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ON EFFICIENT GENERATION OF PULL-BACK OF $T_{\mathbb{P}^n}(-1)$

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This paper is written in honour of Phil Griffith on his retirement. The author would like to take this opportunity to express his respect for Phil's deep insight into homological methods in commutative algebra.

Abstract. Let $f : X \to \mathbb{P}^n$ be a proper map such that dimension of $f(X) \geq 2$. We address the following question: Is $\dim H^o(X,\, f^*(T_{\mathbb{P}^n}(-1))$ $= n+1$? We provide an affirmative answer under standard mild restrictions on X. We also point out that this provides an affirmative answer to a similar question raised via regular alteration of a closed subvariety in a blow-up of a regular local ring at its closed point in the mixed characteristics.

Introduction

The problem that we are going to address in this note came up in the course of studying intersection multiplicity over the blow-up of a regular local ring at its closed point. Let (R, m, K) be a regular local ring of dimension n essentially of finite type over a field or over an excellent discrete valuation ring with residue field $R/m = K$. Assume K to be algebraically closed. Write $X = \operatorname{Spec} R$, $\tilde{X}$ its blow-up at the closed point $s = [m]$. Let q be a prime ideal of R. Let $\tilde{Z}$ denote the blow-up of $Z = \operatorname{Spec}(R/q)$ at its closed point and let $\tilde{Z}_s$ denote the fiber over s. Assume that $\dim Z \geq 3$. Consider the pair $(\tilde{Z},\, \tilde{Z}_s)$. Let (W, W_s) be a regular alteration [J] of $(\tilde{Z},\, \tilde{Z}_s)$, i.e., W is a regular scheme, $\pi : W \to \tilde{Z}$ is a dominant projective morphism, $\dim W = \dim \tilde{Z}$, and $\pi^{-1}(\tilde{Z}_s) = W_s$ is a non-reduced strict normal crossing divisor in W, i.e., reduced part of W_s is a strict normal crossing divisor. Write $\phi = \pi\big|_{(W_s)_{\mathrm{red}}}$, $\mathcal{H} = \Omega_{\mathbb{P}}(1)$, $\mathbb{P} = \mathbb{P}_K^{n-1}$ and let $\mathcal{H}^\vee$ denote the dual of $\mathcal{H}$, i.e., $\mathcal{H}^\vee = T_{\mathbb{P}}(-1)$.

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Consider the exact sequence ($\mathbb{P} = \mathbb{P}^{n-1}$)

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to \bigoplus_{1}^{n} \mathcal{O}_{\mathbb{P}} \to \mathcal{H}^{\vee} \to 0. \tag{1}$$

Write $\pi^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_W(1)$, $(W_s)_{\mathrm{red}} = \widetilde{W}$ and let

$$0 \to \mathcal{O}_{\widetilde{W}}(-1) \to \bigoplus_{1}^{n} \mathcal{O}_{\widetilde{W}} \to \mathcal{H}^{\vee} \otimes \mathcal{O}_{\widetilde{W}} \to 0 \tag{2}$$

denote the pull-back of (1) via ϕ. We want to address the following question.

Question. *Is* $\dim H^0(\widetilde{W}, \mathcal{H}^{\vee} \otimes \mathcal{O}_{\widetilde{W}}) = n$*?*

An affirmative answer to the above question is the main focus of the following theorem.

Theorem 1. *With the above set-up,*

$$H^1\left(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(-1)\right) \to H^1(\widetilde{W}, \bigoplus_{1}^{n} \mathcal{O}_{\widetilde{W}})$$

is injective.

We reduce the proof of the above theorem to the following theorem. The main tool of this reduction is Zariski's Main Theorem (EGA III).

Theorem 2. *Let $\tilde{W}$ be a reduced connected scheme of finite type over an algebraically closed field K and let $g : \tilde{W} \to \mathbb{P}^n = \mathbb{P}$ be a proper map. Let $W_1, \ldots, W_d$ be the irreducible components of $\tilde{W}$. Assume that, for $1 \leq i$, $j \leq d$, (i) $\dim g(W_i) \geq 2$ and (ii) if $W_i \cap W_j \neq \phi$, then $\dim g(W_i \cap W_i) \geq 1$. Consider the exact sequence*

$$o \to \mathcal{O}_{\tilde{W}}(-1) \to \bigoplus_{1}^{n+1} \mathcal{O}_{\tilde{W}} \to \mathcal{H}^{\vee} \otimes \mathcal{O}_{\tilde{W}} \to 0,$$

constructed as above. Then

$$H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \to H^1(\tilde{W}, \bigoplus_{1}^{n+1} \mathcal{O}_{\tilde{W}})$$

is injective.

If $\dim g(\widetilde{W}) = 1$, the above theorem is not valid. For examples of such situations we refer the reader to [He].

I would like to thank L. Ein, R. Hartshorne, M. Hochster, and S. Katz for helpful comments. I usually don't work on this kind of question and I am rather unaware of the chronology of works in this area. I apologize in advance for my short and possibly incomplete list of references.

Section 1

We first prove Theorem 2 as stated above in the following steps.

Proposition 1. *Let $\tilde{W}$ be a variety over K and let $g : \tilde{W} \to \mathbb{P}^n_K = \mathbb{P}$ be a finite map. Assume that $\dim g(\tilde{W}) \geq 2$. Then there exist hyperplane sections $\tilde{W}_L = g^{-1}(L) \cap \tilde{W}$ of $\tilde{W}$ such that $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}_L}) = K$.*

Proof. Suppose first $\operatorname{ch} K = 0$. In this case, by Bertini's Theorem, there exist hyperplane sections L such that $\tilde{W}_L$ is a closed subvariety of $\tilde{W}$. Since g is proper, we have the required result.

Now suppose $\operatorname{ch} K = p > 0$. Let (W', h) be the normalization of $\tilde{W}$ and let g' denote the composition of $W' \xrightarrow{h} \tilde{W} \xrightarrow{g} \mathbb{P}$. Hence g' is finite and $g'^*\mathcal{O}_{\mathbb{P}}(t) = \mathcal{O}_{W'}(t)$ is very ample for $t \gg 0$. Hence W' is projective. Let $q = p^r$, $r \gg 0$. Then $\mathcal{O}_{W'}(q)$ is very ample. Let $f : W' \to W'$ denote the Frobenius map induced by $x \to x^p$ on $\mathcal{O}_{W'}$ and let f^q denote f iterated q times. By Bertini's Theorem, there exist hyperplane sections $\tilde{W}_L$ and W'_L such that they are irreducible. Consider the short exact sequences

$$0 \to \mathcal{O}_{W'}(-1) \to \mathcal{O}_{W'} \to \mathcal{O}_{W'_L} \to 0$$

and

$$0 \to \mathcal{O}_{W'}(-q) \to \mathcal{O}_{W'} \to \mathcal{O}_{W'_{L}q} \to 0,$$

where the bottom one is obtained from the top one via f^q (the bottom sequence is also exact independently by itself). Since W' is a normal projective variety, $H^i(W', \mathcal{O}_{W'}(-q)) = 0$ for $i \leq 1$ and $q \gg 0$; thus $H^0(W', \mathcal{O}_{W'_{L}q}) = K$. Due to normality, $\mathcal{O}_{W'_L} \to \mathcal{O}_{W'_{L}q}$ (induced by f^q) is an injection. Hence $H^0(W', \mathcal{O}_{W'_L}) = K$. Since $h : W' \to \tilde{W}$ is finite and birational, L can be so chosen that $h_L : W'_L \to \tilde{W}_L$ is also the same and $\mathcal{O}_{\tilde{W}_L} \hookrightarrow h_*(\mathcal{O}_{W'_L})$ is an injection (Remark 3.4.11, [F-O-V]). Hence $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}_L}) = K$. □

Proposition 2. *Let $\tilde{W}$ be a variety over K and $g : \tilde{W} \to \mathbb{P}^n_K = \mathbb{P}$ be a finite map. Write $g^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_{\tilde{W}}(1)$. Consider the exact sequence*

$$(*) \qquad 0 \to \mathcal{O}_{\tilde{W}}(-1) \to \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}} \to \mathcal{H}^\vee \otimes \mathcal{O}_{\tilde{W}} \to 0,$$

which is the pull-back via g of the exact sequence

$$(**) \qquad 0 \to \mathcal{O}_{\mathbb{P}}(-1) \to \bigoplus_1^{n+1} \mathcal{O}_{\mathbb{P}} \to \mathcal{H}^\vee \to 0.$$

If $\dim g(\tilde{W}) \geq 2$, then $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \to H^1(\tilde{W}, \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}})$ is injective.

Proof. By the previous Lemma, we have a hyperplane section $\tilde{W}_L$ such that $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}_L}) = K$. Consider the short exact sequence

$$0 \to \mathcal{O}_{\tilde{W}}(-1) \xrightarrow{\ell} \mathcal{O}_{\tilde{W}} \to \mathcal{O}_{\tilde{W}_L} \to 0.$$

Since $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}) = K$, $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \xrightarrow{\tilde{\ell}} H^1(\tilde{W}, \mathcal{O}_{\tilde{W}})$ is injective.

Let L be given by $\sum_{i=0}^{n} a_i X_i = 0$ in $\mathbb{P}$, $a_o \neq 0$. Let $u : K^{n+1} \to K^{n+1}$ denote the isomorphism defined by $u(e_0) = a_0 e_0$, $u(e_i) = e_i + a_i e_0$ for $i > 0$. Let $\varphi : \mathbb{P} \to K$ denote the structure map. Then $\varphi^*(u) : \bigoplus_1^{n+1} \mathcal{O}_{\mathbb{P}} \to \bigoplus_1^{n+1} \mathcal{O}_{\mathbb{P}}$ is an isomorphism and this leads to an isomorphism $\tilde{\varphi}(u) : \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}} \to \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}}$.

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\tilde{W}}(-1) \arrow[r] \arrow[ddr, "\ell"'] & \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}} \arrow[d, "\wr\ \tilde{\varphi}(u)"] \\ & & \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}} \arrow[d, "\text{projection onto the first component.}"] \\ & & \mathcal{O}_{\tilde{W}} \end{tikzcd}$$

Since $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \xrightarrow{\tilde{\ell}} H^1(\tilde{W}, \mathcal{O}_{\tilde{W}})$ is injective, so is $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \to H^1(\tilde{W}, \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}})$.

Proposition 3. *Let $\tilde{W}$ be a reduced connected scheme of finite type over K and let $g : \tilde{W} \to \mathbb{P}^n_K = \mathbb{P}$ be a finite map. Let $W_1, \ldots, W_d$ denote the components of $\tilde{W}$. Assume that* $\dim g(W_i) \geq 2$, $1 \leq i \leq d$ *and* $\dim g(W_i \cap W_j) \geq 1$, *whenever $W_i \cap W_j \neq \phi$.*

Consider the exact sequence (constructed as in (*))

$$0 \to \mathcal{O}_{\tilde{W}}(-1) \to \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}} \to \mathcal{H}^\vee \otimes \mathcal{O}_{\tilde{W}} \to 0.$$

Then $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \to H^1(\tilde{W}, \bigoplus_1^{n+1} \mathcal{O}_{\tilde{W}})$ is injective.

Proof. By the previous Lemma and Bertini's Theorem, there exist generic hyperplane sections L of $\mathbb{P}$ such that W_{iL} is irreducible and $H^0(W_{iL}, \mathcal{O}_{W_{iL}}) = K$ for $1 \leq i \leq d$, and $W_{iL} \cap W_{jL}$ is non-empty whenever $W_i \cap W_j$ is non-empty.

Consider the exact sequence

$$0 \to \mathcal{O}_{(W_i \cup W_j)} \to \mathcal{O}_{W_i} \oplus \mathcal{O}_{W_j} \to \mathcal{O}_{W_i \cap W_j} \to 0.$$

L can be so chosen that

$$0 \to \mathcal{O}_{W_{iL} \cup W_{jL}} \to \mathcal{O}_{W_{iL}} \oplus \mathcal{O}_{W_{jL}} \to \mathcal{O}_{W_{iL} \cap W_{jL}} \to 0$$

is exact. Hence $H^0(\tilde{W}, \mathcal{O}_{W_{iL} \cup W_{jL}}) = K$. Let $t \in \{1, \ldots, d\}$ be such that $t \neq i, j$ and $(W_i \cup W_j) \cap W_t \neq \phi$. Consider the exact sequence

$$0 \to \mathcal{O}_{W_i \cup W_j \cup W_t} \to \mathcal{O}_{W_i \cup W_j} \bigoplus \mathcal{O}_{W_t} \to \mathcal{O}_{(W_i \cup W_j) \cap W_t} \to 0$$

L can be so chosen that

$$0 \to \mathcal{O}_{W_{iL} \cup W_{jL} \cup W_{tL}} \to \mathcal{O}_{W_{iL} \cup W_{jL}} \bigoplus \mathcal{O}_{W_{tL}} \to \mathcal{O}_{(W_{iL} \cup W_{jL}) \cap W_{tL}} \to 0$$

is also exact. Hence $H^0(\tilde{W}, \mathcal{O}_{W_{iL} \cup W_{jL} \cup W_{tL}}) = K$.

Since $\tilde{W}$ is reduced and connected, proceeding in the above manner, after a finite number steps we obtain $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}_L}) = K$.

Now consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{W}}(-1) \to \mathcal{O}_{\tilde{W}} \to \mathcal{O}_{\tilde{W}_L} \to 0$$

and proceed as in the proof of the previous proposition to complete the proof. □

THEOREM 2. *Let $\tilde{W}$ be a reduced connected scheme of finite type over K and let $g : \tilde{W} \to \mathbb{P}^n_K = \mathbb{P}$ be a proper map. Let $W_1, \ldots, W_d$ be the irreducible components of $\tilde{W}$. Assume that, for $1 \leq i, j \leq d$, (i) $\dim g(W_i) \geq 2$ and (ii) if $W_i \cap W_j \neq \phi$, $\dim g(W_i \cap W_j) \geq 1$. Consider the exact sequence*

$$0 \to \mathcal{O}_{\tilde{W}}(-1) \to \bigoplus_{1}^{n+1} \mathcal{O}_{\tilde{W}} \to \mathcal{H}^\vee \otimes \mathcal{O}_{\tilde{W}} \to 0$$

(constructed as in (∗)).

Then $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(-1)) \to H^1(\tilde{W}, \bigoplus_{1}^{n+1} \mathcal{O}_{\tilde{W}})$ is injective.

Proof. Using Grothendieck's Theorem of Connection (Theorem 4.3.1, EGA III) we reduce the problem to the case where g is finite. The proof now follows from Proposition 2. □

Section 2

We are now ready to prove Theorem 1.

Proof of Theorem 1. Note that $\phi : \widetilde{W} \to \tilde{Z}_s$ does not necessarily satisfy the hypothesis in the statement of Theorem 2. Recall that $\pi : W \to \tilde{Z}$ is a regular alteration such that $W_s = \pi^{-1}(\tilde{Z}_s)$ is a non-reduced strict normal crossing divisor with irreducible components $W_1, \ldots, W_d$. This means that for any non-empty subset $\alpha \subset \{1,\dots,d\}$, the closed subscheme $W_\alpha = \bigcap_{i\in\alpha} W_i$ is a regular subscheme and hence a smooth ($K = \overline{K}$) subscheme of codimension $\#\alpha$ in W. Since R is regular local, $\tilde{Z}_s = \operatorname{Proj}(\operatorname{Gr}_m(R/q))$ is equidimensional. Let (W', h) be the normalization of $\tilde{Z}$ in $k(W)$. Hence we have a commutative diagram

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \arrow[dl, "h"] \\ & \tilde{Z} & \end{tikzcd} \tag{3}$$

where h is a finite map (EGA II, 6.3.9). Since π is proper, f is also proper. $\pi_*\mathcal{O}_W = h_* f_* \mathcal{O}_W$. Since $f_*\mathcal{O}_W$ is a coherent $\mathcal{O}_{W'}$ algebra and W' is the normalization of $\tilde{Z}$ in $k(W)$, we have $\mathcal{O}_{W'} = f_*\mathcal{O}_W$. Thus $\pi_*\mathcal{O}_W = h_*\mathcal{O}_{W'}$. By Theorem (4.3.1), EGA III, for all $w' \in W'$, $\overline{f}^{1}(w')$ is connected and non-empty. Since $\dim W = \dim \tilde{Z}$, by Zariski's Main Theorem (Proposition 4.4.1, EGA III), if $V = \{w \in W \mid w \text{ is isolated in } \pi^{-1}(\pi(w))\}$, then V is non-empty open in W and $f\big|_V$ is an isomorphism of V onto an open subset of V' in W'; moreover, $f^{-1}(V') = V$.

Claim. *If $y \in W'$ is such that $\dim \mathcal{O}_{W',y} = 1$, then $y \in V'$.*

Proof. Let x be a closed point in $f^{-1}(y)$. Then, by the dimension formula, $\dim \mathcal{O}_{W,x} = \dim \mathcal{O}_{W,'y} + \operatorname{tr}_{k(W')} k(W) - \operatorname{tr}_{k(y)} k(x) = \dim \mathcal{O}_{W,'y}$ $(k(W') = k(W)) = 1$. Since f is birational and $f^{-1}(y)$ is connected, $f^{-1}(y) = \{x\}$. Thus $y \in V'$.

From (3), by taking the fiber over the closed point in Z, we obtain another commutative diagram

$$\begin{tikzcd} W_s \arrow[rr, "f_s"] \arrow[dr, "\pi_s"'] & & W'_s \arrow[dl, "h_s"] \\ & \tilde{Z}_s & \end{tikzcd}$$

Recall that $\tilde{Z}$ is the blow-up of Z at $\{m/q\}$. We have an exact sequence

$$\mathcal{O} \to \mathcal{O}_{\tilde{Z}}(1) \to \mathcal{O}_{\tilde{Z}} \to \mathcal{O}_{\tilde{Z}_s} \to 0.$$

This leads to exact sequences

$$\mathcal{O} \to \mathcal{O}_W(1) \to \mathcal{O}_W \to \mathcal{O}_{W_s} \to 0$$

and

$$\mathcal{O} \to \mathcal{O}_{W'}(1) \to \mathcal{O}_{W'} \to \mathcal{O}_{W'_s} \to 0;$$

here $\pi^*(\mathcal{O}_{\tilde{Z}}(1)) = \mathcal{O}_W(1)$ and $h^*(\mathcal{O}_{\tilde{Z}}(1)) = \mathcal{O}_{W'}(1)$.

Applying f_* to the top sequence, we obtain an exact sequence

$$0 \to f_*\mathcal{O}_W(1) \to f_*\mathcal{O}_W \to f_*\mathcal{O}_{W_s}.$$

Since $f_*\mathcal{O}_W = \mathcal{O}_{W'}$, we obtain an injection $\mathcal{O}_{W'_s} \hookrightarrow f_*\mathcal{O}_{W_s}$ from the above sequence. Our claim shows that for every irreducible component of W'_s there exists a unique irreducible component of W_s which maps birationally onto it via the proper map f_s induced by f. Since components of W'_s are varieties over K and W_s is equidimensional, this implies that W'_s is equidimensional. Moreover, since W_s is a non-reduced strict normal crossing divisor, if W'_i and W'_j are any two components of W'_s such that $W'_i \cap W'_j \neq \phi$, then $\dim\left(W'_i \cap W'_j\right) \geq 1$. (Recall that $\dim Z \geq 3$.) To see this one may use the following observations: (a) W_s is equidimensional, $\dim(W_i \cap W_j) \geq 1$; and (b) if $\{x_1, \ldots, x_r\}$ is a finite set of points contained in an open subset U of $\operatorname{Spec} A$, then there exists an $f \in A$ s.t. $\{x_1, \ldots, x_n\} \subset \operatorname{Spec} A_f \subset U$. Since h_s is finite, $\dim h_s\left(W'_i \cap W'_j\right) \geq 1$. W_s is connected, hence so is W'_s.

Let $\widetilde{W}$, $\widetilde{W}'$ denote $W_{s\,\mathrm{red}}$ and $W'_{s\,\mathrm{red}}$, respectively, and let $\tilde{f} = f_s\big|_{\widetilde{W}} : \widetilde{W} \to \widetilde{W}'$. Then $\tilde{f}$ is a proper surjective map. We can factorize $\tilde{f}$ as follows (Theorem 4.3.1, EGA III):

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{g} & B \\ & \tilde{f} \searrow & \downarrow h' \\ \theta \downarrow & & \widetilde{W}' \\ & & \downarrow \theta' \\ W_s & \xrightarrow{f_s} & W'_s \end{array}$$

where h' is finite, $\mathcal{O}_B = g_*\mathcal{O}_{\widetilde{W}}$, and θ and θ' are the natural injections (homeomorphisms). For every $y \in \widetilde{W}'$, $\tilde{f}^{-1}(y)$ is connected. Hence, by the Theorem of Connection mentioned above, h' is a homeomorphism. By construction, there exists a dense open subset $U'(= V' \cap \widetilde{W}')$ such that if $\widetilde{U} = \tilde{f}^{-1}(U')$, then $\tilde{f}\big|_{\widetilde{U}} : \widetilde{U} \to U'$ is an isomorphism. By Zariski's Main Theorem (EGA III, Proposition 4.4.1), there exists a dense open subset U of B such that both $g\big|_{\widetilde{U}} : \widetilde{U} \to U$ and $h'\big|_U : U \to U'$ are isomorphisms. Moreover, by construction and our claim, each component of B has a non-empty intersection with U. Thus B is reduced, connected and equidimensional. Note that h', restricted to any component of B, is a birational homeomorphism onto a component of W'_s.

Let B_i be any irreducible component of B and let W_i and W'_i be the corresponding components of $\widetilde{W}$ and W'_s such that $g\big|_{W_i}$ maps W_i birationally onto B_i and B_i is birationally homeomorphic to W'_i (via $\theta' \cdot h'$). Since $\dim h_s(W'_i) \geq 2$, $\dim h_s \cdot \theta' \cdot h'(B_i) \geq 2$ and since $\dim h_s(W'_i \cap W'_j) \geq 1$ whenever $W'_i \cap W'_j \neq \phi$, $\dim h_s \cdot \theta' \cdot h'(B_i \cap B_j) \geq 1$ whenever $B_i \cap B_j \neq \phi$.

Write $\psi = h_s \cdot \theta' \cdot h'$ and $\mathcal{O}_B(1) = \psi^*(\mathcal{O}_{\mathbb{P}}(1))$. Let

$$\mathcal{O} \to \mathcal{O}_B(-1) \to \bigoplus_1^n \mathcal{O}_B \to \mathcal{H}^\vee \otimes \mathcal{O}_B \to 0$$

be the exact sequence obtained by pulling back (1) via ψ. Then, by Theorem 2,

$$H^1(B, \mathcal{O}_B(-1)) \to H^1(B, \bigoplus_1^n \mathcal{O}_B)$$

is injective. This implies that

$$H^0(B, \bigoplus_1^n \mathcal{O}_B) \xrightarrow{\sim} H^0\left(B, \mathcal{H}^\vee \otimes \mathcal{O}_B\right).$$

Since $g_*\left(\mathcal{O}_{\widetilde{W}}\right) = \mathcal{O}_B$ and $g_*\left(\mathcal{H}^\vee \otimes \mathcal{O}_{\widetilde{W}}\right) = \mathcal{H}^\vee \otimes \mathcal{O}_B$, the above isomorphism implies, via (2), that

$$H^0\big(\widetilde{W}, \bigoplus_1^n \mathcal{O}_{\widetilde{W}}\big) \xrightarrow{\sim} H^0\big(\widetilde{W}, \mathcal{H}^\vee \otimes \mathcal{O}_{\widetilde{W}}\big).$$

Hence $H^1\big(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(-1)\big) \to H^1\big(\widetilde{W}, \bigoplus_1^n \mathcal{O}_{\widetilde{W}}\big)$ is injective and the proof is complete. $\square$

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