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SOCLE DEGREES OF FROBENIUS POWERS

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In honor of Phillip Griffith, on the occasion of his retirement

ABSTRACT. Let k be a field of positive characteristic p, R be a Gorenstein graded k-algebra, and $S = R/J$ be an artinian quotient of R by a homogeneous ideal. We ask how the socle degrees of S are related to the socle degrees of $F^e_R(S) = R/J^{[q]}$. If S has finite projective dimension as an R-module, then the socles of S and $F^e_R(S)$ have the same dimension and the socle degrees are related by the formula $D_i = qd_i - (q-1)a(R)$, where $d_1 \le \cdots \le d_\ell$ and $D_1 \le \cdots \le D_\ell$ are the socle degrees of S and $F^e_R(S)$, respectively, and $a(R)$ is the a-invariant of the graded ring R, as introduced by Goto and Watanabe. We prove the converse when R is a complete intersection.

Let $(R, \mathfrak{m})$ be a Noetherian graded algebra over a field of positive characteristic p, with irrelevant ideal $\mathfrak{m}$. We usually let $R = P/C$ with P a polynomial ring, and C a homogeneous ideal. Let J be an $\mathfrak{m}$-primary homogeneous ideal in R. Recall that if $q = p^e$, then the e^{th} Frobenius power of J is the ideal $J^{[q]}$ generated by all i^q with $i \in J$. The basic question is:

QUESTION. *How do the degrees of the minimal generators of $(J^{[q]} : \mathfrak{m})/J^{[q]}$ vary with q?*

The largest of the degrees of a generator of the socle $(J : \mathfrak{m})/J$ will be called the *top socle degree* of R/J. The question of finding a linear bound for the top socle degree of $R/J^{[q]}$ has been considered by Brenner in [3] from a different point of view; his main motivation there is finding inclusion-exclusion criteria for tight closure.

The answer to the Question is well-known (although not explicitly stated in the existing literature) in the case when J has finite projective dimension; see Observation 2.1. We prove that the converse holds when $R = P/C$ is a complete intersection.

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Theorem A. *Let k be a field of positive characteristic p, $q = p^e$ for some positive integer e, P be a positively graded polynomial ring over k, and $R = P/C$ be a complete intersection ring with C generated by a homogeneous regular sequence. Let $\mathfrak{m}$ be the maximal homogeneous ideal of R, J be a homogeneous $\mathfrak{m}$-primary ideal in R, and I be a lifting of J to P. Let ℓ be the dimension of the socle $(J:\mathfrak{m})/J$ of R/J and $d_1, \ldots, d_\ell$ be the degrees of the generators of the socle. Then the following statements are equivalent:*

(a) $\operatorname{pd}_R R/J < \infty$.
(b) *The socle $(J^{[q]}:\mathfrak{m})/J^{[q]}$ of $R/J^{[q]}$ has dimension ℓ and the degrees of the generators are $qd_i - (q-1)a(R)$, for $1 \le i \le \ell$, where $a(R)$ is the a-invariant of R.*
(c) $(C+I)^{[q]} : (C^{[q]} : C) = C + I^{[q]}$.
(d) $I^{[q]} \cap C = (I \cap C)^{[q]} + CI^{[q]}$.

Of course, the following general question remains wide open and very compelling:

Question. *How do the socle degrees of Frobenius powers $J^{[q]}$ encode homological information about the ideals $J^{[q]}$?*

The proof of Theorem A appears in Section 2.

1. Preliminary notions

In this paper, ring means commutative noetherian ring with one. Let k be a field of positive characteristic p. We say that the ring R is a *graded k-algebra* if

(1.1) R is non-negatively graded, $R_0 = k$, and R is finitely generated as a ring over k.

Every ring that we study in this paper is a graded k-algebra. In particular, "Let P be a polynomial ring" means $P = k[x_1, \ldots, x_n]$, for some n, and each variable has positive degree. Every calculation in this paper is homogeneous: all elements and ideals that we consider are homogeneous, all ring or module homomorphisms that we consider are homogeneous of degree zero. If r is a homogeneous element of the ring R, then $|r|$ is the degree of r. The graded k-algebra R has a unique homogeneous maximal ideal

$$\mathfrak{m} = \mathfrak{m}_R = R_+ = \bigoplus_{i>0} R_i;$$

furthermore, R has a unique graded canonical module K_R, which is equal to the graded dual of the graded local cohomology module $\mathrm{H}^d_{\mathfrak{m}}(R)$, where d is the Krull dimension of R; that is,

$$K_R = \operatorname{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(R), E_R),$$

for $E_R = \operatorname{Hom}_k(R, R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$ as a graded R-module. (See, for example, [7, Def. 2.1.2].) The a-invariant of R is defined to be

$$a(R) = -\min\{m \mid (K_R)_m \neq 0\} = \max\{m \mid (\mathrm{H}^d_{\mathfrak{m}}(R))_m \neq 0\}.$$

The definition of the a-invariant is rigged so that if R is a Gorenstein graded k-algebra, then $K_R = R(a(R))$. When the ring R is Cohen-Macaulay, there are many ways to compute $a(R)$. The main tool for these calculations, Proposition 1.2 below, may be found as Proposition 2.2.9 in [7] or Proposition 3.6.12 in [4].

Proposition 1.2. *If $R \to S$ is a graded surjection of graded k-algebras, and R is Cohen-Macaulay, then*

$$K_S = \operatorname{Ext}^c_R(S, K_R),$$

where $c = \dim R - \dim S$. In particular, if $S = R/C$ and the ideal C is generated by the homogeneous regular sequence $f_1, \dots, f_c$, then

$$K_{R/C} = (K_R/CK_R)\left(\sum_{i=1}^{c} |f_i|\right).$$

Corollary 1.3.

(a) *If P is the polynomial ring $k[x_1, \dots, x_n]$, then $a(P) = -\sum_{i=1}^n |x_i|$.*

(b) *If R is the complete intersection ring P/C, where P is the polynomial ring $k[x_1, \dots, x_n]$ and C is the ideal in P generated by the homogeneous regular sequence $f_1, \dots, f_c$, then $a(R) = \sum_{i=1}^c |f_i| - \sum_{i=1}^n |x_i|$.*

(c) *If $R \to S$ is a surjection of graded Cohen-Macaulay k-algebras, and S has finite projective dimension as an R-module, then $a(S) = a(R) + N$, where N is the largest back twist in the minimal homogeneous resolution of S by free R-modules. In other words, if*

$$0 \to \bigoplus_i R(-b_{c,i}) \to \cdots \to \bigoplus_i R(-b_{1,i}) \to R \to S \to 0$$

is the minimal homogeneous resolution of S by free R-modules, then $N = \max_i\{b_{c,i}\}$.

Definition. If S is an artinian graded k-algebra, then the socle of S,

$$\operatorname{soc} S = 0 :\mathfrak{m}_S = \{s \in S \mid s\mathfrak{m}_S = 0\},$$

is a finite dimensional graded k-vector space: $\operatorname{soc} S = \bigoplus_{i=1}^{\ell} k(-d_i)$. We refer to the numbers $d_1 \leq d_2 \leq \cdots \leq d_\ell$ as the *socle degrees of* S.

Observation 1.4. *Let R be an artinian Gorenstein graded k-algebra with socle degree δ, and let J be a homogeneous ideal of R. If the socle degrees of R/J are $\{d_i\}$, then the minimal generators of* $\operatorname{ann} J$ *have degrees $\{\delta - d_i\}$.*

Proof. Choose minimal generators $g_1, \dots, g_s$ of $\operatorname{ann} J$. Gorenstein duality (see Lemma 1.11) implies that

$$\operatorname{ann}(g_1, \dots, \hat{g}_i, \dots, g_s) \not\subseteq \operatorname{ann}(g_i);$$

and thus, for each i, we can choose an element $u_i \in \operatorname{ann}(g_1, \dots, \hat{g}_i, \dots, g_s)$, which represents a generator for the socle of $R/\operatorname{ann}(g_i)$. The ideals J and $\operatorname{ann}(g_1, \dots, g_s)$ are equal and the socle of $R/\operatorname{ann}(g_1, \dots, g_s)$ is minimally generated by $u_1, \dots, u_s$. On the other hand, $u_i g_i$ generates the socle of R, so the degree of u_i is equal to $\delta - |g_i|$. □

Proposition 1.5. *If S is an artinian graded k-algebra and $d_1 \le \dots \le d_\ell$ are the socle degrees of S, then the minimal generators of the canonical module K_S have degrees $-d_\ell \le \dots \le -d_1$.*

Proof. Let $P = k[x_1, \dots, x_n]$ be a polynomial ring which maps onto S. One may compute the degrees of the generators of K_S as well the socle degrees of S in terms of the back twists in the minimal homogeneous resolution of S as a P-module:

$$0 \to \bigoplus_i P(-b_{n,i}) \to \dots \to \bigoplus_i P(-b_{1,i}) \to P \to S \to 0.$$

The canonical module K_S is equal to $\operatorname{Ext}^n_P(S, K_P)$, where $K_P = P(a(P))$ and $a(P) = -\sum_{i=1}^n |x_i|$. It follows that the minimal homogeneous resolution of K_S is

$$0 \to P(a(P)) \to \dots \to \bigoplus_i P(a(P) + b_{n,i}) \to K_S \to 0; \tag{1.6}$$

therefore, the minimal generators of K_S (over either S or P) have degrees $\{-a(P) - b_{n,i}\}$. On the other hand, one may compute $\operatorname{Tor}^P_n(S, P/\mathfrak{m}_P)$ in each coordinate (see, for example, [9, Lemma 1.3]) in order to conclude that

$$\bigoplus_i k(-b_{n,i}) = \operatorname{Tor}^P_n(S, k) = \operatorname{soc} S(a(P)).$$

Thus, the socle degrees of S are equal to $\{a(P) + b_{n,i}\}$. □

Corollary 1.7. *Let $R \to S$ be a surjection of graded k-algebras with S artinian, and R Gorenstein. If $\operatorname{pd}_R S$ finite, then the socle degrees of S are $\{b_i + a(R)\}$, where the back twists in the minimal homogeneous resolution of S by free R-modules are $\{b_i\}$.*

Proof. We know, from Proposition 1.2, that $K_S = \operatorname{Ext}^{\dim R}_R(S, K_R)$, with $K_R = R(a(R))$; therefore,

$$0 \to R(a(R)) \to \dots \to \bigoplus_i R(a(R) + b_i) \to K_S \to 0$$

is a minimal resolution of K_S and the minimal generators of K_S as an R-module, or as an S-module, have degrees $\{-a(R) - b_i\}$. Apply Proposition 1.5. $\square$

Let R be a graded k-algebra. We write ${}^eR$ to represent the ring R endowed with an R-module structure given by the e^{th} iteration of the Frobenius endomorphism $\phi_R : R \to R$. (If r is a scalar in R and s is a ring element in ${}^eR$, then $r \cdot s$ is equal to $r^q s \in {}^eR$, for $q = p^e$.) The Frobenius functor $F_R^e(\_\_) = \_\_ \otimes_R {}^eR$ is base change along the homomorphism ϕ_R^e.

Notation 1.8. Let R be a graded k-algebra. We use the notation $(\ \ )^{[q]}$ in three ways.

(a) If $\boldsymbol{g}$ is a matrix with entries in R, then $\boldsymbol{g}^{[q]}$ is the matrix in which each entry of $\boldsymbol{g}$ is raised to the power q. In particular, if z is an element of the free module $\bigoplus_{i=1}^m R(-b_i)$, then z is an $m \times 1$ matrix and $z^{[q]}$ is the matrix in which each entry of z is raised to the power q.

(b) If $\mathbb{G}_1$ is the free module $\bigoplus_{i=1}^m R(-b_i)$, then $\mathbb{G}_1^{[q]}$ is the free module $\bigoplus_{i=1}^m R(-qb_i)$.

(c) If J is the R-ideal $(a_1, \dots, a_m)$, then $J^{[q]}$ is the R-ideal $(a_1^q, \dots, a_m^q)$.

In particular, if $\mathbb{N}$ is the homogeneous complex of graded free R-modules

$$\cdots \to \mathbb{N}_3 \xrightarrow{\boldsymbol{n}_3} \mathbb{N}_2 \xrightarrow{\boldsymbol{n}_2} \mathbb{N}_1 \xrightarrow{\boldsymbol{n}_1} \cdots,$$

then $\mathbb{N}^{[q]}$ is a very clean way to write the homogeneous complex

$$F_R^e(\mathbb{N}) : \quad \cdots \to \mathbb{N}_3^{[q]} \xrightarrow{\boldsymbol{n}_3^{[q]}} \mathbb{N}_2^{[q]} \xrightarrow{\boldsymbol{n}_2^{[q]}} \mathbb{N}_1^{[q]} \xrightarrow{\boldsymbol{n}_1^{[q]}} \cdots . \tag{1.9}$$

Furthermore, the Frobenius functor is always right exact; so, $F_R^e(R/J) = R/J^{[q]}$.

We conclude this section by gathering a few properties of Gorenstein ideals, Gorenstein duality, and linkage. All of these tricks appear elsewhere in the literature, usually in more generality. We are likely to use them at any moment, without any further ado. Theorem 1.10 is due to Bass [2].

Theorem 1.10. *Let R be a local artinian ring. The following statements are equivalent:*

(1) *R is a Gorenstein ring.*
(2) *The socle of R is principal.*
(3) *The ideal* (0) *is irreducible* (*in the sense that* (0) *is not equal to the intersection of two non-zero ideals*).
(4) *The ring R is self-injective.*

When the conditions of Theorem 1.10 are in effect, then the functor $\mathrm{Hom}_R(\ \ , R) = (\ \ )^*$ is exact and if M is a finitely generated R-module, then the modules M and M^* have the same length.

Lemma 1.11. *Let M be an ideal in the artinian local Gorenstein ring $(R,\mathfrak{m})$.*

(1) *The ideals M and $\operatorname{ann}(\operatorname{ann}(M))$ are equal.*
(2) *If N is an ideal of R with $M \subseteq N$, then $\operatorname{ann}(M)/\operatorname{ann}(N) \cong \operatorname{Hom}(N/M, R)$.*
(3) *If R/M is a Gorenstein ring, then there exists an element y of R with $\operatorname{ann}(y) = M$ and $\operatorname{ann}(M) = (y)$.*
(4) *If y is a non-zero element of R, then $R/\operatorname{ann}(y)$ is a Gorenstein ring.*

Proof. The ideal M is contained in $\operatorname{ann}(\operatorname{ann}(M))$ and the two ideals have the same length because $M^* = R/\operatorname{ann}(M)$. Assertion (1) follows. Apply $\operatorname{Hom}(\_, R)$ to the short exact sequence

$$0 \to N/M \to R/M \to R/N \to 0$$

to obtain (2). If $\operatorname{ann}(M) = (y_1, \dots, y_s)$, then (1) shows that

$$M = \operatorname{ann}(\operatorname{ann}(M)) = \operatorname{ann}(y_1) \cap \cdots \cap \operatorname{ann}(y_s).$$

Under the hypothesis of (3), the ideal M is irreducible and $M = \operatorname{ann}(y_i)$, for some i. Apply (1) again to complete the proof of (3). For (4), it is not difficult to see that the socle of $R/\operatorname{ann}(y)$ is a principal ideal. $\square$

If the conditions of (1) in Proposition 1.12 hold, then the ideal C is called a *Gorenstein ideal*. Notice that when we use this term, we automatically assume the ideal C to have finite projective dimension.

Proposition 1.12. *Let R be a graded k-algebra. Assume that R is a Gorenstein ring. Let C be a homogeneous ideal of R of grade c and finite projective dimension, and $(\mathbb{G}, \boldsymbol{g}_\bullet)$ be the minimal homogeneous resolution of R/C by free R-modules.*

(1) *The following statements are equivalent:*
   (a) *The ring R/C is Gorenstein.*
   (b) *The ring R/C is Cohen-Macaulay and $\operatorname{Ext}^c_R(R/C, R)$ is a cyclic R/C-module.*
   (c) *The ring R/C is Cohen-Macaulay and $\operatorname{Ext}^c_R(R/C, R)$ is isomorphic to a shift of R/C.*
   (d) *The complex $\operatorname{Hom}_R(\mathbb{G}, R)$ is isomorphic to a shift of $\mathbb{G}$.*
(2) *If C is a Gorenstein ideal, then the entries of any matrix representation for $\boldsymbol{g}_c$ form a minimal generating set for C.*
(3) *If the field k has characteristic p and the ideal C is a Gorenstein ideal, then $C^{[p]}$ is a Gorenstein ideal.*

Proof. The equivalence of (a), (b), and (c) is Section 5 of [2]. The ring R/C is Cohen-Macaulay if and only if R/C is a perfect R-module of projective dimension equal to c; see, for example, Section 16.C of [5]. It is now obvious that (c) and (d) are equivalent. Assertion (2) follows from (d). Assertion (3) follows from Theorem 1.7 of [10], which guarantees that $F_R(\mathbb{G})$ is the minimal homogeneous resolution of $R/C^{[p]}$. □

Peskine and Szpiro popularized the concept of linkage by complete intersection ideals. Only slight modifications need be made to [11, Proposition 2.6] in order prove assertion (1) in the following result about linkage by Gorenstein ideals; see, for example, Section 1 of [8].

Proposition 1.13. *Let R be a Gorenstein graded k-algebra, and let $L \subseteq M$ be homogeneous Gorenstein ideals (in the sense of Proposition 1.12) of R of grade c.*

(1) *Let $\mathbb{F}_\bullet$ and $\mathbb{G}_\bullet$ be the minimal homogeneous resolutions of R/L and R/M, respectively, and let $\alpha_\bullet : \mathbb{F}_\bullet \to \mathbb{G}_\bullet$ be a homogeneous map of resolutions which extends the natural map $R/L \to R/M$. The map $\alpha_c : \mathbb{F}_c \to \mathbb{G}_c$ is multiplication by a homogeneous element y of R. Then*
$$L : M = (L, y) \quad \text{and} \quad L : y = M.$$
(2) *If M is generated by the homogeneous regular sequence $f_1, \dots, f_c$ and L is generated by $f_1^{r_1}, \dots, f_c^{r_c}$, then the conclusion of (1) holds for the product $y = f_1^{r_1-1} \cdots f_c^{r_c-1}$.*

Proof. One can prove (2) directly, or deduce it from (1). □

If L, M, and N are ideals in a ring R then a quick calculation yields the remarkably useful formula

$$L : MN = (L : M) : N. \tag{1.14}$$

The proof of the next result may be read from the proof of Proposition 2 in [13].

Proposition 1.15. *Let L and M be homogeneous ideals of a ring R of positive prime characteristic p. If L and $L : M$ have finite projective dimension, then $L^{[p]} : M^{[p]} = (L : M)^{[p]}$.* □

2. The plan of attack and a few examples

We first establish "(a) implies (b)" from Theorem A.

Observation 2.1. *Let k be a field of positive characteristic p, $R \to S$ be a surjection of graded k-algebras in the sense of (1.1), with R Gorenstein and S artinian. If S has finite projective dimension as an R-module, then the socles of S and $F_R^e(S)$ have the same dimension; furthermore, if the socle degrees of S and $F_R^e(S)$ are given by*
$$d_1 \le d_2 \le \cdots \le d_\ell \quad \text{and} \quad D_1 \le D_2 \le \cdots \le D_\ell,$$

respectively, then

$$D_i = qd_i - (q-1)a(R), \tag{2.2}$$

for all i.

Proof. Consider the minimal homogeneous resolution $\mathbb{F}$ of S by free R-modules. We know from [10] that $F_R^e(\mathbb{F}) = \mathbb{F}^{[q]}$ is the minimal homogeneous resolution of $F_R^e(S)$. If the back twists of $\mathbb{F}$ are $\{b_i \mid 1 \le i \le L\}$, then the back twists of $\mathbb{F}^{[q]}$ are $\{qb_i\}$. Use Corollary 1.7 to see that $L = \ell$, $d_i = b_i + a(R)$, and $D_i = qb_i + a(R)$, for all i. □

Remark. The hypothesis that R is Gorenstein is necessary in Observation 2.1. Let $R = S$ be a non-Gorenstein artinian graded k-algebra whose socle lives in at least two distinct degrees. The ring S has finite projective dimension as an R-module, and $d_i = D_i$, for all i, since $F_R^e(R) = R$. The a-invariant of R is equal to the top socle degree of R; so, $a(R) = d_\ell$ and (2.2) holds for $d_i = d_\ell$; but does not hold for $d_i < d_\ell$.

We prove the converse of Observation 2.1 under the assumption that R is a complete intersection. Our main result is the following statement.

Theorem 2.3. *Let k be a field of positive characteristic p, $R \to S$ be a surjection of graded k-algebras in the sense of (1.1), with R a complete intersection and S artinian. Let e be a positive integer, $q = p^e$, and $d_1 \le \cdots \le d_\ell$ be the socle degrees of S. If the socle of $F_R^e(S)$ has the same dimension as the socle of S, and the socle degrees of $F_R^e(S)$ are given by $D_1 \le D_2 \le \cdots \le D_\ell$ as in* (2.2)*, then* $\operatorname{Tor}_1^R(S, {}^eR) = 0$.

Standing notation 2.4. We express $R = P/C$, where P is the polynomial ring $P = k[x_1, \dots, x_n]$, each variable has positive degree, and C is a homogeneous Gorenstein ideal in P of grade c. Let I be a homogeneous $\mathfrak{m}_P$-primary ideal in P, $S = P/(I+C)$, $T = P/I$, and let Z be the $(c-1)$-syzygy of the P-module $K_T(-a(P))$.

Proof of Theorem 2.3. Adopt the notation of 2.4. In Corollary 3.2, we convert numerical information about the socle degrees of S and $F_R^e(S)$ into numerical information about $\operatorname{Tor}_c^P(K_T, R)$ and $\operatorname{Tor}_c^P(K_{F_P^e(T)}, R)$. In Proposition 4.1, the numerical information about Tor_c's is converted into the statement

$$\operatorname{Tor}_1^R(Z \otimes_P R, {}^eR) = 0.$$

This homological statement is expressed as a statement about ideals:

$$(C^{[q]} + I^{[q]}) : (C^{[q]} : C) = C + I^{[q]}$$

in Proposition 5.1. In Proposition 6.1 we deduce

$$I^{[q]} \cap C = (I \cap C)^{[q]} + CI^{[q]}.$$

This result is equivalent to $\operatorname{Tor}_1^R(S, {}^eR) = 0$, as is recorded in Proposition 3.5. □

We would like to prove that the conclusion of Theorem 2.3 continues to hold after one replaces the hypothesis that R is a complete intersection with the weaker hypothesis that R is Gorenstein. Three of our five steps (3.2, 5.1, and 3.5) work when R is Gorenstein. The arguments that we use in the other two steps (4.1 and 6.1) require that R be a complete intersection, although in Proposition 7.4 we prove the ideal theoretic version of (4.1) under the hypothesis that R is Gorenstein and F-pure. At any rate, if R is a complete intersection and the conclusion of Theorem 2.3 holds, then the Theorem of Avramov and Miller [1] (see also [6]) guarantees that S has finite projective dimension as an R-module. We are very curious to know if some form of the Avramov-Miller result,

$$\operatorname{Tor}_1^R(M, {}^eR) = 0 \implies \operatorname{pd}_R M < \infty,$$

for finitely generated R-modules M, can be proven when R is Gorenstein, but not necessarily a complete intersection.

Proof of Theorem A. Adopt the notation of 2.4.

(a)$\Longrightarrow$(b): This is Observation 2.1.

(b)$\Longrightarrow$(c): Assume (b). In Corollary 3.2 and Observation 3.3, we show that if the generator degrees of $\operatorname{Tor}_1^P(Z, R)$ are $\{\gamma_i\}$, then the generator degrees of $\operatorname{Tor}_1^P(F_P^e(Z), R)$ are $\{q\gamma_i\}$. Proposition 4.1 shows that $\operatorname{Tor}_1^R(Z \otimes_P R, {}^eR) = 0$. Proposition 5.1 yields (c).

(c)$\Longrightarrow$(d): This is Proposition 6.1.

(d)$\Longrightarrow$(a): If (d) holds, then Proposition 3.5 shows that $\operatorname{Tor}_1^R(T \otimes_P R, {}^eR) = 0$. The Theorem of Avramov and Miller [1] guarantees that $T \otimes_P R$ has finite projective dimension as an R-module. □

If the hypothesis of Theorem 2.3 is weakened to say only that the socles of S and $F_R^e(S)$ have the same dimension (with no mention about how the socle degrees are related), then the conclusion of Theorem 2.3 fails to hold; see Example 2.9. It is curious, however, that if S is Gorenstein, and the defining ideal of S is contained in a proper ideal of R of finite projective dimension (for example, a parameter ideal of R), then one need only verify that the socles of $F_R^e(S)$ and S both have dimension one in order to conclude that S has finite projective dimension over R.

THEOREM 2.5. *Let $(R, \mathfrak{m})$ be a local k-algebra, with R a complete intersection, and let $J \subset R$ be an $\mathfrak{m}$-primary ideal. Assume that $J \subseteq \mathfrak{b}$ for some proper ideal $\mathfrak{b}$ with $\operatorname{pd}_R \mathfrak{b} < \infty$. If R/J and $R/J^{[q]}$ both are Gorenstein, then $\operatorname{pd}_R J < \infty$.*

Proof. Choose $\mathfrak{a} \subseteq J$ a parameter ideal. Use Lemma 1.11 and (1.14) to write $J = \mathfrak{a} : f$, and write $\mathfrak{b} = J : K = \mathfrak{a} : fK$. Since $\mathfrak{b}$ has finite projective dimension, we know, from Proposition 1.15, that

$$\mathfrak{b}^{[q]} = \mathfrak{a}^{[q]} : f^q K^{[q]}. \tag{2.6}$$

On the other hand, we have

$$\mathfrak{b}^{[q]} \subseteq J^{[q]} : K^{[q]}. \tag{2.7}$$

Since $J^{[q]} \subseteq \mathfrak{a}^{[q]} : f^q$ are both irreducible, we can write $\mathfrak{a}^{[q]} : f^q = J^{[q]} : b$ for some $b \in R$. Plugging this into (2.6) yields

$$\mathfrak{b}^{[q]} = J^{[q]} : bK^{[q]}.$$

Comparing this with (2.7) we get $J^{[q]} : bK^{[q]} = J^{[q]} : K^{[q]}$ (equation (2.7) gives one inclusion; the other inclusion is always true), which is equivalent to $K^{[q]} \subseteq J^{[q]} + bK^{[q]}$. If b is not a unit, this means that $K^{[q]} \subseteq J^{[q]}$, and so $\mathfrak{b}^{[q]} = J^{[q]} : K^{[q]} = R$, which is a contradiction.

Thus, b must be a unit, so

$$J^{[q]} = \mathfrak{a}^{[q]} : f^q. \tag{2.8}$$

Consider the short exact sequence

$$0 \to R/J \to R/\mathfrak{a} \to R/(\mathfrak{a}, f) \to 0.$$

Equation (2.8) implies that its tensorization with ${}^eR$ is exact, and thus

$$\mathrm{Tor}_1^R(R/(\mathfrak{a}, f), {}^eR) = 0,$$

and $(\mathfrak{a}, f)$ has finite projective dimension by the Avramov-Miller result. Consequently, J has finite projective dimension as well. $\square$

Example 2.9. Note that the conclusion of Theorem 2.5 no longer holds without the assumption that J is contained in a proper ideal of finite projective dimension. Consider, for instance, $R = k[x, y, z]/(x^3 + y^3 + z^3)$, $J = (x, y, z^2)$, where k is a field of characteristic $p \equiv 2 \pmod 3$. Clearly J is irreducible, $J^{[p]} = (x^p, y^p)$ is also irreducible, but J does not have finite projective dimension.

3. Convert socle degrees into $\mathrm{Tor}_1^R(Z \otimes_P R, {}^eR)$

Adopt the notation of 2.4. The first three results in this section convert hypothesis (2.2) about the socle degrees of S and $F_R^e(S)$ into a statement about the generator degrees of $\mathrm{Tor}_1^P(Z, R)$ and $\mathrm{Tor}_1^P(F_P^e(Z), R)$. Observation 3.4 shows that $\mathrm{Tor}_1^R(Z \otimes_P R, {}^eR)$ is a quotient of $\mathrm{Tor}_1^P(F_P^e(Z), R)$ by a submodule which is built from the generators of $\mathrm{Tor}_1^P(Z, R)$. The notation that is introduced in the proof of Observation 3.4 will be used again in the proofs of Propositions 4.1 and 5.1.

All of the calculations in Section 3 which we have described so far pertain to the proof of (b)$\Longrightarrow$(c) in Theorem A. It is curious, however, that Observation 3.4 also may be applied (see Proposition 3.5) to give an ideal theoretic interpretation of $\operatorname{Tor}_1^R(T \otimes_P R, {}^eR)$, which is our contribution to the proof of (d)$\Longrightarrow$(a) in Theorem A.

Lemma 3.1. *Adopt the notation of 2.4. If the socle degrees of S are*

$$\{d_i \mid 1 \leq i \leq \ell\},$$

then the minimal generators of $\operatorname{Tor}_c^P(K_T(-a(P)), R)$ have degrees

$$\{a(R) - d_i \mid 1 \leq i \leq \ell\}.$$

Proof. Let $\mathbb{G}$ be the minimal homogeneous resolution of R by free P-modules. Corollary 1.3 (c) tells us that $\mathbb{G}_c = P(a(P) - a(R))$. It follows that

$$\begin{aligned} \operatorname{Tor}_c^P(K_T(-a(P)), R) &= \mathrm{H}_c(K_T(-a(P)) \otimes_P \mathbb{G}) \\ &= \{\alpha \in K_T(-a(R)) \mid C\alpha = 0\} \\ &= \operatorname{Hom}_P(R, K_T(-a(R)). \end{aligned}$$

On the other hand, we have a surjection $T \to S$; so Proposition 1.2 guarantees

$$K_S = \operatorname{Hom}_T(S, K_T) = \operatorname{Hom}_P(R, K_T).$$

Thus,

$$K_S(-a(R)) = \operatorname{Hom}_P(R, K_T(-a(R))) = \operatorname{Tor}_c^P(K_T(-a(P)), R).$$

Apply Proposition 1.5. $\square$

Lemma 3.1 also applies when the ideal I is replaced by the ideal $I^{[q]}$; consequently, if the socle degrees of $F_R^e(S)$ are $\{D_i \mid 1 \leq i \leq L\}$, then the minimal generators of $\operatorname{Tor}_c^P(K_{F_P^e(T)}(-a(P)), R)$ have degrees $\{a(R) - D_i \mid 1 \leq i \leq L\}$. We have established the following conversion of the original hypothesis about socle degrees into a statement about generator degrees of Tor_c.

Corollary 3.2. *Retain the notation of 2.4. Assume that the socles of S and $F_R^e(S)$ have the same dimension. Let*

$$d_1 \leq \cdots \leq d_\ell \quad \textit{and} \quad D_1 \leq \cdots \leq D_\ell$$

be the socle degrees of S and $F_R^e(S)$, respectively, and

$$\gamma_1 \leq \cdots \leq \gamma_\ell \quad \textit{and} \quad \Gamma_1 \leq \cdots \leq \Gamma_\ell,$$

be the minimal generator degrees of

$$\operatorname{Tor}_c^P(K_T(-a(P)), R) \quad \textit{and} \quad \operatorname{Tor}_c^P(K_{F_P^e(T)}(-a(P)), R),$$

respectively. Then

$$D_i = qd_i - (q-1)a(R) \text{ for all } i \iff \Gamma_i = q\gamma_i \text{ for all } i. \qquad \square$$

We interpret the homological objects of Corollary 3.2 as Tor_1 of the appropriate modules. This process involves index shifting and keeping careful track of the twists.

Observation 3.3. *In the notation of 2.4:*

(a) $\operatorname{Tor}_c^P(K_T(-a(P)), R) = \operatorname{Tor}_1^P(Z, R)$, *and*
(b) $\operatorname{Tor}_c^P(K_{F_P^e(T)}(-a(P)), R) = \operatorname{Tor}_1^P(F_P^e(Z), R)$.

Proof. We prove (b). Let $\mathbb{F}$ be the minimal homogeneous resolution of $K_T(-a(P))$ by free P-modules. The functor $F_P^e(\_\_)$ is exact; so, $\mathbb{F}^{[q]}$, which is equal to $F_P^e(\mathbb{F})$ (see (1.9)), resolves $F_P^e(K_T(-a(P)))$. On the other hand, it is not hard to see that $\mathbb{F}^{[q]}$ resolves some twist of $K_{F_P^e(T)}$. Indeed, if $(\ )^* = \operatorname{Hom}_P(\_\_, P)$, then it is clear that $(F_P^e(\mathbb{F}^*))^*$ is equal to a shift of $\mathbb{F}^{[q]}$; furthermore, it is also clear that $(F_P^e(\mathbb{F}^*))^*$ resolves some shift of $K_{F_P^e(T)}$; see, for example, Proposition 1.12. There are many ways to keep track of the shifts. One pain free approach is to apply the technique of (1.6) to K_T and to $K_{F_P^e(T)}$ in order to nail down the fact that $\mathbb{F}^{[q]}$ is the minimal homogeneous resolution of $K_{F_P^e(T)}(-a(P))$; hence,

$$\begin{aligned} F_P^e(K_T(-a(P))) &= K_{F_P^e(T)}(-a(P)), \\ \operatorname{Tor}_i^P(K_{F_P^e(T)}(-a(P)), R) &= \mathrm{H}_i(\mathbb{F}^{[q]} \otimes_P R), \end{aligned}$$

for all i. The beginning of the minimal homogeneous resolution of Z is

$$\cdots \to \mathbb{F}_{c+1} \to \mathbb{F}_c \to \mathbb{F}_{c-1} \to Z \to 0.$$

The functor $F_P^e(\ \ )$ is exact; so,

$$\operatorname{Tor}_1^P(F_P^e(Z), R) = \mathrm{H}_c(\mathbb{F}^{[q]} \otimes_P R) = \operatorname{Tor}_c^P(K_{F_P^e(T)}(-a(P)), R). \qquad \square$$

Observation 3.4. *Let $P \to R$ be a surjection of graded k-algebras, with P a polynomial ring, and let M be a finitely generated graded P-module. Then there is an exact sequence of graded R-modules:*

$$F_R^e\left(\operatorname{Tor}_1^P(M, R)\right) \to \operatorname{Tor}_1^P(F_P^e(M), R) \to \operatorname{Tor}_1^R(M \otimes_P R, {}^eR) \to 0.$$

Proof. Let $(\mathbb{N}, \boldsymbol{n})$ be the minimal homogeneous resolution of M by free P-modules. The functor $F_P^e(\_\_)$ is exact; so, $\mathbb{N}^{[q]}$ is the minimal homogeneous resolution of $F_P^e(M)$ by free P-modules, and $\operatorname{Tor}_1^P(F_P^e(M), R)$ is equal to

$$\mathrm{H}_1(F_P^e(\mathbb{N}) \otimes_P R) = \mathrm{H}_1(F_R^e(\mathbb{N} \otimes_P R)).$$

The functors $F_P^e(\_\_)\otimes_P R$ and $F_R^e(\_\_\otimes_P R)$ are equal because the homomorphisms

$$\begin{array}{ccc} P & \xrightarrow{\phi_P^e} & P \\ {\scriptstyle \text{quot. map}}\big\downarrow & & \big\downarrow{\scriptstyle \text{quot. map}} \\ R & \xrightarrow{\phi_R^e} & R \end{array}$$

commute. Let $^-$ denote the functor $\_\_\otimes_P R$. Select elements $z_1,\dots,z_\ell$ of $\mathbb{N}_1$ so that $\overline{z}_1,\dots,\overline{z}_\ell$ are cycles in $\mathbb{N}\otimes_P R$ and the homology classes $[\overline{z}_1],\dots,[\overline{z}_\ell]$ form a minimal generating set for $\mathrm{H}_1(\mathbb{N}\otimes_P R)=\operatorname{Tor}_1^P(M,R)$. It is clear that $z_i^{[q]}$ is an element of $F_P^e(\mathbb{N}_1)$ with $\overline{z_i^{[q]}}=\overline{z}_i^{[q]}$ a cycle in $F_P^e(\mathbb{N})\otimes_P R=F_R^e(\mathbb{N}\otimes_P R)$, for each i.

The technique of killing cycles (see, for example, Section 2 of [12]) tells us that

$$\mathbb{M}:\quad \overline{\mathbb{N}}_2\oplus\bigoplus_{i=1}^{\ell}R(-|z_i|)\xrightarrow{[\overline{\boldsymbol{n}}_2\,\overline{z}_1\,\dots\,\overline{z}_\ell]}\overline{\mathbb{N}}_1\xrightarrow{\overline{\boldsymbol{n}}_1}\overline{\mathbb{N}}_0\to\overline{M}\to 0$$

is the beginning of a homogeneous resolution of $\overline{M}$ by free R-modules. It follows that

$$\operatorname{Tor}_1^R(\overline{M},{}^eR)=\mathrm{H}_1(\mathbb{M}\otimes_R{}^eR)=\frac{\mathrm{H}_1(F_R^e(\overline{\mathbb{N}}))}{([\overline{z}_1^{[q]}],\dots,[\overline{z}_\ell^{[q]}])}=\frac{\operatorname{Tor}_1^P(F_P^e(M),R)}{([\overline{z}_1^{[q]}],\dots,[\overline{z}_\ell^{[q]}])}.\quad\square$$

The following result is an application of the technique of Observation 3.4.

Proposition 3.5. *Let $R=P/C$ and $T=P/I$, where I and C are homogeneous ideals in the polynomial ring P. Then*

$$\operatorname{Tor}_1^R(T\otimes_P R,{}^eR)=\frac{I^{[q]}\cap C}{(I\cap C)^{[q]}+I^{[q]}C}.$$

Proof. Apply Observation 3.4 to see that

$$\operatorname{Tor}_1^R(T\otimes_P R,{}^eR)=\frac{\mathrm{H}_1(\mathbb{N}^{[q]}\otimes_P R)}{([\overline{z}_1^{[q]}],\dots,[\overline{z}_\ell^{[q]}])},$$

where $(\mathbb{N},\boldsymbol{n})$ is the minimal homogeneous resolution of T by free P-modules, $^-$ is the functor $\_\_\otimes_P R$, and $z_1,\dots,z_\ell$ are homogeneous elements of $\mathbb{N}_1$ with $[\overline{z}_1],\dots,[\overline{z}_\ell]$ a minimal generating set for

$$\mathrm{H}_1(\mathbb{N}\otimes_P R)=\operatorname{Tor}_1^P(P/I,P/C)=(I\cap C)/IC.$$

Observe that $I\cap C=(\boldsymbol{n}_1(z_1),\dots,\boldsymbol{n}_1(z_\ell))+IC$. Observe also, that

$$\mathrm{H}_1(\mathbb{N}^{[q]}\otimes_P R)=\operatorname{Tor}_1^P\left(P/I^{[q]},P/C\right)=(I^{[q]}\cap C)/I^{[q]}C.$$

The isomorphism $\mathrm{H}_1(\mathbb{N}^{[q]}\otimes_P R)\to(I^{[q]}\cap C)/I^{[q]}C$ carries $[\overline{z}_i^{[q]}]$ to the class of $(\boldsymbol{n}_1(z_i))^q$. $\square$

4. Degree considerations concerning Tor_1

Proposition 4.1, followed by [1], is a general statement that says that if the degrees of the minimal generators of

$$\operatorname{Tor}_1^P(M,R) \quad \text{and} \quad \operatorname{Tor}_1^P(F_P^e(M),R)$$

are related in the appropriate manner, then $M \otimes_P R$ has finite projective dimension as an R-module. When the notation of 2.4 and the hypothesis of Theorem 2.3 are in effect, then Corollary 3.2 and Observation 3.3 show that Proposition 4.1 may be applied with $M = Z$.

PROPOSITION 4.1. *Let $P \to R$ be a surjection of graded k-algebras, with P a polynomial ring and R a complete intersection, and let M be a finitely generated graded P-module. Suppose that the minimal generators of* $\operatorname{Tor}_1^P(M,R)$ *have degrees $\{\gamma_i \mid 1 \le i \le \ell\}$. If the minimal generators of* $\operatorname{Tor}_1^P(F_P^e(M),R)$ *have degrees $\{q\gamma_i \mid 1 \le i \le \ell\}$, then* $\operatorname{Tor}_1^R(M \otimes_P R, {}^eR) = 0$.

Proof. Inflation of the base field $k \to K$ gives rise to faithfully flat extensions $P \to P \otimes_k K$ and $R \to R \otimes_k K$. Consequently, we may assume that k is a perfect field. Let C be the ideal in P with $R = P/C$, and let $f_1, \dots, f_c$ be a homogeneous regular sequence in P that generates C. We retain the notation from the proof of Observation 3.4. So,

$$\mathbb{N}: \quad \mathbb{N}_2 \xrightarrow{\boldsymbol{n}_2} \mathbb{N}_1 \xrightarrow{\boldsymbol{n}_1} \mathbb{N}_0 \to M \to 0$$

is the beginning of the minimal homogeneous resolution of M by free P-modules, $\overline{}$ is the functor $\_\_ \otimes_P R$, and $z_1, \dots, z_\ell$ are homogeneous elements of $\mathbb{N}_1$ with $[\overline{z}_1], \dots, [\overline{z}_\ell]$ a minimal generating set for $\mathrm{H}_1(\overline{\mathbb{N}}) = \operatorname{Tor}_1^P(M,R)$. Let Y be the subset $\{[\overline{z}_1^{[q]}], \dots, [\overline{z}_\ell^{[q]}]\}$ of $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]}) = \operatorname{Tor}_1^P(F_P^e(M),R)$. We will prove that

$$Y \text{ generates } \mathrm{H}_1(\overline{\mathbb{N}}^{[q]}). \tag{4.2}$$

As soon as (4.2) is established, then the proof is complete by Observation 3.4.

Fix an integer δ. Define W'_δ and W''_δ to be the P-submodules of $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})$ which are generated by

$$\sum_{\delta<i} \left[\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})\right]_i \quad \text{and} \quad \sum_{i<\delta} \left[\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})\right]_i,$$

respectively. Let V_δ be the vector space $V_\delta = \mathrm{H}_1(\overline{\mathbb{N}}^{[q]})/(W'_\delta + W''_\delta)$. Let X be a homogeneous minimal generating set for the P-module $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})$. Let X_δ be the subset of X which consists of those generators which have degree equal to δ. Notice that the set X_δ is a basis for the vector space V_δ over k. Let

$$Y_\delta = \{[\overline{z}_i^{[q]}] \in Y \mid |z_i^{[q]}| = \delta\}.$$

Our hypothesis guarantees that the dimension of the vector space V_δ is exactly equal to the number of elements in Y_δ. We prove that

$$\text{the elements of } Y_\delta \text{ are linearly independent in } V_\delta. \tag{4.3}$$

Once (4.3) is established, then we will know that Y_δ is a basis for V_δ and that the elements of Y_δ are part of a minimal generating set of $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})$. We repeat this calculation for all δ to see that Y is part of a minimal generating set for $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})$. The hypothesis tells us that the elements of Y have the correct degrees to be a complete minimal generating set for $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})$. We conclude that Y is a complete minimal generating set for $\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})$. At that point the proof will be complete by (4.2).

We establish (4.3) by induction on δ. When δ is small, then Y_δ is the empty set, V_δ is zero, and everything is fine. Now we work on the inductive step. There is nothing to do unless δ is a multiple of q. Relabel the z_i, if necessary, and select the integer λ so that $z_1, \dots, z_\lambda$ have degree less than δ/q, and $z_{\lambda+1}, \dots, z_\ell$ have degree at least δ/q. Consider a non-trivial k-linear combination of the elements of Y_δ. The field k is closed under the taking of q^{th} roots; so this linear combination is equal to $[\overline{z}^{[q]}]$, where z is a non-trivial k-linear combination of those z_i that have degree equal to δ/q. So

$$[\overline{z}] \text{ is not zero in } \frac{\mathrm{H}_1(\overline{\mathbb{N}})}{P([\overline{z}_1], \dots, [\overline{z}_\lambda])}. \tag{4.4}$$

We assume that

$$[\overline{z}^{[q]}] = 0 \text{ in } \frac{\mathrm{H}_1(\overline{\mathbb{N}}^{[q]})}{W''_\delta}, \tag{4.5}$$

and we will show that this assumption leads to a contradiction. Keep in mind that the induction hypothesis guarantees that

$$W''_\delta = P([\overline{z}_1^{[q]}], \dots, [\overline{z}_\lambda^{[q]}]). \tag{4.6}$$

We know that $\overline{z}$ and $\overline{z}_1, \dots, \overline{z}_\lambda$ all are cycles in $\overline{\mathbb{N}}$; so

$$\boldsymbol{n}_1 z \in C\mathbb{N}_0 \quad \text{and} \quad \boldsymbol{n}_1 z_i \in C\mathbb{N}_0 \text{ for } 1 \le i \le \lambda. \tag{4.7}$$

We introduce a notational convenience. Let

$$\mathbb{G}_2 = \mathbb{N}_2 \oplus \bigoplus_{i=1}^{\lambda} P(-|z_i|)$$

and let $\boldsymbol{g}_2 : \mathbb{G}_2 \to \mathbb{N}_1$ be the map

$$\boldsymbol{g}_2 = \begin{bmatrix} \boldsymbol{n}_2 & z_1 & \dots & z_\lambda \end{bmatrix}.$$

Of course, the map

$$\boldsymbol{g}_2^{[q]} : \mathbb{G}_2^{[q]} \to \mathbb{N}_1^{[q]}$$

also now has meaning. Assumption (4.5), together with the induction hypothesis (4.6), tells us that

$$z^{[q]} \in \operatorname{im} \boldsymbol{g}_2^{[q]} + C\mathbb{N}_1^{[q]},$$

which is the base case for the following induction. We will prove that if $1 \le t \le c(q-1)$, then

$$z^{[q]} \in \operatorname{im} \boldsymbol{g}_2^{[q]} + C^{[q]}\mathbb{N}_1^{[q]} + C^t\mathbb{N}_1^{[q]} \implies z^{[q]} \in \operatorname{im} \boldsymbol{g}_2^{[q]} + C^{[q]}\mathbb{N}_1^{[q]} + C^{t+1}\mathbb{N}_1^{[q]}. \tag{4.8}$$

As soon as (4.8) is established, then

$$z^{[q]} \in \operatorname{im} \boldsymbol{g}_2^{[q]} + C^{[q]}\mathbb{N}_1^{[q]}$$

because $C^{c(q-1)+1} \subseteq C^{[q]}$. Then the proof will be complete because we may apply Observation 4.10 to

$$z^{[q]} \in \operatorname{im} \begin{bmatrix} \boldsymbol{g}_2 & f_1 I & \cdots & f_c I \end{bmatrix}^{[q]}$$

to conclude that

$$z \in \operatorname{im} \begin{bmatrix} \boldsymbol{g}_2 & f_1 I & \cdots & f_c I \end{bmatrix},$$

and this violates (4.4).

We prove (4.8). For each c-tuple $\alpha = (\alpha_1, \dots, \alpha_c)$ of non-negative integers, with $\alpha_i < q$ for all i, and $\sum \alpha_i = t$, there exists $y_\alpha \in \mathbb{N}_1^{[q]}$ such that

$$z^{[q]} - \sum_\alpha f_1^{\alpha_1} \cdots f_c^{\alpha_c} y_\alpha \in C^{[q]}\mathbb{N}_1^{[q]} + \operatorname{im} \boldsymbol{g}_2^{[q]}. \tag{4.9}$$

Each element y_α is homogeneous of degree $\delta - t$. Fix a c-tuple α. Apply $\boldsymbol{n}_1^{[q]}$ to (4.9) and use (4.7) to see that

$$f_1^{\alpha_1} \cdots f_c^{\alpha_c} \boldsymbol{n}_1^{[q]} y_\alpha \in (f_1^{\alpha_1+1}, \dots, f_c^{\alpha_c+1})\mathbb{N}_0^{[q]}.$$

It follows, from the fact that $f_1, \dots, f_c$ is a regular sequence, that

$$\boldsymbol{n}_1^{[q]} y_\alpha \in C\mathbb{N}_0^{[q]}.$$

So, $\overline{y}_\alpha$ is a one-cycle, of degree less than δ, in $\mathbb{N}^{[q]}$. The induction hypothesis (4.6) tells us that

$$y_\alpha \in \operatorname{im} \boldsymbol{g}_2^{[q]} + C\mathbb{N}_1^{[q]},$$

and (4.8) is established. □

We close this section with a quick application of the flatness of the Frobenius functor for regular rings. The result is well known and is a critical step in the proof of the previous result.

Observation 4.10. *Let P be a polynomial ring, $\boldsymbol{g} : \mathbb{G}_2 \to \mathbb{G}_1$ be a homomorphism of graded P-modules, and z be an element of $\mathbb{G}_1$. If $z^{[q]}$ is in the image of $\boldsymbol{g}^{[q]}$, then z is in the image of $\boldsymbol{g}$.*

Proof. Let $\widetilde{\mathbb{G}}_2$ be the graded free module $\mathbb{G}_2 \oplus P(-|z|)$ and $\widetilde{\boldsymbol{g}}$ be the map of graded free modules

$$\widetilde{\boldsymbol{g}} = \begin{bmatrix} \boldsymbol{g} & z \end{bmatrix} : \widetilde{\mathbb{G}}_2 \to \mathbb{G}_1.$$

The hypothesis ensures the existence of $h \in \mathbb{G}_2^{[q]}$ with $\begin{bmatrix} -h & 1 \end{bmatrix}^{\mathrm{T}}$ in the kernel of $\widetilde{\boldsymbol{g}}^{[q]}$, where $(\ )^{\mathrm{T}}$ means "transpose". The Frobenius functor $\_\_ \otimes_P {}^eP$ is flat; so, there exist $\begin{bmatrix} t_i & b_i \end{bmatrix}^{\mathrm{T}}$ in $\ker \widetilde{\boldsymbol{g}}$ and $a_i \in P$ such that $\sum_i a_i \begin{bmatrix} t_i^{[q]} & b_i^q \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -h & 1 \end{bmatrix}^{\mathrm{T}}$. Degree considerations tell us that b_i is a unit, for some i. For this i, $\boldsymbol{g}(t_i) + b_i z = 0$ and $\boldsymbol{g}(-t_i/b_i) = z$. □

5. We interpret Tor_1 of the syzygy in terms of ideals

Recall, from the beginning of Section 4, that if the notation of 2.4 and the hypotheses of Theorem 2.3 are in effect, then $\mathrm{Tor}_1^R(Z \otimes_P R, {}^eR) = 0$. In this section, we interpret this Tor-module in terms of ideals. Our interpretation holds even when the hypotheses of Theorem 2.3 are not in effect.

PROPOSITION 5.1. *Adopt the notation of 2.4. If $A \subseteq I$ is any homogeneous $\mathfrak{m}_P$-primary Gorenstein ideal and N is equal to $a(P/A^{[q]}) - a(R)$, then*

$$\mathrm{Tor}_1^R(Z \otimes_P R, {}^eR) = \mathrm{Hom}_P\left(\frac{(C^{[q]} + I^{[q]}) : (C^{[q]} : C)}{C + I^{[q]}}, \frac{P}{A^{[q]}}(N)\right); \tag{5.2}$$

furthermore,

$$\mathrm{Tor}_1^R(Z \otimes_P R, {}^eR) = 0 \iff \frac{(C^{[q]} + I^{[q]}) : (C^{[q]} : C)}{C + I^{[q]}} = 0.$$

Proof. In our argument, we assume that $A \subseteq I$ are perfect ideals of the same grade without assuming that these ideals are $\mathfrak{m}_P$-primary, and we prove that

$$\mathrm{Tor}_1^R(Z \otimes_P R, {}^eR) = \frac{A^{[q]} : (C + I^{[q]})}{(A : (C + I))^{[q]}(C^{[q]} : C) + A^{[q]}}(N). \tag{5.3}$$

Once (5.3) is established, then, when A is an $\mathfrak{m}_P$-primary ideal, Gorenstein duality (see, for example, Lemma 1.11 and (1.14)) may be employed to show that the right side of (5.3) and the right side of (5.2) are both equal to

$$\frac{A^{[q]} : (C + I^{[q]})}{A^{[q]} : \big((C + I)^{[q]} : (C^{[q]} : C)\big)}(N).$$

Let $(\mathbb{G}, \boldsymbol{g}_\bullet)$ be the minimal homogeneous resolution of R by free P-modules and $\boldsymbol{y}_\bullet : \mathbb{G}^{[q]} \to \mathbb{G}$ be a map of complexes which lifts the natural quotient map $P/C^{[q]} \to R$. The ideal C is Gorenstein of grade c; so $\mathbb{G}_c = P(a(P) - a(R))$. The map $\boldsymbol{y}_c : \mathbb{G}_c^{[q]} \to \mathbb{G}_c$ is multiplication by y, for some element y in P of degree $(q-1)(a(R)-a(P))$. We know, from linkage theory (see Proposition 1.13), that

$$C^{[q]}:C=(y,C^{[q]}). \tag{5.4}$$

Use the surjections $P/A \to T$ and $P/A^{[q]} \to F_P^e(T)$ to calculate

$$K_T=\frac{A:I}{A}(a(P/A)) \quad \text{and} \quad K_{F_P^e(T)}=\frac{A^{[q]}:I^{[q]}}{A^{[q]}}(a(P/A^{[q]})).$$

It follows that $\mathrm{H}_c(K_T(-a(P))\otimes_P \mathbb{G})$ is equal to

$$\left\{\alpha\in\frac{A:I}{A}(a(P/A)-a(R))\,\middle|\,\alpha C=0\right\}=\frac{A:(I+C)}{A}\,(a(P/A)-a(R))$$

and

$$\mathrm{H}_c(K_{F_P^e(T)}(-a(P))\otimes_P\mathbb{G})=\frac{A^{[q]}:(I^{[q]}+C)}{A^{[q]}}(N).$$

Let $(\mathbb{F},\boldsymbol{f})$ be the minimal homogeneous resolution of $K_T(-a(P))$ by free P-modules. We saw in Observation 3.4 that

$$\mathrm{Tor}_1^R(Z\otimes_P R,{}^eR)=\frac{\mathrm{H}_c(\mathbb{F}^{[q]}\otimes_P R)}{([\overline{z}_1^{[q]}],\dots,[\overline{z}_\ell^{[q]}])},$$

where $z_1,\dots,z_\ell$ are elements in $\mathbb{F}_c$ with $[\overline{z}_1],\dots,[\overline{z}_\ell]$ a minimal generating set for $\mathrm{H}_c(\mathbb{F}\otimes_P R)$. We use the isomorphisms

$$\begin{aligned}\mathrm{H}_c(\mathbb{F}\otimes_P R)&=\mathrm{H}_c(\mathrm{Tot}(\mathbb{F}\otimes_P\mathbb{G}))=\mathrm{H}_c(K_T(-a(P))\otimes_P\mathbb{G})\\&=\frac{A:(C+I)}{A}(a(P/A)-a(R))\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}\mathrm{H}_c(\mathbb{F}^{[q]}\otimes_P R)=\mathrm{H}_c(\mathrm{Tot}(\mathbb{F}^{[q]}\otimes_P\mathbb{G}))\quad&=\mathrm{H}_c(K_{F_P(T)}(-a(P))\otimes_P\mathbb{G})\\=\frac{A^{[q]}:(C+I^{[q]})}{A^{[q]}}(N)&\end{aligned}\tag{5.6}$$

to re-express $\mathrm{Tor}_1^R(Z\otimes_P R,{}^eR)$ as a subquotient of $P(N)$. First, we give an explicit description of the isomorphism (5.5). Let w_c be an element of $\mathbb{F}_c$ with $\boldsymbol{f}_c(w_c)\in C\mathbb{F}_{c-1}$. Each row and column of the double complex $\mathbb{F}\otimes\mathbb{G}$ is acyclic; therefore, for each i, with $0\le i\le c$, there exists $w_i\in\mathbb{F}_i\otimes\mathbb{G}_{c-i}$ such that

$$(\boldsymbol{f}_i\otimes 1)(w_i)=(1\otimes\boldsymbol{g}_{c-i+1})(w_{i-1}). \tag{5.7}$$

In particular, w_0 is an element of $\mathbb{F}_0\otimes\mathbb{G}_c$. The isomorphism (5.5) sends the homology class $[\overline{w}_c]$ to the image of w_0 in

$$\frac{A:(C+I)}{A}(a(P/A)-a(R)).$$

In a similar manner, if W_c is an element of $\mathbb{F}_c^{[q]}$ with $\boldsymbol{f}_c^{[q]}(W_c) \in C\mathbb{F}_{c-1}^{[q]}$, then the isomorphism (5.6) sends the homology class $[\overline{W}_c]$ in $\mathrm{H}_c(\mathbb{F}^{[q]} \otimes_P R)$ to the image of W_0 in

$$\frac{A^{[q]} :(C + I^{[q]})}{A^{[q]}}(N),$$

where $W_i \in \mathbb{F}_i^{[q]} \otimes \mathbb{G}_{c-i}$ and

$$(\boldsymbol{f}_i^{[q]} \otimes 1)(W_i) = (1 \otimes \boldsymbol{g}_{c-i+1})(W_{i-1}). \tag{5.8}$$

We finish the argument by showing that the submodule $([\overline{z}_1^{[q]}], \dots, [\overline{z}_\ell^{[q]}])$ of $\mathrm{H}_c(\mathbb{F}^{[q]} \otimes_P R)$ is mapped onto the submodule

$$\frac{(A :(C + I))^{[q]}y + A^{[q]}}{A^{[q]}}(N) \quad \text{of} \quad \frac{A^{[q]} :(C + I^{[q]})}{A^{[q]}}(N)$$

under the isomorphism (5.6). Let $\overline{w}_c$ be a cycle in $\mathbb{F} \otimes R$, for some element w_c of $\mathbb{F}_c$. We are given the family $\{w_i\}$ with $w_i \in \mathbb{F}_i \otimes \mathbb{G}_{c-i}$ such that (5.7) holds. If $w_i = \sum_j u_{i,j} \otimes v_{c-i,j}$ with $u_{i,j} \in \mathbb{F}_i$ and $v_{c-i,j} \in \mathbb{G}_{c-i}$, then let

$$W_i = \sum_j u_{i,j}^{[q]} \otimes \boldsymbol{y}_{c-i}(v_{c-i,j}^{[q]}) \in \mathbb{F}_i^{[q]} \otimes \mathbb{G}_{c-i}.$$

A short calculation shows that (5.8) holds for $\{W_i\}$ and we conclude that if a in

$$\frac{A :(C + I)}{A}(a(P/A) - a(R))$$

is the image of the homology class $[\overline{w}_c]$ under the isomorphism (5.5), then ya^q in

$$\frac{A^{[q]} :(C + I^{[q]})}{A^{[q]}}(N)$$

is the image of the homology class $[\overline{w}_c^{[q]}]$ under the isomorphism (5.6). $\square$

6. The key calculation

Retain the notation of 2.4. In Section 4, we proved that if the socle hypothesis (2.2) is in effect, then $\operatorname{Tor}_1^R(Z \otimes_P R, {}^eR) = 0$. Our goal in Theorem 2.3 is to prove that $\operatorname{Tor}_1^R(T \otimes_P R, {}^eR) = 0$. Homological arguments in Sections 3 and 5 connect these Tor-modules to quotients of ideals. In the present section we show how information about the Tor-module of Z gives information about the Tor-module of T, when R is a complete intersection.

Proposition 6.1. *Let P be a regular ring of positive characteristic p, and let C and I be ideals in P. Assume that C is generated by the regular sequence $f_1, \dots, f_c$ and that*

$$(C + I)^{[q]} :y = C + I^{[q]}, \tag{6.2}$$

where $y = (f_1 \cdots f_c)^{q-1}$. *Then*

$$I^{[q]} \cap C = (I \cap C)^{[q]} + CI^{[q]}.$$

Proof. Recall, from Proposition 1.13, that $C^{[q]} : C = (y, C^{[q]})$. Take $\xi \in I^{[q]} \cap C$. We prove that if $1 \le t \le c(q-1)$, then

$$\xi \in C^t + C^{[q]} + CI^{[q]} \implies \xi \in C^{t+1} + C^{[q]} + CI^{[q]}. \tag{6.3}$$

Of course, we know that the hypothesis of (6.3) holds for $t = 1$. Once we have established (6.3), then, since $C^{c(q-1)+1} \subseteq C^{[q]}$, we know that

$$\xi \in I^{[q]} \cap C^{[q]} + CI^{[q]} = (I \cap C)^{[q]} + CI^{[q]},$$

because the Frobenius functor on P is flat.

Now we prove (6.3). Write ξ as an element of $C^{[q]} + CI^{[q]}$ plus

$$\sum_{\alpha} b_\alpha f_1^{\alpha_1} \cdots f_c^{\alpha_c},$$

where $\alpha = (\alpha_1, \dots, \alpha_c)$ varies over all c-tuples of non-negative integers with $\alpha_i < q$ for all i and $\sum_{i=1}^{c} \alpha_i = t$. Fix an index α. Observe that

$$f_1^{q-1-\alpha_1} \cdots f_c^{q-1-\alpha_c} \xi$$

is equal to $b_\alpha y$ plus an element of $C^{[q]} + I^{[q]}$. Hypothesis (6.2) tells us that b_α is in $C + I^{[q]}$; (6.3) is established, and the proof is complete. $\square$

7. The Gorenstein F-pure case

The question of whether the conclusion of Theorem 2.3 holds when R is Gorenstein is still open. In this section, we include partial results in this direction. Recall that the ring R of positive prime characteristic p is F-pure if whenever J is an ideal of R and x is an element of R with $x \notin J$, then $x^q \notin J^{[q]}$ for all $q = p^e$.

First note that the top socle degree (tsd) of a Frobenius power is always at least equal to the "expected" top socle degree, in the sense of Observation 2.1:

PROPOSITION 7.1. *Let k be a field of positive characteristic p, $R \to S$ be a surjection of graded k-algebras with S artinian. Assume that either R is a complete intersection or R is Gorenstein and F-pure. If d is the top socle degree of S, then the top socle degree of $F_R^e(S)$ is at least $qd - (q-1)a(R)$.*

Proof. Write $S = R/J$, with $J \subset R$ an $\mathfrak{m}$-primary ideal, where $\mathfrak{m}$ is the unique homogeneous maximal ideal of R.

We first assume that R is Gorenstein and F-pure. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R, generated by a regular sequence, with $\mathfrak{a} \subset J$. Let $g_1, \dots, g_s$, with $|g_1| \le \cdots \le |g_s|$, be elements in R which represent a minimal generating set for $(\mathfrak{a} : J)/\mathfrak{a}$. The hypothesis that R is F-pure ensures that

$g_i^q \notin (g_1^q, \dots, \widehat{g_i^q}, \dots, g_s^q, \mathfrak{a}^{[q]})$ for all i; therefore, $g_1^q, \dots, g_s^q$ represents a minimal generating set for $(g_1^q, \dots, g_s^q, \mathfrak{a}^{[q]})/\mathfrak{a}^{[q]}$. It follows that the minimum generator degree (min gen degree) of $(g_1^q, \dots, g_s^q, \mathfrak{a}^{[q]})/\mathfrak{a}^{[q]}$ is $q|g_1|$. Apply Observation 1.4 to the ideal $J/\mathfrak{A}$ of the ring $R/\mathfrak{a}$ to see that

$$\begin{aligned}
\operatorname{tsd} R/J &= \operatorname{tsd}\left(\frac{R/\mathfrak{a}}{J/\mathfrak{a}}\right)\\
&= \text{socle degree}\, R/\mathfrak{a} - \text{min gen degree}\,(\operatorname{ann}(J/\mathfrak{a}))\\
&= \text{socle degree}\, R/\mathfrak{a} - \text{min gen degree}\,((\mathfrak{a}:J)/\mathfrak{a})\\
&= \text{socle degree}\, R/\mathfrak{a} - |g_1|.
\end{aligned}$$

Duality (see Lemma 1.11) gives

$$\mathfrak{a}^{[q]} : (\mathfrak{a}^{[q]} : (g_1^q, \dots, g_s^q)) = (g_1^q, \dots, g_s^q, \mathfrak{a}^{[q]})$$

in R; so the annihilator of the ideal

$$\frac{\mathfrak{a}^{[q]} : (g_1^q, \dots, g_s^q)}{\mathfrak{a}^{[q]}} \tag{7.2}$$

in the ring $R/\mathfrak{a}^{[q]}$ is $(g_1^q, \dots, g_s^q, \mathfrak{a}^{[q]})/\mathfrak{a}^{[q]}$. Apply Observation 1.4 to the ideal (7.2) to see that

$$\begin{aligned}
\operatorname{tsd}&\frac{R}{\mathfrak{a}^{[q]} : (g_1^q, \dots, g_s^q)} = \operatorname{tsd} \frac{R/\mathfrak{a}^{[q]}}{(\mathfrak{a}^{[q]} : (g_1^q, \dots, g_s^q))/\mathfrak{a}^{[q]}}\\
&= \text{socle degree}\, \frac{R}{\mathfrak{a}^{[q]}} - \text{min gen degree}\left(\operatorname{ann} \frac{\mathfrak{a}^{[q]} : (g_1^q, \dots, g_s^q)}{\mathfrak{a}^{[q]}}\right)\\
&= \text{socle degree}\, \frac{R}{\mathfrak{a}^{[q]}} - \text{min gen degree}\left(\frac{(g_1^q, \dots, g_s^q, \mathfrak{a}^{[q]})}{\mathfrak{a}^{[q]}}\right)\\
&= \text{socle degree}\, \frac{R}{\mathfrak{a}^{[q]}} - q|g_1|.
\end{aligned}$$

The R-module $R/\mathfrak{a}$ has finite projective dimension; so Observation 2.1 yields

$$\text{socle degree}\, \frac{R}{\mathfrak{a}^{[q]}} = q\, \text{socle degree}\, \frac{R}{\mathfrak{a}} - (q-1)a(R).$$

Duality gives $J = \mathfrak{a} : (g_1, \dots, g_s)$. It follows that $J^{[q]} \subseteq \mathfrak{a}^{[q]} : (g_1^q, \dots, g_s^q)$; therefore,

$$\begin{aligned}\operatorname{tsd}\frac{R}{J^{[q]}} &\ge \operatorname{tsd}\frac{R}{\mathfrak{a}^{[q]}:(g_1^q,\dots,g_s^q)} = \text{socle degree}\,\frac{R}{\mathfrak{a}^{[q]}} - q|g_1| \\ &= \text{socle degree}\,\frac{R}{\mathfrak{a}^{[q]}} - q\left(\text{socle degree}\,\frac{R}{\mathfrak{a}} - \operatorname{tsd}\frac{R}{J}\right) \\ &= q\operatorname{tsd}\frac{R}{J} + \left(\text{socle degree}\,\frac{R}{\mathfrak{a}^{[q]}} - q\,\text{socle degree}\,\frac{R}{\mathfrak{a}}\right) \\ &= q\operatorname{tsd}\frac{R}{J} - (q-1)a(R).\end{aligned}$$

The proof is complete if R is Gorenstein and F-pure. Throughout the rest of the argument, R is a complete intersection. We begin by reducing to the case where J is an irreducible ideal. Assume, for the time being, that the result has been established for irreducible ideals. Let $J = J_1 \cap \dots \cap J_n$, with each J_i irreducible. Recall that $\operatorname{tsd} R/J$ is the largest integer d with $R_d \not\subseteq J$. It follows that the $\operatorname{tsd} R/J$ is equal to the maximum of the set $\{\operatorname{tsd} R/J_k\}$. Fix a subscript k with $\operatorname{tsd} R/J = \operatorname{tsd} R/J_k$. We know that $J^{[q]} \subseteq J_k^{[q]}$; therefore,

$$q\operatorname{tsd}\frac{R}{J} - (q-1)a(R) = q\operatorname{tsd}\frac{R}{J_k} - (q-1)a(R) \le \operatorname{tsd}\frac{R}{J_k^{[q]}} \le \operatorname{tsd}\frac{R}{J^{[q]}}.$$

Henceforth, the ideal J is irreducible. Write $R = P/C$, where P is a polynomial ring and the ideal C is generated by the homogeneous regular sequence $f_1,\dots,f_c$. Let I be the pre-image of J in P. In particular, $C \subseteq I$. The rings $R/J = P/I$ and $P/I^{[q]}$ are Gorenstein, so Observation 1.4 gives

$$\text{socle degree}\,\frac{P}{I^{[q]}} - M = \operatorname{tsd}\frac{P}{I^{[q]}+C},$$

where M is the least degree among homogeneous non-zero elements of $(I^{[q]} : C)/I^{[q]}$. The P-module P/I has finite projective dimension; so Observation 2.1 yields

$$q\,\text{socle degree}(P/I) - (q-1)a(P) = \text{socle degree}(P/I^{[q]}).$$

Recall the formula $a(P/C) = a(P) + \sum_{i=1}^{c}|f_i|$. The inequality

$$q\,\text{socle degree}(P/I) - (q-1)a(P/C) \le \operatorname{tsd}\left(\frac{P}{I^{[q]}+C}\right)$$

is equivalent to the inequality

$$M \le (q-1)(|f_1| + \dots + |f_c|). \tag{7.3}$$

We establish (7.3). There exists an integer t, with $0 \le t \le c(q-1)$, such that $C^t \not\subseteq I^{[q]}$; but $C^{t+1} \subseteq I^{[q]}$. Thus, some element $f_1^{t_1}\cdots f_c^{t_c}$ of C^t, with $\sum t_i = t$ and $0 \le t_i \le q-1$ for all i, represents a non-zero element of $(I^{[q]} : C)/I^{[q]}$; therefore,

$$M \le |f_1^{t_1}\cdots f_c^{t_c}| \le (q-1)(|f_1| + \dots + |f_c|). \qquad \square$$

The next result shows that we can get most of the way through the proof of Theorem 2.3 under the assumption that R is Gorenstein and F-pure. The conclusion of Proposition 7.4 is exactly the same as the conclusion that is obtained when Proposition 5.1 is used in the proof of Theorem 2.3. Only the last step (the analogue of Proposition 6.1) is still missing.

PROPOSITION 7.4. *Let k be a field of positive characteristic p, $R \to S$ be a surjection of graded k-algebras with R Gorenstein and S artinian. Assume, in addition, that R is F-pure. Assume that $d_1 \le \cdots \le d_\ell$ are the socle degrees of S and the socle of $F^e_R(S)$ has the same dimension as the socle of S, with degrees of the generators given by $D_1 \le D_2 \le \cdots \le D_\ell$, with*

$$D_i = qd_i - (q-1)a(R),$$

for all i. Let $R = P/C$, and $S = R/IR$, with P a polynomial ring, $I \subset P$. Then we have

$$(C^{[q]} + I^{[q]}) : (C^{[q]} : C) = C + I^{[q]}.$$

Proof. Let $A = C + (x_1, \dots, x_d)$, where the images of $x_1, \dots, x_d$ are a system of parameters in R. Let $K = A : (I + C)$, so that we also have $I + C = A : K$. We have

$$(I^{[q]} + C^{[q]}) : (C^{[q]} : C) = (A^{[q]} : K^{[q]}) : (C^{[q]} : C) = (A^{[q]} : (C^{[q]} : C)) : K^{[q]}.$$

We claim that $A^{[q]} : (C^{[q]} : C) = A^{[q]} + C$. We see this by looking at the comparison map of resolutions induced by the projection $P/A^{[q]} \to P/(A^{[q]} + C)$. If $\mathbb{F}$ is the resolution of P/C, and $\mathbb{K}$ is the Koszul complex on $x_1, \dots, x_d$, then the resolution of $P/A^{[q]}$ is given by $\mathbb{F}^{[q]} \otimes \mathbb{K}^{[q]}$, the resolution of $P/(A^{[q]} + C)$ is given by $\mathbb{F} \otimes \mathbb{K}^{[q]}$, and the comparison map between them is given by the comparison map $\mathbb{F}^{[q]} \to \mathbb{F}$, tensored with $\mathbb{K}^{[q]}$. Thus, the last map is multiplication by an element of P which represents the generator of $(C^{[q]} : C)/C^{[q]}$.

It follows that $(I^{[q]} + C^{[q]}) : (C^{[q]} : C) = (A^{[q]} + C) : K^{[q]}$. It is clear that

$$(A^{[q]} + C) : K^{[q]} \supseteq I^{[q]} + C.$$

We next show that the rings defined by these ideals have the same socle degrees. Let δ and Δ be the socle degrees of the Gorenstein rings P/A and $P/(A^{[q]} + C)$, respectively. The P/C-module P/A has finite projective dimension; so

$$\Delta = q\delta - (q-1)a(P/C).$$

Let $g_1, \dots, g_s$ be elements of K which represent a minimal generating set for K/A. Observation 1.4 gives that the socle degrees of $P/(I + C)$ are $\{\delta - |g_i|\}$. So, our hypothesis tells us that the socle degrees of $P/(I^{[q]} + C)$ are

$$\{q(\delta - |g_i|) - (q-1)a(P/C)\} = \{\Delta - q|g_i|\}.$$

It is clear that $g_1^q, \ldots, g_s^q$ represents a generating set for $(K^{[q]} + C)/(A^{[q]} + C)$. Apply the hypothesis that P/C is F-pure to each of the ideals $(g_1^q, \ldots, \widehat{g_i^q}, \ldots, g_s^q, A^{[q]})$ of P/C to conclude that $(K^{[q]} + C)/(A^{[q]} + C)$ is minimally generated by $g_1^q, \ldots, g_s^q$. Observation 1.4 yields that the socle degrees of

$$\frac{P}{(A^{[q]} + C) : K^{[q]}}$$

are exactly the same as the socle degrees of $P/(I^{[q]} + C)$; therefore, Lemma 1.1 of [9] shows that

$$I^{[q]} + C = (A^{[q]} + C) : K^{[q]} = (I^{[q]} + C^{[q]}) : (C^{[q]} : C). \qquad \square$$

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