatorname{ht}(\mathfrak{n}/\mathfrak{q}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{q}_1)\\
&\quad = \operatorname{ht}_{M\otimes_R A}(\mathfrak{n}) - \operatorname{ht}_{M\otimes_R A}(\mathfrak{q}) + \operatorname{ht}_{M\otimes_R A}(\mathfrak{q}) - \operatorname{ht}_{M\otimes_R A}(\mathfrak{q}_1)\\
&\quad = \operatorname{ht}(\mathfrak{n}/\mathfrak{q}_1)\\
&\quad = \dim(A/\mathfrak{q}_1)\\
&\quad = \dim(A/\mathfrak{p}A),
\end{aligned}$$

where the last equality follows from (c). Consequently,

$$\operatorname{ht}(\mathfrak{q}/\mathfrak{p}A) + \dim(A/\mathfrak{q}) = \dim(A/\mathfrak{p}A).$$

This completes the proof of (iii).

Conversely, assume that the conditions (i), (ii) and (iii) hold. In order to show that $M \otimes_R A$ is $\mathcal{S}$-Cohen–Macaulay, in view of Theorem 3.5(ii), we need to prove the following claims:

(d) $(M \otimes_R A)_{\mathfrak{q}}$ is Cohen–Macaulay for all $\mathfrak{q} \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$.
(e) $\operatorname{ht}_{M\otimes_R A}(\mathfrak{q}) + \dim(A/\mathfrak{q}) = \dim(M \otimes_R A)$ for all $\mathfrak{q} \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$.

The claim (d) follows immediately from (i) and (ii). Now, we shall achieve (e). Let $\mathfrak{q} \in \mathcal{S} - \operatorname{Supp}_A(M \otimes_R A)$. Set $\mathfrak{p} := f^{-1}(\mathfrak{q})$. Then we have

$$\begin{aligned}
&\dim(A/\mathfrak{q}) + \operatorname{ht}_{M\otimes_R A}(\mathfrak{q})\\
&\quad = \dim(A/\mathfrak{q}) + \operatorname{ht}_M(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}A)\\
&\quad = \dim(A/\mathfrak{q}) + \dim M - \dim(R/\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}A)\\
&\quad = \dim(A/\mathfrak{q}) + \dim(M \otimes_R A) - \dim(A/\mathfrak{m}A) - \dim(R/\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}A)\\
&\quad = \dim(A/\mathfrak{q}) + \dim(M \otimes_R A) - \dim(A/\mathfrak{p}A) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}A)\\
&\quad = \dim(M \otimes_R A),
\end{aligned}$$

where the last equality follows from (iii). □

For the ring extension $(R, \mathfrak{m}) \longrightarrow (\widehat{R}, \mathfrak{m}\widehat{R})$, Theorem 3.9 becomes much simpler.

Proposition 3.10. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. If $\mathcal{S}$ is a Serre class of $\widehat{R}$-modules and $M \otimes_R \widehat{R}$ is $\mathcal{S}$-Cohen–Macaulay, then M is $\mathcal{S}^c$-Cohen–Macaulay.*

Proof. Let $\underline{x} := x_1, \ldots, x_\ell$ be a system of parameters for M. So $\underline{x}$ is a system of parameters for $M \otimes_R \widehat{R}$. Therefore, $((x_1, \ldots, x_{i-1})M \otimes_R \widehat{R} :_{M\otimes_R \widehat{R}}$

$x_i)/(x_1,\dots,x_{i-1})(M\otimes_R \widehat{R})\in\mathcal{S}$. The isomorphism

$$\frac{((x_1,\dots,x_{i-1})M\otimes_R \widehat{R}:_{M\otimes_R\widehat{R}} x_i)}{(x_1,\dots,x_{i-1})(M\otimes_R \widehat{R})}\cong \frac{((x_1,\dots,x_{i-1})M:_M x_i)}{(x_1,\dots,x_{i-1})M}\otimes_R \widehat{R}$$

completes the proof. □

Let $(R,\mathfrak{m})$ be a local ring and M a d-dimensional finitely generated R-module. If $d=0$ or $M=0$, we set $\mathfrak{a}(M)=R$. In the other case, we set $\mathfrak{a}(M):=\mathfrak{a}_0(M)\cdots\mathfrak{a}_{d-1}(M)$, where $\mathfrak{a}_i(M)=\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ for all $i=0,\dots,d-1$.

THEOREM 3.11. *Let $(R,\mathfrak{m})$ be a local ring and M a d-dimensional finitely generated R-module. Consider the following conditions*:

(i) $R/\mathfrak{a}(M)\in\mathcal{S}$.

(ii) *M is an $\mathcal{S}$-Cohen–Macaulay R-module.*

Always (i) *implies* (ii) *and if R is a quotient of a Gorenstein local ring, then* (ii) *implies* (i).

Proof. In the cases $\dim M=0$ and $M=0$, the desired claims are trivial. Therefore, without loss of generality, we can assume that $\dim M>0$.

(i) ⇒ (ii) Let $\underline{x}:=x_1,\dots,x_d$ be a system of parameters for M. Set $I:=\underline{x}R$ and consider

$$\tau_{\underline{x}}(M):=\bigcap \operatorname{Ann}_R\left(\frac{((x_1^t,\dots,x_{i-1}^t)M:_M x_i^t)}{(x_1^t,\dots,x_{i-1}^t)M}\right),$$

where the intersection is taken over all $t\in\mathbb{N}$ and $1\le i\le d$. We denote $\operatorname{Ann}_R(H^i_I(M))$ by $\mathfrak{a}^i_I(M)$. By [Sch, Theorem 3(a)], we have

$$\mathfrak{a}^0_I(M)\cdots\mathfrak{a}^{d-1}_I(M)\subseteq\tau_{\underline{x}}(M).$$

Note that $\sqrt{I+\operatorname{Ann}_R M}=\mathfrak{m}$, since $\underline{x}$ is a system of parameters for M. The independence theorem for local cohomology modules implies that $H^i_{\mathfrak{m}}(M)\cong H^i_I(M)$ and consequently $\mathfrak{a}^i_I(M)=\mathfrak{a}_i(M)$. Therefore,

$$\mathfrak{a}(M)\subseteq\operatorname{Ann}_R\left(\frac{((x_1,\dots,x_{i-1})M:_M x_i)}{(x_1,\dots,x_{i-1})M}\right)$$

for all $1\le i\le d$. The condition $R/\mathfrak{a}(M)\in\mathcal{S}$ implies that

$$R/\operatorname{Ann}_R\left(\frac{((x_1,\dots,x_{i-1})M:_M x_i)}{(x_1,\dots,x_{i-1})M}\right)\in\mathcal{S}$$

for all $1\le i\le d$. On the other hand, the R-modules

$$R/\operatorname{Ann}_R\left(\frac{((x_1,\dots,x_{i-1})M:_M x_i)}{(x_1,\dots,x_{i-1})M}\right)$$

and

$$\frac{((x_1,\dots,x_{i-1})M:_M x_i)}{(x_1,\dots,x_{i-1})M}$$

have same supports for all $1 \leq i \leq d$. Therefore, it is enough to apply Lemma 2.1 to obtain that $\underline{x}$ is a weak M-sequence with respect to $\mathcal{S}$.

(ii) $\Rightarrow$ (i) Suppose the contrary that, $R/\mathfrak{a}(M) \notin \mathcal{S}$. In view of Lemma 2.1, we can find $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}(M))$ such that $R/\mathfrak{p} \notin \mathcal{S}$. Next, we show that

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) < \dim(M).$$

So in view of Theorem 3.5(ii), we get a contradiction.

Now, we do this. We have $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}_i(M))$ for some $i < \dim M$. Let $(R', \mathfrak{n})$ be a Gorenstein ring of dimension r' for which there exists a surjective ring homomorphism $f : R' \longrightarrow R$. The local duality theorem [BS, Theorem 11.2.6] implies that

$$H^i_\mathfrak{m}(M) \cong \operatorname{Hom}_R\big(\operatorname{Ext}^{r'-i}_{R'}(M, R'), E_R(R/\mathfrak{m})\big),$$

where $E_R(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. Hence, $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Ext}^{r'-i}_{R'}(M, R'))$. Let $t := \dim R/\mathfrak{p}$ and $\mathfrak{p}' := f^{-1}(\mathfrak{p})$. Now $R'_{\mathfrak{p}'}$ is a Gorenstein local ring and $t = \dim(R'/\mathfrak{p}')$. Since R' is a Gorenstein ring, $\dim R'_{\mathfrak{p}'} = \dim R' - \dim(R'/\mathfrak{p}') = r' - t$.

Let $f' : R'_{\mathfrak{p}'} \longrightarrow R_\mathfrak{p}$ be the surjective ring homomorphism, which is induced by f. There is an $R_\mathfrak{p}$-isomorphism,

$$\operatorname{Ext}^{r'-i}_{R'_{\mathfrak{p}'}}(M_\mathfrak{p}, R'_{\mathfrak{p}'}) \cong (\operatorname{Ext}^{r'-i}_{R'}(M, R'))_\mathfrak{p}.$$

Again the local duality theorem implies that

$$H^{i-t}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}\big(\operatorname{Ext}^{r'-i}_{R'_{\mathfrak{p}'}}(M_\mathfrak{p}, R'_{\mathfrak{p}'}), E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})\big),$$

as $R_\mathfrak{p}$-modules. Therefore, $H^{i-t}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. Consequently,

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq i - t < \dim M - t = \dim M - \dim(R/\mathfrak{p}). \qquad \square$$

The following example shows that the assumption, R is a quotient of a Gorenstein local ring, in Theorem 3.11 is needed.

Example 3.12. By [NM, Example 3.4], there exists a 3-dimensional local ring $(R, \mathfrak{m})$ such that $\dim(R/\mathfrak{a}(R)) = 3$. Denote the class of all finitely generated R-modules of Krull dimension less than 2, by $\mathcal{S}$. It is a Serre class of R-modules. The assumption $\dim(R/\mathfrak{a}(R)) = 3$ implies that $R/\mathfrak{a}(R) \notin \mathcal{S}$. Also, the example shows that R is a generalized f-ring, i.e., $\mathcal{S}$-Cohen–Macaulay.

Denote by $\mathcal{NCM}(M)$ the non-Cohen–Macaulay locus of M, i.e.

$$\mathcal{NCM}(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \text{ is not Cohen–Macaulay}\}.$$

Assume that R is a quotient of a Cohen–Macaulay ring. It is well known that the non-Cohen–Macaulay locus of M is a closed subset of $\operatorname{Spec} R$ with respect to the Zariski topology, i.e., $\mathcal{NCM}(M) = V(\mathfrak{a}_M)$ for some ideal $\mathfrak{a}_M$ of R. Therefore, such ideals are unique up to radical.

THEOREM 3.13. *Let $(R,\mathfrak{m})$ be a local ring which is a quotient of a Cohen–Macaulay ring and M a finitely generated R-module. Then the following are equivalent*:

(i) *M is an $\mathcal{S}$-Cohen–Macaulay R-module.*
(ii) *$R/\mathfrak{a}_M \in \mathcal{S}$ and $\dim(R/\mathfrak{p}) = \dim M$ for all prime ideals $\mathfrak{p} \in \min(\mathcal{S} - \mathrm{Supp}_R(M))$.*

Proof. (i) $\Rightarrow$ (ii) In order to prove $R/\mathfrak{a}_M \in \mathcal{S}$, it is enough to show that $R/\mathfrak{p} \in \mathcal{S}$ for all $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}_M)$. Let $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}_M)$. Since $M_\mathfrak{p}$ is not Cohen–Macaulay, Theorem 3.5(ii) implies that $R/\mathfrak{p} \in \mathcal{S}$.

Let $\mathfrak{p} \in \min(\mathcal{S} - \mathrm{Supp}_R(M))$. Then by the implication (i) $\Rightarrow$ (vi) of Theorem 3.5, we have $\dim(R/\mathfrak{p}) = \dim M$.

(ii) $\Rightarrow$ (i) Let $\mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M)$. In particular, $R/\mathfrak{p} \notin \mathcal{S}$. In view of Lemma 2.1 and $R/\mathfrak{a}_M \in \mathcal{S}$, we have $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a}_M)$. So $M_\mathfrak{p}$ is a Cohen–Macaulay $R_\mathfrak{p}$-module.

Set $t := \dim(R/\mathfrak{p})$ and $s := \mathrm{ht}_M(\mathfrak{p})$. Hence, we have following saturated chains of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{m},$$
$$\mathfrak{q} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_s = \mathfrak{p}.$$

By concatenating these chains, we get the following saturated chain of prime ideals:

$$\mathfrak{q} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_s = \mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{m}.$$

The epimorphism $R/\mathfrak{q}_0 \longrightarrow R/\mathfrak{p}$ shows that $R/\mathfrak{q}_0 \notin \mathcal{S}$ and consequently $\mathfrak{q}_0 \in \min(\mathcal{S} - \mathrm{Supp}_R(M))$. So $\dim(R/\mathfrak{q}_0) = \dim M$. On the other hand, R is a quotient of a Cohen–Macaulay ring. This implies that $\mathrm{Supp}(R/\mathfrak{q}_0)$ is catenary. Therefore,

$$\dim M = \dim(R/\mathfrak{q}_0) = s + t = \mathrm{ht}_M(\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Consequently, Theorem 3.5(ii) implies that M is an $\mathcal{S}$-Cohen–Macaulay R-module. ☐

REFERENCES

[AM] M. Aghapournahr and L. Melkersson, *Local cohomology and Serre subcategories*, J. Algebra **320** (2008), 1275–1287. MR 2427643

[A] K. Ahmadi-Amoli, *Local cohomology and cohomology of certain generalized Hughes complexes*, Comm. Algebra **26** (1998), 1305–1317. MR 1612169

[BN] M. Brodmann and L. T. Nhan, *A finiteness result for associated primes of certain ext-modules*, Comm. Algebra **36** (2008), 1527–1536. MR 2406607

[BS] M. Brodmann and R. Y. Sharp, *Local cohomology, an algebraic introduction with geometric applications*, Cambridge Stud. Adv. Math., vol. 60, Cambridge Univ. Press, Cambridge, 1998. MR 1613627

[BH] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Stud. Adv. Math., vol. 39, rev. ed., Cambridge Univ. Press, Cambridge, 1998. MR 1251956

[CHK] N. T. Cuong, N. V. Hoang and P. H. Khanh, *Asymptotic stability of certain sets of assuciated prime ideals of local cohomology modules*, available at `arXiv:0804.0964v1`.

[CST] N. T. Cuong, P. Schenzel and N. V. Trung, *Verallgemeinerte CohenMacaulay moduln*, Math. Nachr. **85** (1978), 57–73. MR 0517641

[GR] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Math., vol. 1800, Springer, 2003. MR 2004652

[LT] R. Lu and Z. Tang, *The f-depth of an ideal on a module*, Proc. Amer. Math. Soc. **130** (2002), 1905–1912. MR 1896021

[Ma] A. Mafi, *Some results on local cohomology modules*, Arch. Math. **87** (2006), 211–216. MR 2258920

[Mel1] L. Melkersson, *Some applications of a criterion for Artinianness of a module*, J. Pure Appl. Algebra **101** (1995), 291–303. MR 1348571

[Mel2] L. Melkersson, *Modules cofinite with respect to an ideal*, J. Algebra **285** (2005), 649–668. MR 2125457

[Na] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, vol. 13, 1962. MR 0155856

[N] L. T. Nhan, *On generalized regular sequences and the finiteness for associated primes of local cohomology modules*, Comm. Algebra **33** (2005), 793–806. MR 2128412

[NM] L. T. Nhan and M. Morales, *Generalized f-modules and the associated primes of local cohomology modules*, Comm. Algebra **34** (2006), 863–878. MR 2208103

[Q] N. Quoc Chinh, *On reducing sequences and an application to local cohomology modules*, Bull. Iranian Math. Soc. **31** (2) (2005), 1–11. MR 2267066

[RSS] P. Roberts, A. K. Singh and V. Srinivas, *Annihilators of local cohomology in characteristic zero*, Illinois J. Math. **51** (2007), 237–254. MR 2346196

[Sch] P. Schenzel, *Cohomological annihilators*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 345–350. MR 0654081

[St] B. Stenström, *Rings of quotients*, Die Grundlehren der Mathematischen Wissenschaften, vol. 217, Springer, New York–Heidelberg, 1975. MR 0389953

[Str] J. R. Strooker, *Homological questions in local algebra*, London Mathematical Lecture Note Series, vol. 145, Cambridge University Press, Cambridge, 1990. MR 1074178

[SV] J. Stückrad and W. Vogel, *Buchsbaum rings and applications*, Web Deutsecher verlag der wissenschaften, Berlin, 1986. MR 0881220

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