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FINE BEHAVIOR OF SYMBOLIC POWERS OF IDEALS

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This paper is written in appreciation of the many contributions of Phil Griffith to commutative algebra.

Abstract. A fundamental property connecting the symbolic powers and the usual powers of ideals in regular rings was discovered by Ein, Lazarsfeld, and Smith in 2001, and later extended by Hochster and Huneke in 2002. In this paper we give further generalizations which give better results in case the quotient of the regular ring by the ideal is F-pure or F-pure type. Our methods also give insight into a conjecture of Eisenbud and Mazur concerning the existence of evolutions. The methods used come from tight closure and reduction to positive characteristic.

1. Introduction

All given rings in this paper are commutative, associative with identity, and Noetherian. In [3], L. Ein, R. Lazarsfeld, and K. Smith discovered the following remarkable fact about the behavior of symbolic powers of ideals in affine regular rings of equal characteristic 0: if c is the largest height[1] of an associated prime of I, then $I^{(cn)} \subseteq I^n$ for all $n \geq 0$. Here, if W is the complement of the union of the associated primes of I, $I^{(t)}$ denotes the contraction of $I^t R_W$ to R, where R_W is the localization of R at the multiplicative system W. Their proof depended on the theory of multiplier ideals (see [1], [16], [17], [23], and [24] for background), including an asymptotic version, and, in particular, needed resolution of singularities as well as vanishing theorems.

Stronger results were obtained in [11, Theorem 1.1] with proofs by methods that were, in some ways, more elementary. The results of [11] are valid in both equal characteristic 0 and in positive prime characteristic p, but depend on reduction to characteristic p. The paper [11] used tight closure methods

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[1]See Discussion 1.1 and Theorem 2.1.

and, in consequence, needed neither resolution of singularities nor vanishing theorems that may fail in positive characteristic.

In this paper we use ideas related to those of [11] to prove further results in this direction that are more subtle.

E.g., in the local case the conclusion may be that a certain symbolic power of I is contained in the product of the maximal ideal with a certain other symbolic power of I. It appears to be very difficult to establish these theorems without tight closure theory or some correspondingly intricate method: we do not know whether the techniques of [3] can be used to obtain the main theorems here even in equal characteristic 0.[2]

The papers [2], [12], and [13] study the existence of evolutions, which is equivalent to the question of whether, for a prime P in a regular local ring $(R, \mathfrak{m})$, $P^{(2)} \subseteq \mathfrak{m}P$. Eisenbud and Mazur [2] ask whether this is always true in a regular local ring of equal characteristic 0. Kunz gave a counterexample in characteristic 2 which is extended to all positive characteristics in [2]: these counterexamples are codimension three primes in regular local rings of dimension four.[3] Our Theorems 3.5 and 4.2 prove that in a regular local ring containing a field, if P is a prime of codimension three, then $P^{(4)} \subseteq \mathfrak{m}P$. In dimension 3 regular local rings, codimension 2 primes are in the linkage class of a complete intersection (this is true in regular local rings of any dimension whenever the quotient by the codimension two ideal is Cohen-Macaulay), and the fact that $P^{(2)} \subseteq \mathfrak{m}P$ is established for primes in the linkage class of a complete intersection in [2]. Our Theorems 3.5 and 4.2 prove that in a regular local ring of arbitrary dimension containing a field, if P is a prime of codimension two, then $P^{(3)} \subseteq \mathfrak{m}P$. In general we are able to prove that if P is a prime of codimension c in a regular local ring containing a field, then $P^{(c+1)} \subseteq \mathfrak{m}P$ (see Theorem 3.5).

We note that to prove the most basic form of our results, all that we need to know about tight closure is the definition and the fact that, in a regular ring, every ideal is tightly closed. The results of Section 4 require more of the theory, and we refer the reader to [7]–[10], [14], and [22] for further background on tight closure theory.

Although all of our proofs initially take place in positive characteristic, relatively standard methods show that whenever, roughly speaking, the statements "make sense," corresponding results hold for rings containing a field of characteristic 0. We need a definition before stating our results.

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[2]Since we finished this research, Shunsuke Takagi [26] has announced that he can obtain similar results, and in some cases stronger results, using an interpretation of multiplier ideals via tight closure.

[3]Specifically, map $K[[x_1, x_2, x_3, x_4]] \twoheadrightarrow K[[t]]$ by sending x_1, x_2, x_3, x_4 to t^a, t^b, t^c, t^d resp. Let P be the kernel. If K has characteristic $p > 0$ and the values of a, b, c, d are p^2, $p(p+1)$, $p(p+1)+1$, and $(p+1)^2$, resp., then $f = x_1^{p+1}x_2 - x_2^{p+1} - x_1x_3^p + x_4^p \in P^{(2)} \setminus \mathfrak{m}P$.

Discussion 1.1. Our results are typically expressed in terms of a number associated with I which we call the *key number* of I. We shall give here several possible definitions of this term: the reader may choose any one of them. Our reason for proceeding in this way is that the results are sharpest for a rather technical version of "key number" based on analytic spread, but are correct and a lot less technical if one simply uses instead the largest number of generators after localizing at an associated prime of the ideal (or the largest height of an associated prime of the ideal). We note that all of the definitions that we give agree for radical ideals in regular rings.

Thus, in the sequel, the reader may use any of the following as the definition of the key number of I: the number obtained will, in general, depend on which definition is used, but the theorems will be correct for any of these numbers (or any larger number).

(1) The *key number* of the ideal I is the largest number of generators of IR_P for any associated prime P of I.
(2) The *key number* of the ideal I is the largest height (or codimension) of any associated prime P of I, where the height of P is the Krull dimension of R_P.
(3) The *key number* of the ideal I is the largest analytic spread[4] of IR_P for any associated prime P of I.

Because the analytic spread of I in a local ring R is bounded both by the Krull dimension of R and by the least number of generators of I, the notion of key number given in (3) is never larger than either of the other two. As already mentioned, the three notions coincide in the case of a radical ideal of a regular ring.

We note that by the main results of [11], if c is the key number for I, then $I^{(cn)} \subseteq I^n$ for all nonnegative integers n: see Theorem 2.1 of Section 2. On first perusal of this paper the reader may well want to focus on the case where I is radical or even prime.

It is a natural question to ask whether the result that $I^{(cn)} \subseteq I^n$ for all $n \geq 1$ can be improved, perhaps by assuming more about the singularities of R/I. The following example, which we learned from L. Ein, shows that one will not be able to improve this result too much.

Example 1.2. In $K[x_1, \ldots, x_n]$, consider the primes (x_i, x_j) for $j \neq i$. Let I denote their intersection, which is generated by all monomials consisting of

[4]For a discussion of *analytic spread* and related ideas we refer the reader to [25], and [19]. A summary of what is needed here is given in [11], §2.3. We note that when $I \subseteq (R, \mathfrak{m})$ with $(R, \mathfrak{m})$ local, the analytic spread of I is the same as the Krull dimension of the ring $(R/\mathfrak{m}) \otimes_R \mathrm{gr}_I R$, where $\mathrm{gr}_I R = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$ is the associated graded ring of R with respect to the I-adic filtration. When $R/\mathfrak{m}$ is infinite, this is the same as the smallest number of generators of an ideal $J \subseteq I$ such that I is contained in the integral closure of the ideal J.

products of $n-1$ of the variables. This is a radical ideal of pure height 2 with key number $c = 2$.

For every integer k, $x_1^k \cdots x_n^k \in I^{(2k)}$, but if $k < n-1$, it is not in I^{k+1}: since each generator omits one variable, for a product of $k+1 < n$ generators there must be some variable that occurs in all $k+1$ factors, and that variable will have an exponent of at least $k+1$ in the product. Note that the main result of [11] implies that $I^{(2k)} \subseteq I^k$.

In giving proofs in Section 3 we first give the argument in the case where the key number c is defined as in (1). We then later explain the modifications in the arguments needed for case (3). The point in case (3) is that by the introduction of an additional fixed multiplier, we can, in essence, replace I, in certain localizations, by an ideal with c generators on which it is integrally dependent. We do not need to give an argument for the case where the key number is defined as in (2), since the number defined in (3) is never larger.

The result that follows, Theorem 1.3, is a composite of Theorems 3.5, 3.6, and 4.2 containing our main results.

THEOREM 1.3. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field and let I be a proper ideal of R with key number c.*

(1) *For every $s \geq 1$, $I^{(cs+1)} \subseteq \mathfrak{m}I^{(s)}$.*
(2) *If R is characteristic p and R/I is F-pure, then $I^{(rc-1)} \subseteq II^{(r-1)}$.*

We shall also discuss a version of part (2) for local rings of finitely generated algebras over a field of characteristic 0 in Section 3: one needs to replace the notion of "F-pure" by a suitable characteristic 0 notion ("F-pure type").

We do not know how to prove (1) or (2) by elementary means, even when $(R, \mathfrak{m})$ is local and has dimension 3, $s = 1$, and $I = P$ is a prime of codimension 2: in that case (1) gives that $P^{(3)} \subseteq \mathfrak{m}P$.

Our results take a simpler form when $s = 1$ (in (1)) or $r = 2$ (in (2)) because $I^{(1)} = I$, and we make this explicit:

THEOREM 1.4. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field and let I be a proper ideal of R with key number c.*

(1) *$I^{(c+1)} \subseteq \mathfrak{m}I$.*
(2) *If R is characteristic p and R/I is F-pure, then $I^{(2c-1)} \subseteq I^2$.*

In the next section we recall the main results of [11]. Section 3 contains the proofs of our results for regular rings in positive prime characteristic, and Section 4 contains the equal characteristic 0 versions of our results.

2. Prior comparison results

The main results of [11] in all characteristics are summarized in the following theorem. Note that I^* denotes the tight closure of the ideal I. The characteristic zero notion of tight closure used in this paper is the *equational tight closure* of [10] (see, in particular, Definition (3.4.3) and the remarks in (3.4.4) of [10]). This is the smallest of the characteristic zero notions of tight closure, and therefore gives the strongest result.

Theorem 2.1. *Let R be a Noetherian ring containing a field. Let I be any ideal of R, and let c be the key number[5] of I.*

(1) *If R is regular, $I^{(cn+kn)} \subseteq (I^{(k+1)})^n$ for all positive n and nonnegative k. In particular, $I^{(cn)} \subseteq I^n$ for all positive integers n.*
(2) *If I has finite projective dimension, then $I^{(cn)} \subseteq (I^n)^*$ for all positive integers n.*
(3) *If R is finitely generated, geometrically reduced (in characteristic 0, this simply means that R is reduced) and equidimensional over a field K, and locally I is either 0 or contains a nonzerodivisor (this is automatic if R is a domain), then, with J equal to the Jacobian ideal of R over K, for every nonnegative integer k and positive integer n, we have that $J^n I^{(cn+kn)} \subseteq ((I^{(k+1)})^n)^*$ and $J^{n+1} I^{(cn+kn)} \subseteq (I^{(k+1)})^n$. In particular, we have that $J^n I^{(cn)} \subseteq (I^n)^*$ and $J^{n+1} I^{(cn)} \subseteq I^n$ for all positive integers n.*

The following result is implicit but not explicit in [11].

Theorem 2.2. *Let R be a regular ring of positive prime characteristic p. Let I be an ideal of R.*

(1) *Let c be the largest number of generators of I after localizing at an associated prime of I. Then $I^{(cnq)} \subseteq (I^{(n)})^{[q]}$ for every $n \in \mathbb{N}$ and $q = p^e$.*
(2) *Let c be the key number of I in any of the senses of Discussion 1.1. Then there is a positive integer s such that $I^s I^{(cnq)} \subseteq (I^{(n)})^{[q]}$ for every $n \in \mathbb{N}$ and $q = p^e$.*

Proof. Let W be the multiplicative system that is the complement of the union of the associated primes of I. The elements of W are not zerodivisors on any symbolic power of I, by construction of the symbolic powers, nor on any bracket power of a symbolic power of I, since the Frobenius endomorphism is flat: see, for example, Lemma 2.2(d) of [11] (and [5], [15], [20] for several related results).

(1) Thus, it will suffice to show that $I^{(cnq)} R_W \subseteq (I^{(n)})^{[q]} R_W$, i.e., that $I^{cnq} R_W \subseteq (I^n)^{[q]} R_W$. Since R_W is semilocal, it suffices to show this after localizing at a maximal ideal, and this will be a (maximal) associated prime P of I. Thus, we need only show that $(I_P)^{cnq} \subseteq \big((I_P)^n\big)^{[q]}$. Since I_P has at most c generators, say $f_1, \ldots, f_c$ (we may take some of these to be 0), by

[5]See Discussion 1.1.

definition of c, any monomial of degree cnq in these must be such that at least one of the f_j has exponent $\geq nq$, which means that it is in $\bigl((I_P)^n\bigr)^{[q]}$, and the result follows.

(2) We replace R by $R[t]$, where t is an indeterminate, and I by its expansion to this ring. The new associated primes of I are the expansions of the original associated primes. The crucial effect of this trick is that now the localization of the ring at any associated prime of the ideal has infinite residue field. We go back to our original notation and call the ring R. Then for each associated prime P of I we may choose an ideal J of R_P with c generators such that IR_P is integral over J. Then there is an integer s_P such that $I^{s_P} I^N R_P \subseteq J^N$ for every $N \in \mathbb{N}$. Choose s to be at least the maximum of these finitely many s_P. We shall show that $I^s I^{(cnq)} \subseteq (I^{(n)})^{[q]}$. Since no element of W is a zerodivisor on $(I^{(n)})^{[q]}$, it suffices to show this after localizing at W and hence after localizing at each of the associated primes P of I. We replace R by R_P and I by IR_P. But then all we need to show is that $I^s I^{cnq} \subseteq (I^n)^{[q]}$, and we have that $I^s I^{cnq} \subseteq J^{cnq} \subseteq (J^n)^{[q]}$ (exactly as in part (1), since J has c generators), and this is contained in $(I^n)^{[q]}$. □

3. The main results for characteristic p regular rings

Let R be a regular domain of positive characteristic $p > 0$, let I be an ideal of R, let c be the key number for I, and let r be an integer with $r \geq 2$. We define a sequence of ideals as follows:

Definition 3.1. Fix an integer k, $0 \leq k \leq r(c-1)$. Set $I_{0,k} = I$, and inductively set $I_{n,k} = \bigl(I_{n-1} I^{(r-1)} : I^{(rc-k)}\bigr)$. Set $J_k = \bigcup_j I_{j,k}$. We will usually simplify notation when k is fixed and write $I_n = I_{n,k}$.

Observe that I_j form an increasing sequence of ideals (depending on k). Since $k \leq r(c-1)$ it follows that $rc - k \geq rc - r(c-1) = r$ and thus $I_{n-1} I^{(rc-k)} \subseteq I_{n-1} I^{(r-1)}$, and $I_{n-1} \subseteq I_{n-1} I^{(rc-k)} : I^{(r-1)} = I_n$. Hence $J_k = I_N$ for all large N.

Theorem 3.2. *Let R be a regular domain of positive characteristic $p > 0$, let I be an ideal of R, let c be the key number for I, and let r be an integer with $r \geq 2$. For all $q = p^e$, and any integer k, $0 \leq k \leq r(c-1)$,*

$$I^{(q((n+1)k-nc))} \subseteq I_{n+1}^{[q]}.$$

Proof. We first assume that key number is defined as in Discussion 1.1(1) and prove the case $n = 0$. We need to prove that

$$I^{(kq)} \subseteq \bigl(II^{(r-1)} : I^{(rc-k)}\bigr)^{[q]}.$$

Since we are in a regular ring,

$$\bigl(II^{(r-1)} : I^{(rc-k)}\bigr)^{[q]} = (II^{(r-1)})^{[q]} : (I^{(rc-k)})^{[q]}.$$

Suppose that $u \in I^{(qk)}$ and $v \in I^{(rc-k)}$. It will suffice to show that $uv^q \in (II^{(r-1)})^{[q]}$. Evidently, we may assume that $v \neq 0$.

Say $b \geq rc - k$. Fix any power Q of p. We shall show that $v^b(uv^q)^Q \in \big((II^{(r-1)})^{[q]}\big)^{[Q]}$. Since b is independent of Q, this shows that $uv^q \in \big((II^{(r-1)})^{[q]}\big)^*$ (the tight closure), and since R is regular, this will establish that $uv^q \in (II^{(r-1)})^{[q]}$.

By the division algorithm, $qQ = (a-1)(rc-k) + \rho$, where $a - 1 \in \mathbb{N}$ and $0 \leq \rho < rc - k$. Then $\rho + b \geq rc - k$. Clearly

$$(*) \qquad a(rc-k) \geq qQ.$$

Then

$$v^b(uv^q)^Q = u^Q v^{b+qQ} \in (u^Q v^{ac} v^{a\left((r-1)c-k\right)}) = (v^{ac})(u^Q v^{a((r-1)c-k)}).$$

Now, since $v \in I^{(rc-k)}$, we have that

$$v^{ac} \in I^{((rc-k)ac)} \subseteq I^{(qQc)} \subseteq I^{[qQ]}$$

by Theorem 2.2, and since $u \in I^{(qk)}$, we find that

$$u^Q v^{a((r-1)c-k)} \subseteq I^{(qQk+(rc-k)a((r-1)c-k))},$$

and, using $(*)$, the exponent is $\geq qQk + qQ\big((r-1)c-k\big) = qQ(r-1)c$. Thus,

$$u^Q v^{a((r-1)c-k)} \subseteq I^{(qQ(r-1)c)} \subseteq (I^{(r-1)})^{[qQ]}$$

by Theorem 2.2. Multiplying, we have that

$$v^b(uv^q)^Q \in I^{[qQ]}(I^{(r-1)})^{[qQ]} = (II^{(r-1)})^{[qQ]} = \big((II^{(r-1)})^{[q]}\big)^{[Q]},$$

as required.

We next describe the changes needed in the argument in the case where the key number is defined as in Discussion 1.1(3). If $I = (0)$, there is nothing to prove. If not, choose s as in Theorem 2.2(b) and let y be any nonzero element of I. The argument is almost the same, but we show instead that $y^{2s}v^b(uv^q)^Q \in \big((II^{(r-1)})^{[q]}\big)^{[Q]}$ for all Q, which again allows us to conclude that $uv^q \in (II^{(r-1)})^{[q]})^*$. In the final paragraph of the argument we get that $y^s v^{ac} \subseteq y^s I^{(qQc)} \subseteq I^{[qQ]}$ by Theorem 2.2(b), and, similarly, that $y^s u^Q v^{a((r-1)c-k)} \subseteq I^{(qQ(r-1)c)} \subseteq (I^{(r-1)})^{[qQ]}$ by Theorem 2.2(b). We can multiply: the argument is otherwise unchanged. This completes the case $n = 0$.

We now assume the result for $n - 1$ and prove it for n. This induction works for every choice of key number as defined in Discussion 1.1.

We need to prove that

$$I^{(q((n+1)k-nc))} \subseteq I_{n+1}^{[q]} = (I_n I^{(r-1)} : I^{(rc-k)})^{[q]}.$$

Suppose that $u \in I^{(q((n+1)k-nc)}$ and $v \in I^{(rc-k)}$. It will suffice to show that $uv^q \in (I_n I^{(r-1)})^{[q]}$. Evidently, we may assume that $v \neq 0$.

First suppose that $c \le k$. Then $(n+1)k - nc = k + n(k-c) \ge k$ and the result follows from the case $n = 0$, since by that case

$$I^{(q((n+1)k-nc))} \subseteq I^{(qk)} \subseteq (II^{(r-1)} : I^{(rc-k)})^{[q]} \subseteq (I_n I^{(r-1)} : I^{(rc-k)})^{[q]}.$$

We can therefore assume that $c > k$. Fix any power Q of p. By the division algorithm,

$$(c-k)qQ = (a-1)(rc-k) + \rho,$$

where $a - 1 \in \mathbb{N}$ and $0 \le \rho < rc - k$. We shall show that $v^\rho (uv^q)^Q \in \big((I_n I^{(r-1)})^{[q]}\big)^{[Q]}$ for all Q. Since ρ is independent of Q, this shows that $uv^q \in \big((I_n I^{(r-1)})^{[q]}\big)^*$ (the tight closure), and since R is regular, this will establish that $uv^q \in (I_n I^{(r-1)})^{[q]}$. Note that $a(rc-k) \ge (c-k)qQ$. We break the product into two terms:

$$v^\rho (uv^q)^Q = (u^Q v^a) \cdot v^{\rho + qQ - a}.$$

Now, since $v \in I^{(rc-k)}$, we have that

$$v^a \in I^{((rc-k)a)} \subseteq I^{((c-k)qQ)}$$

and hence

$$u^Q v^a \in I^{((c-k)qQ)} I^{(qQ((n+1)k-nc))} \subseteq I^{(qQ(nk-(n-1)c)} \subseteq I_n^{[qQ]}.$$

The last step follows from our induction.

Observe that

$$\begin{aligned}(\rho + qQ - a)(rc-k) &\ge qQ(rc-k) - ((a-1)(rc-k) + \rho)) \\ &= qQ(rc-k) - (c-k)qQ = qQ(r-1)c,\end{aligned}$$

so that

$$v^{\rho+qQ-a} \in I^{((r-1)cqQ)} \subseteq (I^{(r-1)})^{[qQ]}$$

by Theorem 2.2.

Hence

$$v^\rho (uv^q)^Q = (u^Q v^a) \cdot v^{\rho+qQ-a} \in (I_n I^{(r-1)})^{[qQ]},$$

which proves that $uv^q \in (I_n I^{(r-1)})^*$ and finishes the proof of the theorem. ☐

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic $p > 0$, let I be an ideal of R, and let c be the key number.*[6] *Then either $I^{(rc-1)} \subseteq II^{(r-1)}$ for every $r \ge 2$ or $I^{(q)} \subseteq \mathfrak{m}^{[q]}$ for every $q = p^e$.*

Proof. Take $k = 1$ and apply Theorem 3.2 to the ideal I_1. Either $I_1 = R$, in which case $I^{(rc-1)} \subseteq II^{(r-1)}$, or else it is a proper ideal, in which case $I^{(q)} \subseteq I_1^{[q]} \subseteq \mathfrak{m}^{[q]}$. ☐

[6] See Discussion 1.1.

Remark 3.4. The second of the alternative conclusions is a very strong form of the Zariski-Nagata theorem[7], which asserts that $P^{(N)} \subseteq \mathfrak{m}^N$ for every prime of the regular local ring $(R, \mathfrak{m})$. For $q = p^e$, $\mathfrak{m}^{[q]}$ tends to be quite a bit smaller than $\mathfrak{m}^q$. Note that the second of the alternate conclusions does imply that the Zariski-Nagata theorem holds even when N is not a power of p. For suppose $u \in P^{(N)}$. For any $q \geq N$ we have $q = aN + b$, where $a = \lfloor \frac{q}{N} \rfloor$ and $0 \leq b < N$. Then $u^b u^a \in P^{(q)} \subseteq \mathfrak{m}^q$, which shows that $(b+a)\operatorname{ord}_{\mathfrak{m}} u \geq Na+b$ for all $q \geq N$, and so $\operatorname{ord}_{\mathfrak{m}} u \geq (Na+b)/(b+a) = (N + \frac{b}{a})/(1 + \frac{b}{a})$. Since $b < N$ while $a \longrightarrow \infty$ as $q \longrightarrow \infty$, this is $> N - 1$ for $q \gg 0$. Thus, $\operatorname{ord}_{\mathfrak{m}} u \geq N$, i.e., $u \in \mathfrak{m}^N$.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic p, let I be an ideal of R, and let c be the key number for I. For all $s \geq 1$,*

$$I^{(cs+1)} \subseteq \mathfrak{m} I^{(s)}.$$

Proof. We claim that $J_{c-1} = R$. By Theorem 3.2,

$$I^{(q((n+1)(c-1)-nc))} \subseteq I_{n+1}^{[q]}$$

for all n. But for all large n, $(n+1)(c-1) - nc = c - n + 1 < 0$, and hence $I_{n+1} = R$. Since $I_{n+1} \subseteq J_{c-1}$ we obtain $J_{c-1} = R$.

Choose n least with the property that $I_n = R$ (for $k = c - 1$). Note that $I_0 = I \neq R$, so that $I_{n-1} \neq R$. But then $R = I_n = I_{n-1} I^{(r-1)} : I^{(rc-c+1)}$ implies that

$$I^{(rc-c+1)} \subseteq I_{n-1} I^{(r-1)} \subseteq \mathfrak{m} I^{(r-1)}.$$

Setting $s = r - 1$ gives the conclusion of the theorem. □

Theorem 3.6. *Let R be a regular ring of positive characteristic $p > 0$, let I be a proper ideal of R, let $r \geq 2$ be an integer, and let c be the key number for I. If R/I is F-pure, then $I^{(rc-1)} \subseteq I I^{(r-1)}$.*

Proof. The proof reduces at once to the local case, so we may assume that $(R, \mathfrak{m})$ is a regular local ring. If $I^{(c-1)} \subseteq I I^{(r-1)}$ we are done, since $rc - 1 \geq c - 1$. Thus, we may assume that $I^{(c-1)} \not\subseteq I I^{(r-1)}$.

By Fedder's criterion for F-purity (cf. [4]), $I^{[q]} : I \not\subseteq \mathfrak{m}^{[q]}$. If $I^{(rc-1)} \not\subseteq I I^{(r-1)}$, then we may set $k = 1$ in Theorem 3.2. But then $I^{(kq)} = I^{(q)} \subseteq (I I^{(r-1)} : I^{(rc-1)})^{[q]} \subseteq \mathfrak{m}^{[q]}$, a contradiction, for then $I^{(q)} \subseteq \mathfrak{m}^{[q]}$, while $I^{[q]} : I \not\subseteq \mathfrak{m}^{[q]}$. □

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[7] The theorem is equivalent to Theorem 1 of [6], where Zariski's proof is given; for Nagata's proof see [18, p. 143].

4. The main results in characteristic 0

In this section we give the extensions of the various positive characteristic results to the equal characteristic case. As mentioned in Section 2, the notion of tight closure that we use here is that of equational tight closure from [10, Sections 3.4.3–4]. The main results of this section are contained in Theorem 4.2 below. We need to do some groundwork before we can prove that theorem, however. The proof of the main results depends on three steps: one is to localize and complete, the second is to descend from the complete case to the affine case, and the third is to use reduction to positive characteristic in the affine case. We follow the same path as was done in detail in [11]. To state the main theorem we need the definition of F-pure type.

Definition 4.1. Let R be a ring which is finitely generated over a field k of characteristic 0. The ring R is said to be of *F-pure type* if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq k$ and a finitely generated A-algebra R_A which is free over A such that $R \cong R_A \otimes_A K$ and such that for all maximal ideals μ in a Zariski dense subset of $\operatorname{Spec}(A)$, with $\kappa = A/\mu$, the fiber rings $R_A \otimes_A \kappa$ are F-pure.

Theorem 4.2. *Let R be a regular ring containing a field of characteristic* 0*, and let I be a proper ideal of R with key number c.*

(1) *If R is local with maximal ideal $\mathfrak{m}$, then for all $s \geq 1$,*

$$I^{(cs+1)} \subseteq \mathfrak{m} I^{(s)}.$$

(2) *If R is of finite type over a field of equal characteristic* 0 *and R/I is of F-pure type, then $I^{(rc-1)} \subseteq II^{(r-1)}$.*

Proof. We prove a stronger statement to prove (1), which does not require the ring to be local. Recall the definition of I_j, which makes sense in all characteristics: Fix an integer k, $0 \leq k \leq r(c-1)$. Set $I_0 = I$, and inductively set $I_n = \left(I_{n-1}I^{(r-1)} : I^{(rc-k)}\right)$. Set $J_k = \bigcup_j I_j$.

Recall that I_j form an increasing sequence of ideals (depending on k), since $I_{n-1}I^{(rc-k)} \subseteq I_{n-1}I^{(r-1)}$ as $k \leq (r-1)c$, so that $rc - k \geq rc - r(c-1) = r$. Hence $J_k = I_N$ for all large N.

We replace (1) by

$$J_{c-1} = R. \tag{$1'$}$$

We first prove that ($1'$) implies (1). Assume ($1'$) and assume that R is local with maximal ideal $\mathfrak{m}$. The proof is entirely similar to the proof of Theorem 3.5 in positive characteristic. We let $k = c-1$, and consider the ascending chain of ideal $\{I_j\}$. By ($1'$), for some large n, $I_n = R$. Choose n least with this property. Note that $I_0 = I \neq R$, so that $I_{n-1} \neq R$ and is a proper ideal. But then $R = I_n = I_{n-1}I^{(r-1)} : I^{(rc-c+1)}$ implies that

$$I^{(rc-c+1)} \subseteq I_{n-1}I^{(r-1)} \subseteq \mathfrak{m}I^{(r-1)}.$$

Setting $s = r-1$ gives (1).

We prove ($1'$) for finitely generated algebras over a field K of characteristic 0, and at the same time prove (2). Assume there is a counterexample in either of these cases. We use the standard descent theory of Chapter 2 of [10] to replace the field K by a finitely generated Z-subalgebra A, so that we have a counterexample in an affine algebra R_A over A with $R_A \subseteq R$ and $R \cong K \otimes_A R_A$. In particular, R_A will be reduced. In doing so we descend I to an ideal I_A of R_A as well as the ideals and their prime radicals in its primary decomposition. We define the ideals I_{jA} and J_{kA} in a similar manner. In case ($1'$), we have the stable ideal J_{kA}, which is the union of the ideals I_{jA}, and we are assuming that $1 \notin J_{(c-1)A}$. In case (3) we have an element u_A that fails to satisfy the containment we are trying to prove. Since R is regular, we can localize at a nonzero element of A to make R_A smooth over A. In either case, we can localize at a nonzero element of A to make A smooth over $\mathbb{Z}$.

For variable maximal ideals of A, we denote the residue field by κ. Notice that as we pass to fibers $\kappa \longrightarrow R_\kappa = R_A \otimes_A \kappa$ we may assume that each minimal prime P_A of I_A becomes a radical ideal whose minimal primes in R_κ are all of the same height as the original. Thus, in the fiber, the primary decomposition of I_κ may have more components, but each of these will be obtained from the image of one of the original components by localization. The biggest analytic spread after localizing at an associated prime will not change. The ring R_κ will be regular, and in part (2) we know that for a dense set of maximal ideals of A, the fiber R_κ/I_κ is F-pure. Both (2) and ($1'$) now follow since both results hold in characteristic p for a dense set of closed fibers.

We now consider the general case for (1). The problem reduces to the local case, and then it suffices to prove the required containment after completion. Although $I\widehat{R}$ may have more associated primes, Discussion 1.1 shows that the biggest analytic spread as one localizes at these cannot increase.

Since R is regular, note that for all integers j, $\widehat{I}^{(j)} = I^{(j)}\widehat{R}$, so that $(\widehat{I}^{(j)})^n = (I^{(j)})^n\widehat{R}$. (The associated primes of $\widehat{R}/I^{(j)}\widehat{R}$ are among those associated to $\widehat{R}/P\widehat{R}$ for some associated prime P of I, by Proposition 15 in IV B.4 of [21], since any associated prime of $I^{(j)}$ must be an associated prime of I, and by another application of Proposition 15 in IV B.4 of [21] these in turn are associated primes of $I\widehat{R}$.) Thus,

$$I^{(cs+1)} \subseteq \widehat{I}^{(cs+1)} \subseteq \widehat{\mathfrak{m}}\widehat{I}^{(s)} \cap R = \mathfrak{m}I^{(s)}\widehat{R} \cap R = \mathfrak{m}I^{(s)},$$

as required, by the faithful flatness of $\widehat{R}$ over R.

We may now use Theorem 4.3 of [11] to descend to a suitable affine algebra over a coefficient field for the complete local ring, and the results follow from what we have already proved in the affine case. This reduction is entirely similar to the reduction done in [11], and we refer the reader to that paper for full details. □

Remark 4.3. We still do not know the best possible conclusion, even in codimension two. For example, in $K[x, y, z]$, for homogeneous primes $P_1, \dots, P_s$ of height two, we do not know whether or not $(P_1 \cap \cdots \cap P_s)^{(3)} \subseteq (P_1 \cap \cdots \cap P_s)^2$. All evidence suggests this is always true, but we have not been able to extend our methods to decide this question.

It is also possible that for regular local rings $(R, \mathfrak{m})$ in all characteristics $I^{(cs)} \subseteq \mathfrak{m}I$, where c is the key number for I. If so, this would prove the Eisenbud-Mazur conjecture (even in positive characteristic) for prime ideals of codimension two. It would also show that the counterexamples in positive characteristic are sharp.

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