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APOLARITY AND COVARIANT FORMS

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Dedicated to Phillip Griffith on the occasion of his retirement

ABSTRACT. This paper examines the existence of apolar covariants of order greater than or equal to n. These are shown to exist for all $n > 2$.

1. Introduction

This paper examines the concept of the apolarity of binary forms. A form g of order m is apolar to the form f of order $n \leq m$ if g is in the kernel of the linear mapping from forms of order m to forms of order $m - n$ defined by the n-th transvectant (Überschiebung) of the form f. The transvectant of two covariant forms (forms invariant under the action of the special linear group of 2×2-matrices) is again a form that is a covariant. This construction is central to the determination by Gordan [9] of the finite generation of the rings of covariant binary forms and to new approaches to finite generation [3].

The vanishing of the transvectant and more generally the value of the transvectant of two binary n-ics has been a subject of renewed interest in recent years. As can be seen from Theorem 3.1, it is closely connected with the Waring problem for forms [4], [6], [7], [11], [12], [13]. The question of the existence of apolar covariant forms is one that has not been adequately addressed by previous work on the topic. Examples of apolar covariant forms that were explicitly known in the classical literature are limited. A form of odd degree is apolar to itself [8, §207]. A form of odd order $(2n-1)$ is apolar to its canonizant which is of order n [8, §206–207]. Lastly the binary form of order 4 is apolar to its covariant of order 6 [10, §89].

This note focuses on forms of order $m \geq n$ that are apolar to the standard form of order n. We call such forms *major apolar covariants* to distinguish them from the covariants of smaller order. These arose from considerations of the construction of covariants of the binary n-ic from those of the binary $(n-1)$-ic (see [2]).

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2. Definitions

The transvectant has played an important role in classical invariant theory [5], [8], [14] at least since the middle of the 19th century. It is related to the Poisson bracket.

In this paper $\Bbbk$ is a field of characteristic zero. Suppose that A is an algebra over the field $\Bbbk$. A form in $A[x, y]$ is a homogeneous polynomial in $A[x, y]$.

Definition 2.1. The Poisson bracket of two forms $f, g \in A[x, y]$ is denoted by $P(f, g)$ and is defined by

$$P(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Additional notations used for this bracket are $T(f, g), (f, g), [f, g]$, among others. Interest will be focused on the iterations of the operator. These are denoted variously as $P^{(n)}(f, g), T^{(n)}(f, g), (f, g)_n, [f, g]_n$.

For fixed f the bracket is a derivation in the second variable and it is linear in each variable.

There is an important expansion that will be needed in the computations that follow.

Proposition 2.2. *Suppose f, g are forms in $A[x, y]$. Then*

$$P^{(r)}(f,g) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{\partial^r f}{\partial x^{r-j}\partial y^j} \frac{\partial^r g}{\partial x^j \partial y^{r-j}}$$

This can be demonstrated by induction. The expansion is found in many places—for example in [14].

In particular, it follows that

Proposition 2.3.

$$P^{(r)}(f,g) = (-1)^r P^{(r)}(g,f).$$

If f is a form of degree n and g is a form of degree m, then $P^{(r)}(f, g)$ is a form of degree $n + m - 2r$. In the literature on invariant theory, this form is called the rth *transvectant* of the forms f, g.

The results to be found below will be focused on some very specific expressions of representations of the additive group.

Definition 2.4. The *standard form* (of degree n) in the polynomial ring $\Bbbk[a_0, a_1, \ldots, a_n, x, y]$ is the form

$$f_n(a_0,\ldots,a_n,x,y) = \sum_{i=0}^{n} \binom{n}{i} a_i x^{n-i} y^i. \tag{2.1}$$

Suppose

$$g(x,y) = B_0 x^m + \binom{m}{1} B_1 x^{m-1} y + \cdots + \binom{m}{j} B_j x^{m-j} y^j + \cdots + B_m y^m$$

with coefficients $B_i \in \Bbbk[a_0, \dots, a_n]$ and where $m \geq n$. Then

$$\begin{aligned} P^{(n)}(f_n, g) &= \frac{n!m!}{(m-n)!} \sum_{j=0}^{n} \sum_{s=0}^{m-n} (-1)^j \binom{n}{j} \binom{m-n}{s} a_j B_{s+n-j} x^{m-n-s} y^s \\ &= \frac{n!m!}{(m-n)!} \sum_{s=0}^{m-n} \binom{m-n}{s} \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_j B_{n+s-j} \right) x^{m-n-s} y^s. \end{aligned}$$

The n-th transvectant of the standard form f_n with

$$g = (x - by)^m$$

can be computed using Proposition 2.2. Because

$$\frac{\partial^n (x-by)^m}{\partial x^j \partial y^{n-j}} = \frac{m!}{(m-n)!} (-1)^{n-j} b^{n-j} (x - by)^{m-n}$$

one obtains using Proposition 2.2

$$\begin{aligned} P^{(n)}(f_n, (x-by)^m) &= (-1)^n \frac{n!m!}{(m-n)!} \left(\sum_{j=0}^{n} \binom{n}{j} a_j b^{n-j} \right) (x-by)^{m-n} \\ &= (-1)^n \frac{n!m!}{(m-n!)} f_n(a_0, a_1, \dots, a_n, b, 1)(x-by)^{m-n}. \end{aligned}$$

In particular, if b is a root of f_n, then the transvectant

$$P^{(n)}(f_n, (x-by)^m) = 0. \tag{2.2}$$

Definition 2.5. A form $g \in \Bbbk[a_0, \dots, a_n, x, y]$ of degree m is *apolar* to the form f of degree n (with $n \leq m$) if

$$P^{(n)}(f, g) = 0.$$

Definition 2.6. A covariant form g of degree $m \geq n$ that is apolar to the standard form f_n will be called a *major apolar covariant form.*

In particular, the canonizant of a form of odd order is not to be considered as an major apolar covariant form in this sense.

3. Apolarity and sums of powers

The key result that characterizes the forms apolar to the form f is given by the following theorem whose proof has its foundation in the proof of [10].

THEOREM 3.1. *Let $f = \sum_{i=1}^{n} \binom{n}{i} a_i x^{n-i} y^i$ be a form of degree n with roots $(\xi_1 : 1), \dots, (\xi_n : 1)$ in a splitting field $\Bbbk[\xi_1, \dots, \xi_n]$ of the form f. A form g of degree m greater than n with coefficients in $k[\xi_1, \dots, \xi_n]$ is apolar to f (i.e., the transvectant $(f, g)_n = 0$) if and only if there exist $c_i \in k[\xi_1, \dots, \xi_n]$ such that*

$$g = \sum_{i=1}^{n} c_i (x - \xi_i y)^m.$$

Proof. First consider the forms $(x - \xi_i y)^m$. By (2.2) these forms are all apolar to the standard form f_n. It suffices to show that these form a basis of the vector space of apolar forms of degree m.

From the fullness of rank of the matrix

$$\begin{pmatrix} a_0 & -\binom{n}{1}a_1 & \cdots & (-1)^n a_n & 0 & 0 & 0 \\ 0 & a_0 & -\binom{n}{1}a_1 & \cdots & (-1)^n a_n & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & -\binom{n}{1}a_1 & \cdots & (-1)^n a_n \end{pmatrix},$$

the dimension of apolar forms of degree m is exactly n so it will suffice to show that the forms $(x - \xi_i y)^m$ are linearly independent. This follows immediately from the distinctness of the roots of f and the consequent non-vanishing of the associated Vandermonde determinant. □

4. Theorems and examples

Although the condition of being both a covariant form and an apolar form may seem rare, there are several families of examples, both classical and new, that should be noted. The first example follows immediately from Proposition 2.3.

PROPOSITION 4.1. *If n is odd, then the standard form is an apolar covariant form.*

Simple computations show that for the standard cubic there is no other apolar covariant form. Considering the quintic, there are *two* independent forms that are apolar: the form f_5 and the Hessian (which is up to a scalar $P^{(2)}(f_5, f_5)$).

PROPOSITION 4.2. *The Hessian of the standard form f_5 is a major apolar covariant of f_5.*

Proof. The Hessian of the standard quintic is the covariant of degree 2 and order 6. The coefficient of x^6 of the Hessian is the $\mathbb{G}_a$-invariant $a_0 a_2 - a_1^2$. Consider the 5-th transvectant of the standard quintic and the Hessian. If this transvectant is a non-zero form, it is a form of degree 3 and order 1. But the quintic has no covariant forms of degree smaller than 5 of order 1, as can be seen in the table found in [10, §116]. □

This does however not lead to a general family of major apolar covariants, as is the consequence of the following result which was the starting place for our investigation of major apolar covariants.

Proposition 4.3 ([2]). *The Hessian $P^{(2)}(f_n, f_n)$ is not an apolar covariant for any form of order $n > 5$.*

Proof. It suffices to show that $E_n = P^{(n)}(f_n, P^{(2)}(f_n, f_n)) \neq 0$ for the special case that $f_n = x^n + xy^{n-1}$. By a direct computation using Proposition 2.2, one obtains

$$E_n = -2\,(n!)^2(n-1)^2\left[\binom{2n-4}{n-4} + (-1)^n(n-3)(n-2)\right]y^{n-4}.$$

It is thus clear that $E_n \neq 0$ for n even and greater than 4.

For odd $n \geq 5$, let

$$\alpha_n = \binom{2n-4}{n-4}, \qquad \beta_n = (n-3)(n-2).$$

For $n = 5$, either direct computation or the use of Proposition 4.2 gives the ratio $\alpha_5/\beta_5 = 1$.

Observe that

$$\frac{\alpha_{n+2}/\beta_{n+2}}{\alpha_n/\beta_n} = 16\frac{(n-\frac{1}{2})}{n+1}\frac{(n-\frac{3}{2})}{n+2} \geq 16 \cdot \frac{3}{4} \cdot \frac{1}{2} = 6.$$

It follows that if n is odd and greater than 5, then $\alpha_n/\beta_n > 1$. Hence $E_n \neq 0$ for $n > 5$. □

This will be the subject of further discussion in [2]. These were the only major apolar covariants of forms of odd order that were noticed classically. That there are others is the content of the following proposition.

Proposition 4.4. *If $n \geq 7$ is odd, then the covariant $\chi_n = P^{(2)}(f_n, P^{(n-3)}(f_n, f_n))$ is a non-zero major apolar covariant of order $n+2$ and degree three.*

Proof. The transvectant $P^{(n)}(\chi, f_n)$ is nominally a covariant of degree four and order 2 of the binary n-ic. It is required to show that this is zero. This is done by showing that the dimension of the vector space of covariants of degree four and order 2 the binary n-ic is zero. By Hermite reciprocity [8, §131] this is equivalent to showing that the dimension of the vector space of covariants of degree n and order 2 of the binary quartic is zero. The generators of the ring of covariants of the binary quartic have orders zero, four, and six [10, §89]. Therefore the binary quartic has no covariants of order two.

It is left as an exercise to the reader to confirm that $\chi_n \neq 0$ for $n \geq 7$ by evaluating the special case where $f_n = x^n + x^2y^{n-2}$. □

The situation is even less well known for standard forms for even degree. Classically the only major apolar covariant that appears to have been (explicitly) known is the sextic covariant of the quartic [10, §89 p. 94]. This form is (up to sign) denoted by t in [10] and by G in [8]. This covariant is the Jacobian (first transvectant) of the standard form f_4 with the Hessian. The form has as the coefficient of x^6 the $\mathbb{G}_a$-invariant $a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$.

For every even $n = 2k > 2$, there is a covariant of degree 3 and order $n+2$ whose leading coefficient is of the form $a_0^2a_{n-1} + \cdots$ [8, §165(bis)]. For the standard sextic form, this covariant is called $C_{3,8}$ in [8, §244] or the Jacobian of f_6 and the fourth transvectant of f_6 with itself in [10, §134].

Proposition 4.5. *For n even and greater than 2, the covariant of degree 3 and order $n+2$ of the standard form f_n associated with the $\mathbb{G}_a$-invariants arising from the minimal degree protomorphs described in* [8, §165(bis)] *is a major apolar covariant.*

Proof. This covariant ξ_n is the one obtained as $\psi = P^{(1)}(P^{(n-2)}(f_n, f_n), f_n)$. The transvectant $P^{(n)}(\xi_n, f_n)$ is by construction a covariant of degree 4 and order 2 of the binary n-ic. By the same reasoning as in the proof of Proposition 4.4 the transvectant vanishes. □

There are additional apolar covariants that are not of the type described in Proposition 4.5 as can be seen from the following result.

Proposition 4.6. *The binary octavic has a major apolar covariant of degree four and order ten.*

Proof. From either the description of the covariants of the octavic in [1] or from the Cayley-Sylvester formula given in [15, Corollary 4.2.8], there are the following facts. The dimension of the vector space $V_{8;4,10}$ of covariants of the binary octavic of degree four and order ten is two. The dimension of the vector space $V_{8;5,2}$ of the binary octavic of covariants of degree five and order two is one.

The 8-th transvectant with the standard form of order eight defines a transformation from $V_{8;4,10}$ to $V_{8;5,2}$. This mapping therefore has a non-trivial kernel and hence there is a major apolar covariant of degree four and order ten. □

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