Illinois Journal of Mathematics
Volume 53, Number 1, Spring 2009, Pages 25–38
S 0019-2082

CONFORMAL METRICS AND BOUNDARY ACCESSIBILITY

TOMI NIEMINEN

ABSTRACT. We study conformal metrics on the unit ball of Euclidean space. We prove an extension of a theorem originally due to Gerasch on the broadly accessibility of the boundary points of a domain quasiconformally equivalent to a ball. We also show that our result is close to optimal. Our abstract approach leads to new results also for the boundary behavior of (quasi)conformal mappings.

1. Introduction

We continue the study of conformal metrics on the unit ball $\mathbb{B}^n$ of Euclidean space. Thus, given a continuous density $\rho : \mathbb{B}^n \to \mathbb{R}_+$, we define a conformal metric d_ρ by setting

$$\text{length}_\rho(\gamma) = \int_\gamma \rho(z)|dz|$$

for a curve γ in $\mathbb{B}^n$, and

$$d_\rho(x,y) = \inf_\gamma \text{length}_\rho(\gamma) \quad \text{for } x,y \in \mathbb{B}^n,$$

where the infimum is taken over all curves joining x and y in $\mathbb{B}^n$. We also define a measure μ_ρ by setting

$$\mu_\rho(E) = \int_E \rho^n dm_n \quad \text{for a Borel set } E \subset \mathbb{B}^n,$$

where m_n denotes the n-dimensional Lebesgue measure.

Further, we assume that the density ρ satisfies a Harnack inequality, i.e., there exists a constant $A \geq 1$ so that

$$\frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A$$

Received March 25, 2008; received in final form July 14, 2008.
2000 *Mathematics Subject Classification.* Primary 30C65.

whenever $x, y \in B(z, \frac{1}{2}(1-|z|))$ for some $z \in \mathbb{B}^n$. We also assume that the density ρ satisfies a volume growth condition: there exists a constant $B > 0$ so that

$$\mu_\rho(B_\rho(x,r)) \leq Br^n \quad \text{for all } x \in \mathbb{B}^n, r > 0.$$

Here, $B_\rho(x,r)$ denotes an open ball with center x and radius r in the metric d_ρ. The motivation for conformal metrics arises primarily from the theory of (quasi)conformal mappings. Recall that the derivative $|f'(x)|$ of a conformal mapping f is a prime example of a density satisfying the above conditions, see [2] for more information and examples.

In this paper, we study the accessibility of the boundary points $\xi \in \partial\mathbb{B}^n$ in the d_ρ-metric. Recall that a boundary point y of a domain $\Omega \subset \mathbb{R}^n$ is called *broadly accessible*, if there is a sequence of balls in Ω, converging to y, so that the center of each ball can be joined to y by an arc in Ω whose length is only slightly larger than the radius of the ball. Gerasch [4] proved that for almost every point $\xi \in \partial\mathbb{B}^2$ the radial limit $f(\xi)$ under a conformal mapping $f : \mathbb{B}^2 \to f(\mathbb{B}^2)$ is a broadly accessible boundary point of the domain $f(\mathbb{B}^2)$. Martio and Näkki [10] then established the same result for quasiconformal mappings $f : \mathbb{B}^n \to f(\mathbb{B}^n)$, $n \geq 2$. This result was further extended by Koskela and Rohde [8, Theorem 4.1], who considered exceptional sets of smaller size. The next theorem, which is a combination of Theorem 5.2 and Lemma 7.5 in [2], can be considered as a generalization of the results mentioned above to the setting of conformal metrics.

Theorem A ([2]). *Let $0 < \alpha \leq n-1$. Then there exists a set $E \subset \partial\mathbb{B}^n$ with $H^\alpha(E) = 0$ such that, for all $\xi \in \partial\mathbb{B}^n \setminus E$, there is a sequence of points $(x_k) \to \xi$ (in the Euclidean sense) with*

$$\xi \in B_\rho(x_k, \lambda r_{x_k}) \tag{1.1}$$

for all $k \in \mathbb{N}$. Here, $\lambda = \lambda(\alpha, n) \to \infty$ as $\alpha \to 0$.

Here, we write $r_z = \rho(z)(1-|z|)$; recall that this quantity is comparable to the ρ-distance of z to the boundary, see [2, Proposition 6.2]. We shall extend Theorem A by further reducing the size of the exceptional set E.

Theorem 1.1. *Let $s > 1$ and let*

$$\varphi(t) = \exp\left(-\left(\log\frac{1}{t}\right)^{1/s}\right).$$

Then there is a set $E \subset \partial\mathbb{B}^n$ with $H^\varphi(E) = 0$ such that, for all $\xi \in \partial\mathbb{B}^n \setminus E$, there is a sequence of points $(x_k) \to \xi$ (in the Euclidean sense) so that

$$\xi \in B_\rho\left(x_k, Cr_{x_k}\left(\log\frac{1}{r_{x_k}}\right)^{s-1}\right) \tag{1.2}$$

for all $k \in \mathbb{N}$. Here, $C = C(A, n, s) > 0$. Moreover, the exponent $s-1$ in (1.2) is the best possible.

As a corollary, we obtain the following result on the boundary behavior of (quasi)conformal mappings. We denote by $\delta(y)$ the Euclidean distance of a point $y \in \Omega$ to the boundary $\partial\Omega$.

COROLLARY 1.2. *Let $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ be a K-quasiconformal mapping. Let $s > 1$ and let*

$$\varphi(t) = \exp\left(-\left(\log\frac{1}{t}\right)^{1/s}\right).$$

Then there is a set $E \subset \partial\mathbb{B}^n$ with $H^\varphi(E) = 0$ such that, for all $\xi \in \partial\mathbb{B}^n \setminus E$, there is a sequence of points $(y_k) \to f(\xi)$ in Ω so that

$$f(\xi) \in B\left(y_k, C\delta(y_k)\left(\log\frac{1}{\delta(y_k)}\right)^{s-1}\right) \tag{1.3}$$

for all $k \in \mathbb{N}$. Here, $C = C(K, n, s) > 0$. Moreover, the exponent $s-1$ in (1.3) is the best possible.

Note that if $f : \mathbb{B}^2 \to \Omega \subset \mathbb{R}^2$ in Corollary 1.2 is a conformal mapping, then one can even use the internal metric in the image domain Ω instead of the Euclidean metric, see [2, p. 639].

In the next theorem, we consider exceptional sets of even smaller scale. Again, the result contains new information even in the classical setting of (quasi)conformal mappings of the unit ball.

THEOREM 1.3. *Let $s > n-1$ and let*

$$\varphi(t) = \frac{1}{(\log\frac{1}{t})^s}.$$

Then there is a set $E \subset \partial\mathbb{B}^n$ with $H^\varphi(E) = 0$ such that, for all $\xi \in \partial\mathbb{B}^n \setminus E$, there is a sequence of points $(x_k) \to \xi$ (in the Euclidean sense) so that

$$\xi \in B_\rho(x_k, Cr_{x_k}^\beta) \tag{1.4}$$

for all $k \in \mathbb{N}$. Here, $C = C(A, n, s) > 0$ and $\beta \le \frac{s-(n-1)}{s+1}$.

This result is optimal at least asymptotically: if $s \le n-1$, then the assertion of Theorem 1.3 can fail with any positive exponent β. Moreover, in the case $n = 2$ and $s > 1$, the assertion of Theorem 1.3 can fail if $\beta > (s-1)/s$. See Section 3 for a more detailed discussion on the sharpness of Theorems 1.1 and 1.3.

In this paper, we use the *generalized Hausdorff φ-measure*, denoted by H^φ, to estimate the size of sets. Recall that this measure is defined by

$$H^\varphi(E) = \lim_{r\to 0}\left(\inf\left\{\sum \varphi(\operatorname{diam} B_i) : E \subset \bigcup B_i, \operatorname{diam}(B_i) \le r\right\}\right),$$

where the dimension gauge function φ is required to be continuous and increasing with $\varphi(0) = 0$. In particular, if $\varphi(t) = t^\alpha$ with some $\alpha > 0$, then H^φ

is the usual *α-dimensional Hausdorff measure* denoted also by H^α. See [13] or [3] for more information on the generalized Hausdorff measure.

Throughout the paper, we will assume that the gauge function φ is a doubling weight function satisfying

$$\int_0 \frac{\varphi(t)^{1/(n-1)}}{t}\, dt < \infty. \tag{1.5}$$

This condition turns out to be the critical one for the results of this paper. This is related to the fact that if $H^\varphi(E)=0$ with a dimension gauge φ failing to satisfy (1.5), then E has zero conformal (n-)capacity, see [1].

The Theorems 1.1 and 1.3 are consequences of our more general main theorem formulated below (Theorem 1.5). For the proof of this theorem, we will need the next lemma, which is perhaps of some interest on its own.

LEMMA 1.4. *Let φ be a doubling weight function satisfying (1.5) and let ψ be a function satisfying*

$$\left(\int_0^r \frac{\varphi(s)^{1/(n-1)}}{s} ds\right)^{\frac{n-1}{n}} = O(\psi(r)) \quad \text{as } r \to 0. \tag{1.6}$$

Then there is a set $E \subset \partial\mathbb{B}^n$ with $H^\varphi(E)=0$ such that, for all $\xi \in \partial\mathbb{B}^n \setminus E$, there exists a sequence $(t_k) \to 1$ so that

$$\text{length}_\rho([t_k\xi, \xi)) \le \psi(1-t_k) \tag{1.7}$$

for all $k \in \mathbb{N}$.

In the following, we shall assume in addition to (1.6), that for all sufficiently small $t>0$, $\psi(t) \ge t$ is an increasing and differentiable weight function, so that for $u=\psi^{-1}$,

$$\frac{u(t)}{u'(t)} \text{ is increasing} \tag{1.8}$$

and

$$\log\frac{1}{u(t)} \le c\log\frac{1}{u(2t)} \tag{1.9}$$

with some constant $c>0$ depending only on φ. Note that these technical assumptions are harmless in the sense that in all interesting situations we can choose ψ so that these conditions are satisfied. See for example the proofs of the Theorems 1.1 and 1.3 below. Our main result is the following.

THEOREM 1.5. *Let φ be a doubling weight function satisfying (1.5). Let ψ be a weight function satisfying (1.6) in addition to the technical assumptions (1.8) and (1.9), and denote $u=\psi^{-1}$. Then there is a set $E \subset \partial\mathbb{B}^n$ with $H^\varphi(E)=0$ such that, for all $\xi \in \partial\mathbb{B}^n \setminus E$, there exists a sequence of points $(x_k) \to \xi$ (in the Euclidean sense) so that*

$$\xi \in B_\rho(x_k, C_1\lambda(r_{x_k}))$$

for all $k \in \mathbb{N}$. *Here* λ *is the inverse function of* $C_2 u(t)/u'(t)$ *and* $C_1, C_2 > 0$ *depend only on the given data* A, n *and* φ.

Observe that Theorem 1.5 is sharp at least in the following sense: if the dimension gauge φ fails to satisfy (1.5), then there may exist a set $E \subset \partial\mathbb{B}^n$ so that $H^\varphi(E) > 0$ and $\text{length}_\rho([0,\xi)) = \infty$ for all $\xi \in E$, see [6, Section 3] for an example of such a situation in the plane. This implies by the Gehring–Hayman theorem [2, Theorem 3.1] that the condition (1.5) is crucial for any result of this kind to hold: if it fails, then the assertion of Theorem 1.5 can fail with any (finite) function λ. Consequently, the condition $s > n-1$ in Theorem 1.3 is also critical in this sense.

Let us also point out that if $\varphi(t) = t^\alpha$ with $0 < \alpha \le n-1$, then we can take λ to be a linear function, and thus we recover Theorem A.

2. Proofs of the results

The results of this paper can not be obtained simply by refining the classical proofs. Namely, by extending [2, Theorem 5.2] one can only obtain

$$\rho(t\xi) = o\left(\frac{\varphi(1-t)^{1/n}}{(1-t)}\right) \quad \text{as } t \to 1,$$

for H^φ-a.e. $\xi \in \partial\mathbb{B}^n$, which implies a considerably weaker integrability of ρ on the radii than Lemma 1.4.

Instead, our proof of Lemma 1.4 follows the ideas of [6] and [12] with some modifications. In the proof of Theorem 1.5, we will apply an efficient method of counting Whitney cubes in an averaged sense. A similar technique was used also in [11] as a tool for establishing a sharp dimension estimate for the boundaries of generalized Hölder domains and John domains.

Proof of Lemma 1.4. Let $\mathcal{W}$ be a Whitney decomposition of $\mathbb{B}^n$, i.e., $\mathcal{W}$ is a collection of closed dyadic cubes $Q \subset \mathbb{B}^n$ with pairwise disjoint interiors such that

$$\bigcup_{Q \in \mathcal{W}} Q = \mathbb{B}^n$$

and that $\text{diam}(Q) \le \text{dist}(Q, \partial\mathbb{B}^n) \le 4\,\text{diam}(Q)$. See [14] for the existence of such a decomposition. Further, for a point $\xi \in \partial\mathbb{B}^n$ and a number $i \in \mathbb{N}$ let $\mathcal{W}_i(\xi)$ consist of all the cubes $Q \in \mathcal{W}$ which intersect the radial segment $[(1-2^{-i})\xi, \xi)$. Finally, denote by $\mathcal{W}_i$ the ith generation of Whitney cubes, i.e. all the cubes $Q \in \mathcal{W}$ with side length 2^{-i}.

Let us write $E_\infty = \{\xi \in \partial\mathbb{B}^n : \text{length}_\rho([0,\xi)) = \infty\}$. Then $\text{cap}_n(E_\infty) = 0$ and, moreover, $H^\varphi(E_\infty) = 0$ because of the condition (1.5), see e.g., [12, Remark 1.3].

For $j,k\in\mathbb{N}$, define $G_j=\{\xi\in\partial\mathbb{B}^n:\ \mathrm{length}_\rho([0,\xi))\le j\}$ and

$$F_j^k=\bigcup_{\xi\in G_j}\bigcup\{Q\in\mathcal{W}_0(\xi):\ \mathrm{diam}(Q)\le 2^{-k}\}$$

and write $F_j=F_j^0$. Then F_j is open and $\mathrm{diam}_\rho(F_j)<\infty$ by the Harnack inequality. Thus, also $\mu_\rho(F_j)<\infty$ by the volume growth condition, and moreover,

$$\mu_\rho(F_j^k)\to 0\quad\text{as } k\to\infty. \tag{2.1}$$

Let E consist of all the points $\xi\in\partial\mathbb{B}^n$ for which the assertion (1.7) fails and write $E_j=E\cap G_j$. Thus,

$$E_j=\{\xi\in G_j:\ \mathrm{length}_\rho([t\xi,\xi))>\psi(1-t)\text{ for all } t\ge t_\xi\},$$

where $t_\xi<1$ depends on the point ξ. Then define for each $k\in\mathbb{N}$ a set

$$E_j^k=\{\xi\in G_j:\ \mathrm{length}_\rho([t\xi,\xi))>\psi(1-t)\text{ for all } t\ge 1-2^{-k}\}.$$

Observe that $E_j^1\subset E_j^2\subset E_j^3\subset\cdots$, and $E_j=\bigcup_k E_j^k$. Also note that $E=E_\infty\cup\bigcup_j E_j$ and hence, by the subadditivity of the Hausdorff measure, it suffices to show that $H^\varphi(E_j)=0$ for all $j\in\mathbb{N}$ in order to prove the theorem.

Fix $j\in\mathbb{N}$. Let us assume towards a contradiction that $H^\varphi(E_j)>0$. Then, by the subadditivity of the Hausdorff measure, $H^\varphi(E_j^{k_0})>0$ for some $k_0\in\mathbb{N}$. Thus $H^\varphi(E_j^k)>0$ for all $k\ge k_0$ since $E_j^{k_0}\subset E_j^k$. Hence, by Frostman's lemma [9, Theorem 8.8], for each $k\ge k_0$ there exists a Radon measure ν supported in E_j^k so that $\nu(B(x,r))\le\varphi(r)$ for all $x\in\partial\mathbb{B}^n$ and $r>0$ and that

$$\nu(E_j^k)\ge CH_\infty^\varphi(E_j^k)\ge CH_\infty^\varphi(E_j^{k_0})>0. \tag{2.2}$$

Here $H_\infty^\varphi(E_j^k)=\inf\{\sum_i\varphi(\mathrm{diam}(B_i)):\ E_j^k\subset\bigcup_i B_i\}$ is the usual Hausdorff φ-content of E_j^k and the constant $C>0$ depends only on n.

Let us define $u_j(x)=\rho(x)^n$ for $x\in F_j$ and $u_j(x)=0$ elsewhere. We denote by $S(Q)$ the "shadow" of a cube $Q\in\mathcal{W}$, i.e., $S(Q)$ consists of all points $\xi\in\partial\mathbb{B}^n$ for which the radius $[0,\xi)$ intersects the cube Q. Since $\mathrm{length}_\rho([(1-2^{-k})\xi,\xi))>\psi(2^{-k})$ for all $\xi\in E_j^k$, we deduce by the inequalities of Harnack and Hölder that

$$\begin{aligned}\nu(E_j^k)\psi(2^{-k})&<\int_{\partial\mathbb{B}^n}\mathrm{length}_\rho\big([(1-2^{-k})x,x)\big)\,d_\nu x\\&\le\int_{\partial\mathbb{B}^n}\sum_{Q\in\mathcal{W}_k(x)}\mathrm{diam}_\rho(Q)\,d_\nu x\\&\le\sum_{\{Q\in\mathcal{W}_i:i\ge k\}}\nu(S(Q))\,\mathrm{diam}_\rho(Q)\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&\le c_0 \sum_{\{Q\in\mathcal{W}_i : i\ge k\}} \nu(S(Q))\Big(\int_Q \rho^n\,dm\Big)^{1/n}\\
&\le c_0\Big(\sum_{\{Q\in\mathcal{W}_i : i\ge k\}} \int_Q u_j\,dm\Big)^{1/n}\\
&\quad\times\Big(\sum_{\{Q\in\mathcal{W}_i : i\ge k\}} \nu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}\\
&\le c_0\mu_\rho(F_j^k)^{1/n}\Big(\sum_{i\ge k}\sum_{Q\in\mathcal{W}_i} \nu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.
\end{aligned}
$$

Here and throughout the proof, we denote by c_i positive constants depending at most on A, n and the doubling constant of φ.

On the other hand, we have that

$$
\begin{aligned}
(2.4)\qquad &\Big(\sum_{i\ge k}\sum_{Q\in\mathcal{W}_i} \nu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}\\
&\le \Big(\sum_{i\ge k}\max_{Q\in\mathcal{W}_i} \nu(S(Q))^{\frac{1}{n-1}}\sum_{Q\in\mathcal{W}_i}\nu(S(Q))\Big)^{\frac{n-1}{n}}\\
&\le c_1\Big(\sum_{i\ge k}\max_{Q\in\mathcal{W}_i} \nu(S(Q))^{\frac{1}{n-1}}\nu(E_j^k)\Big)^{\frac{n-1}{n}}.
\end{aligned}
$$

Moreover, since $\nu(S(Q))\le\varphi(\operatorname{diam}(S(Q)))$ and $\operatorname{diam}(S(Q))\le C2^{-i}$ for each $Q\in\mathcal{W}_i$ with some constant $C>0$ depending only on n, it follows that

$$\nu(S(Q))\le\varphi(C2^{-i})\le c_2\varphi(2^{-i})$$

for all cubes $Q\in\mathcal{W}_i$, where the last inequality follows from the doubling condition of φ. By combining this with (2.3) and (2.4), we obtain

$$
\begin{aligned}
(2.5)\qquad \nu(E_j^k)^{1/n}\psi(2^{-k}) &\le c_3\mu_\rho(F_j^k)^{1/n}\Big(\sum_{i\ge k}\varphi(2^{-i})^{\frac{1}{n-1}}\Big)^{\frac{n-1}{n}}\\
&\le c_4\mu_\rho(F_j^k)^{1/n}\Big(\int_0^{2^{-k}}\frac{\varphi(s)^{\frac{1}{n-1}}}{s}\,ds\Big)^{\frac{n-1}{n}}.
\end{aligned}
$$

We now conclude by the estimates (2.5), (2.1), and the assumption (1.6) that $\nu(E_j^k)$ tends to zero as k tends to infinity, but this is a contradiction with (2.2). It follows that $H^\varphi(E_j)=0$ and thus also $H^\varphi(E)=0$ by the subadditivity of the Hausdorff measure. □

Proof of Theorem 1.5. Let $\mathcal{W}$ be a Whitney decomposition of $\mathbb{B}^n$. Let $E\subset\partial\mathbb{B}^n$ be as in Lemma 1.4 and let $\xi\in\partial\mathbb{B}^n\setminus E$. For an integer $i\in\mathbb{N}$, denote by $\gamma_i(\xi)$ the line segment $[a\xi, b\xi)$, where $\text{length}_\rho([a\xi,\xi)) = 2^{-i+1}$ and $\text{length}_\rho([b\xi,\xi)) = 2^{-i}$. Then define $\chi_i(\xi) = 1$ if there exist at most

$$\frac{2^{-i}u'(2^{-i})}{c_1 u(2^{-i})}.$$

Whitney cubes $Q \in \mathcal{W}$ intersecting the line segment $\gamma_i(\xi)$, and $\chi_i(\xi) = 0$ otherwise.

We show first that with a small enough constant $c_1 > 0$ there is an increasing sequence of integers $(i_k) \to \infty$ such that $\chi_{i_k}(\xi) = 1$ for all $k \in \mathbb{N}$. To that end, suppose that this assertion fails. Thus, $\chi_i(\xi) = 0$ for all $i \geq j_0$ with some integer j_0.

Recall that the quasihyperbolic distance $k_{\mathbb{B}^n}(x_0, x_1)$ between two points $x_0, x_1 \in \mathbb{B}^n$ is defined by

$$\inf_\gamma \int_\gamma \frac{ds}{d(x, \partial\mathbb{B}^n)},$$

where the infimum is taken over all rectifiable curves joining x_0 to x_1 in $\mathbb{B}^n$. Notice that, for $x_0 = 0$ and $x_1 = t\xi$ sufficiently close to the boundary, the quasihyperbolic distance $k_{\mathbb{B}^n}(0, t\xi) = \log\frac{1}{1-t}$ is comparable to the number of Whitney cubes intersecting the line segment $[0, t\xi]$. Hence, for sufficiently large $j \in \mathbb{N}$ and $t \in \gamma_j(\xi)$, we have that

$$\log\frac{1}{1-t} \geq C \sum_{i=j_0}^{j-1} \left(\frac{2^{-i}u'(2^{-i})}{c_1 u(2^{-i})} - 1 \right). \tag{2.6}$$

On the other hand, by Lemma 1.4, we know that there is a sequence $(t_k) \to 1$ so that

$$\text{length}_\rho([t_k\xi, \xi)) \leq \psi(1 - t_k)$$

for all $k \in \mathbb{N}$. This implies that

$$2^{-j} \leq \psi(1 - t_k)$$

or equivalently

$$u(2^{-j}) \leq 1 - t_k \tag{2.7}$$

for $t_k \in \gamma_j(\xi)$. By combining (2.6) and (2.7), we obtain the following chain of inequalities for an arbitrarily large integer j:

$$\begin{aligned} \log\frac{1}{u(2^{-j})} &\geq C \sum_{i=j_0}^{j-1} \left(\frac{2^{-i}u'(2^{-i})}{c_1 u(2^{-i})} - 1 \right) \\ &\geq \frac{C}{c_1} \int_{2^{-j+1}}^{2^{-j_0}} \frac{u'(t)}{u(t)}\, dt - j \\ &\geq \frac{C}{c_1} \left(\log\frac{1}{u(2^{-j+1})} - \log\frac{1}{u(2^{-j_0})} \right) - j. \end{aligned} \tag{2.8}$$

But since $\varphi(t) \geq t$ for small t, then $t \geq u(t)$ for $u = \varphi^{-1}$ and it follows that

$$j \leq C \log \frac{1}{u(2^{-j})}$$

for sufficiently large j. Hence, by the assumption (1.9), the inequality (2.8) is a contradiction when we choose j large enough and the constant $c_1 > 0$ small enough depending on φ and n. Thus, we conclude that there is an increasing sequence of integers $(i_k) \to \infty$ so that $\chi_{i_k}(\xi) = 1$ for all $k \in \mathbb{N}$.

Let us then consider a line segment γ_k with $\chi_k(\xi) = 1$. We deduce that since $\text{length}_\rho(\gamma_k(\xi)) = 2^{-k}$ and there are no more than $\frac{2^{-k} u'(2^{-k})}{c_1 u(2^{-k})}$ Whitney cubes intersecting the segment $\gamma_k(\xi)$, some of these cubes must have a large ρ-diameter. More precisely, denote by $\mathcal{W}_k(\xi)$ all the Whitney cubes intersecting $\gamma_k(\xi)$ and observe that if all the cubes $Q \in \mathcal{W}_k(\xi)$ satisfy

$$\text{diam}_\rho(Q) < \frac{c_1 u(2^{-k})}{u'(2^{-k})},$$

then

$$\text{length}_\rho(\gamma_k(\xi)) \leq \sum_{Q \in \mathcal{W}_k(\xi)} \text{diam}_\rho(Q) < \frac{2^{-k} u'(2^{-k})}{c_1 u(2^{-k})} \cdot \frac{c_1 u(2^{-k})}{u'(2^{-k})} = 2^{-k},$$

which is a contradiction. Therefore, there is at least one cube $Q_k \in \mathcal{W}_k(\xi)$ satisfying

$$\text{diam}_\rho(Q_k) \geq \frac{c_1 u(2^{-k})}{u'(2^{-k})}.$$

By the Harnack inequality, we know that $\text{diam}_\rho(Q_k)$ is comparable to r_{x_k}, where x_k is the center of Q_k. Thus,

$$r_{x_k} \geq \frac{c_2 u(2^{-k})}{u'(2^{-k})}$$

with $c_2 > 0$ depending only on A, n and φ. Hence by choosing $C_2 = c_2$, we obtain

$$\lambda(r_{x_k}) \geq 2^{-k},$$

because λ is increasing by (1.8). It follows from the Harnack inequality that

$$\xi \in B_\rho(x_k, C_1 \lambda(r_{x_k}))$$

when $C_1 > 0$ is chosen large enough depending only on A and n. Clearly $(x_k) \to \xi$ in the Euclidean sense, and thus the proof is complete. □

REMARK 2.1. Note that the only place in the proofs, where we used the volume growth condition, was in the beginning of the proof of Lemma 1.4, where we deduced that $H^\varphi(E_\infty) = 0$ and $\mu_\rho(F_j) < \infty$. Very recently it was shown that these are true even with a relaxed volume growth assumption [12]. Hence, by applying the results of [12], one can show that Lemma 1.4 and

Theorem 1.5 also hold with a weaker volume growth condition depending on the dimension gauge φ.

Proof of Theorem 1.1. For the convenience of the reader, we shall write the detailed calculations. Here, we denote by c_i positive constants depending at most on A, n, and s. Notice that φ satisfies the condition (1.5), and we may take

$$\psi(t) = \exp\left(-\frac{1}{2n}\left(\log\frac{1}{t}\right)^{1/s}\right)$$

for all sufficiently small $t > 0$, whence ψ is increasing and differentiable for all small t and it also satisfies the condition (1.6). Then we have that

$$u(t) = \psi^{-1}(t) = \exp\left(-c_1\left(\log\frac{1}{t}\right)^{s}\right)$$

and

$$\frac{u(t)}{u'(t)} = c_2\frac{t}{(\log\frac{1}{t})^{s-1}},$$

and thus the conditions (1.8) and (1.9) are also satisfied. The inverse of $C_2u(t)/u'(t)$ at r is at most

$$\lambda(r) = c_3 r\left(\log\frac{1}{r}\right)^{s-1}$$

for all sufficiently small $r > 0$. The claim now follows by Theorem 1.5. The second part of the theorem (the sharpness of the exponent $s-1$) follows from Theorem 3.1 below. □

Proof of Theorem 1.3. Notice that we may take

$$\begin{aligned}\psi(t) &= \left(\int_0^t \frac{\varphi(r)^{1/(n-1)}}{r}\,dr\right)^{\frac{n-1}{n}}\\ &= \left(\int_0^t \frac{1}{r(\log\frac{1}{r})^{\frac{s}{n-1}}}\,dr\right)^{\frac{n-1}{n}}\\ &= c_1\left(\log\frac{1}{t}\right)^{-\frac{s-(n-1)}{n}}\end{aligned}$$

for all sufficiently small $t > 0$, whence ψ is increasing and differentiable for all small t and it obviously satisfies the condition (1.6). Moreover,

$$u(t) = \psi^{-1}(t) = \exp\left(-c_2 t^{-\frac{n}{s-(n-1)}}\right)$$

and

$$\frac{u(t)}{u'(t)} = c_3 t^{\frac{s+1}{s-(n-1)}},$$

and thus the conditions (1.8) and (1.9) are satisfied. Hence, by Theorem 1.5, we can take λ to be the inverse of $C_2u(t)/u'(t)$ or

$$\lambda(r) = c_4 r^{\frac{s-(n-1)}{s+1}}.$$

□

3. Sharpness of the results

In this section, we show the essential sharpness of the Theorems 1.1 and 1.3 in the plane. Recall that if $f : \mathbb{B}^2 \to f(\mathbb{B}^2) = \Omega \subset \mathbb{R}^2$ is a conformal mapping, then $\rho(x) = |f'(x)|$ is a continuous density satisfying the Harnack inequality and the volume growth condition, see [2, p. 639]. In this case, d_ρ corresponds to the internal Euclidean metric in the image domain Ω. Moreover, the quantity $r_z = \rho(z)(1-|z|)$ for a point $z \in \mathbb{B}^2$ is comparable to $\operatorname{dist}(f(z), \partial\Omega)$ by an absolute constant. Hence, it suffices for us to give an example of a conformal mapping f, which maps a set $E \subset \partial\mathbb{B}^2$ of positive φ-measure to a "sufficiently inaccessible" set on the boundary of Ω.

More precisely, we prove the following theorems.

Theorem 3.1. *Let $s > 1$ and let*

$$\varphi(t) = \exp\left(-\left(\log\frac{1}{t}\right)^{1/s}\right).$$

There exists a set $E \subset \partial\mathbb{B}^2$ and a conformal mapping $f : \mathbb{B}^2 \to \Omega \subset \mathbb{R}^2$ so that $H^\varphi(E) > 0$ and for any $\beta < s-1$ and $C > 0$ we have for all $\xi \in E$ that

$$f(\xi) \notin B\left(y, C\operatorname{dist}(y,\partial\Omega)\left(\log\frac{1}{\operatorname{dist}(y,\partial\Omega)}\right)^\beta\right)$$

for all $y \in \Omega$ sufficiently close to the radial limit $f(\xi) \in \partial\Omega$.

Theorem 3.2. *Let $s > 1$ and let*

$$\varphi(t) = \frac{1}{(\log\frac{1}{t})^s}.$$

There exists a set $E \subset \partial\mathbb{B}^2$ and a conformal mapping $f : \mathbb{B}^2 \to \Omega \subset \mathbb{R}^2$ so that $H^\varphi(E) > 0$ and for any $\beta > \frac{s-1}{s}$ and $C > 0$ we have for all $\xi \in E$ that

$$f(\xi) \notin B(y, C\operatorname{dist}(y,\partial\Omega)^\beta) \tag{3.1}$$

for all $y \in \Omega$ sufficiently close to the radial limit $f(\xi) \in \partial\Omega$.

Proof of Theorem 3.1. Let us first construct a simply connected domain $\Omega \subset \mathbb{R}^2$ in the following way. Let $c < 1$ and set $\alpha(0) = \frac{c}{2}$ and

$$\alpha(i) = \min\left\{ci^{s-1}2^{-i}, \frac{c}{2}\right\}$$

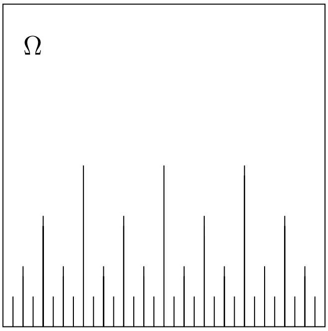

FIGURE 1.

for $i \in \mathbb{N}$. Starting with the open unit square $\Omega_0 = (0,1)^2$, remove a closed vertical line segment T_{01} of length $\alpha(0)$ standing at the point $(2^{-1}, 0)$. We set $\Omega_1 = \Omega_0 \setminus T_{01}$. We then iterate this process: given a domain Ω_i for $i \in \mathbb{N}$, remove 2^i closed vertical line segments T_{ik}, $k = 1, \dots, 2^i$, of length $\alpha(i)$ so that T_{ik} stands at the point $(2^{-i-1} + (k-1)2^{-i}, 0)$. We define

$$\Omega_{i+1} = \Omega_i \Big\backslash \bigcup_{k=1}^{2^i} T_{ik}$$

and

$$\Omega = \bigcap_{i=1}^{\infty} \Omega_i .$$

Then Ω is a simply connected domain and there exists a conformal mapping $f : \mathbb{B}^2 \to \Omega$. See Figure 1 for an illustration of the domain Ω.

Let $\beta < s - 1$, $C > 0$ and choose $x_0 = (\frac{1}{2}, \frac{3}{4})$. Observe that every point $x \in (0,1) \times \{0\}$ belongs to the boundary of Ω and also the internal distance between x and x_0 is finite. Moreover,

$$x \notin B\left(y, C \operatorname{dist}(y, \partial\Omega) \left(\log \frac{1}{\operatorname{dist}(y, \partial\Omega)}\right)^{\beta}\right)$$

for all $y \in \Omega$ sufficiently close to x. Thus, it only remains to estimate the size of the set $E \subset \partial\mathbb{B}^2$ of points ξ for which the radial limit $f(\xi)$ belongs to the segment $(0,1) \times \{0\}$.

Denote by k_Ω the quasihyperbolic metric in Ω. A straightforward calculation shows that Ω satisfies the growth condition

$$k_\Omega(x, x_0) \le C_1 \left(\log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)}\right)^s \tag{3.2}$$

for all $x \in \Omega$ sufficiently close to the boundary, where C_1 depends only on c and s. In particular, we can make C_1 arbitrarily small by choosing c

small enough in the construction of Ω. Now, by [5, Theorem 1.2], we know that f is uniformly continuous with a modulus of continuity $\psi(t) = C_2\exp(-C_3(\log\frac{1}{t})^{1/s})$. Here $C_3 = C_4C_1^{-1/s}$, and hence, we can take $C_3 = 1$ by choosing c small enough in the construction of Ω. Thus, we have that

$$|f(x) - f(y)| \leq C_5 \exp\left(-\left(\log\frac{1}{|x-y|}\right)^{1/s}\right) = C_5\varphi(|x-y|) \tag{3.3}$$

for all $x, y \in \mathbb{B}^2$ sufficiently close to each other.

Observe that since the internal diameter of Ω is finite, the radial limit of f exists for all points $\xi \in \partial\mathbb{B}^2$. This follows from the Gehring–Hayman theorem (cf. [2, Remark 4.5]). Let E consist of those points $\xi \in \partial\mathbb{B}^2$ for which the radial limit $f(\xi)$ belongs to the segment $(0,1)\times\{0\}$. Suppose that $H^\varphi(E) = 0$. Then for any $\varepsilon > 0$ there is a collection of balls B_i such that $E \subset \bigcup_i B_i$ and $\sum_i \varphi(\operatorname{diam}(B_i)) < \varepsilon/C_5$. But now the union $\bigcup_i f(B_i)$ covers the segment $(0,1)\times\{0\}$ and the diameter of $f(B_i)$ is at most $C_5\varphi(\operatorname{diam}(B_i))$ by the inequality (3.3). Hence $H^1((0,1)\times\{0\}) \leq \sum_i C_5\varphi(\operatorname{diam}(B_i)) < \varepsilon$, but this is a contradiction. It follows that $H^\varphi(E) > 0$ and the proof is complete. □

Proof of Theorem 3.2. The proof is similar to the one of Theorem 3.1, but the situation is more delicate. Namely, the modulus of continuity of f implied by [5, Theorem 1.2] is no longer good enough. Indeed, we must equip Ω with the internal metric instead of the Euclidean metric and use [7, Theorem 1.1] in order to obtain the asymptotically sharp estimate of Theorem 3.2.

In the construction of Ω, we now choose $p = \frac{s-1}{s}$ and

$$\alpha(i) = \min\left\{c2^{-pi}, \frac{c}{2}\right\}$$

for $i \in \mathbb{N}$. Then we choose $\beta > p$. It follows that any $f(\xi)$ on the line segment $(0,1)\times\{0\}$ satisfies (3.1) for all $y \in \Omega$ sufficiently close to $f(\xi)$.

On the other hand, Ω now satisfies the growth condition

$$k_\Omega(x, x_0) \leq C_1\left(\frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)}\right)^{1-p}$$

for all $x \in \Omega$ sufficiently close to the boundary with a constant $C_1 > 0$ depending on c and s. By [7, Theorem 1.1], this implies that

$$\delta_\Omega(f(x), f(y)) \leq C_2\left(\log\frac{1}{|x-y|}\right)^{-\frac{p}{1-p}} = C_2\varphi(|x-y|)^p \tag{3.4}$$

for all $x, y \in \mathbb{B}^2$ sufficiently close to each other. Here $\delta_\Omega(f(x), f(y))$ denotes the internal distance of $f(x)$ and $f(y)$ in Ω, i.e., the infimum of the lengths of curves in Ω joining $f(x)$ and $f(y)$. The assertion $H^\varphi(E) > 0$ now follows essentially as in the proof of Theorem 3.1 above. However, one needs to use (3.4) to estimate the internal diameter of the sets $f(B_i)$ in Ω. The claim then follows by observing that the internal Hausdorff dimension (i.e., the Hausdorff dimension with respect to the metric δ_Ω) of the set $(0,1) \times \{0\}$ is at least $1/p$. □

Acknowledgment. The author is grateful to Professor Pekka Koskela for useful discussions.

References

[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Springer, Berlin, 1996. MR 1411441

[2] M. Bonk, P. Koskela and S. Rohde, *Conformal metrics on the unit ball in euclidean space*, Proc. London Math. Soc. (3) **77** (1998), 635–664. MR 1643421

[3] H. Federer, *Geometric measure theory*, Springer, New York, 1969. MR 0257325

[4] T. E. Gerasch, *On the accessibility of the boundary of a simply connected domain*, Michigan Math. J. **33** (1986), 201–207. MR 0837578

[5] S. Hencl and P. Koskela, *Quasihyperbolic boundary conditions and capacity: Uniform continuity of quasiconformal mappings*, J. Anal. Math. **96** (2005), 19–35. MR 2177180

[6] P. Koskela and T. Nieminen, *Homeomorphisms of finite distortion: Discrete length of radial images*, Math. Proc. Cambridge Philos. Soc. **144** (2008), 197–205. MR 2388241

[7] P. Koskela and T. Nieminen, *Uniform continuity of quasiconformal mappings and conformal deformations*, Conform. Geom. Dyn. **12** (2008), 10–17. MR 2372760

[8] P. Koskela and S. Rohde, *Hausdorff dimension and mean porosity*, Math. Ann. **309** (1997), 593–609. MR 1483825

[9] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Univ. Press, 1995. MR 1333890

[10] O. Martio and R. Näkki, *Boundary accessibility of a domain quasiconformally equivalent to a ball*, Bull. London. Math. Soc. **36** (2004), 114–118. MR 2011985

[11] T. Nieminen, *Generalized mean porosity and dimension*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 143–172. MR 2210114

[12] T. Nieminen and T. Tossavainen, *Boundary behavior of conformal deformations*, Conform. Geom. Dyn. **11** (2007), 56–64. MR 2314242

[13] C. A. Rogers, *Hausdorff measures*, Cambridge Univ. Press, 1970. MR 0281862

[14] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970. MR 0290095

Tomi Nieminen, Department of Mathematics and Statistics, P.O. Box 35, FIN-40014 University of Jyväskylä, Finland

E-mail address: tominiem@maths.jyu.fi