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THE LOEWNER AND HADAMARD VARIATIONS

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ABSTRACT. We give an explicit formula relating the infinitesimal generators of the Loewner differential equation and the Hadamard variation. This is applied to establish an extension of the Hadamard variation to the case of arbitrary simply-connected domains and to prove the existence of Loewner chains with arbitrary smooth initial generator starting at an arbitrary univalent function which is sufficiently smooth up to the boundary. As another application of this method, we show that every subordination chain f_t is differentiable almost everywhere and satisfies a Loewner equation, without assuming that $f_t'(0)$ is continuous.

1. Introduction

The Hadamard (or Julia) variational formula for Green's function is obtained by varying the boundary of a sufficiently smooth domain along its normal by an amount of fixed sign which varies from point to point. One thus obtains a chain of domains of increasing or decreasing size, and a corresponding variational formula. On the other hand, in the case of simply connected domains the Loewner differential equation describes continuously increasing or decreasing families of domains using subordination chains of the corresponding normalized conformal maps [5]. Since Hadamard variation can reach essentially arbitrary nearby domains, it is natural to expect a relation between the two variational methods.

In this paper, we relate the Hadamard and Loewner variations in an explicit way in terms of their infinitesimal generators. For instance, we obtain a connection between these two variational methods for the case that the Loewner chain is sufficiently smooth, see Theorem 1 below. For the proof, we use a generalization of the Hadamard variational formula to arbitrary homotopies which was recently derived in [9].

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This explicit relation between the two variational methods makes it possible to study Hadamard variation from the viewpoint of Loewner's theory and vice versa. In Section 2.2, we focus on one direction of this relation and use the Loewner equation to give a generalization of the Hadamard formula to the case of arbitrary simply-connected domains and a wider class of perturbations. In Section 3 and Section 4, we take the opposite point of view and investigate the Loewner differential equation using Hadamard variation. In Section 3, we derive in this way an existence theorem for the Loewner partial differential equation, which says that given sufficiently smooth initial function f_0 and initial generator p of positive real part, there is a Loewner chain f_t so that in the Loewner partial differential equation

$$\frac{\partial f_t}{\partial t}(z) = z p_t(z) \frac{\partial f_t}{\partial z}(z)$$

we have that $p_0 = p_t$. In other words, one can specify not just the initial function (as is well known) but also the initial direction. In the final Section 4, we prove a strengthening of Pommerenke's extension [6], [7] of the Loewner method, where we do not require continuity of the first derivative of the mappings f_t. This is achieved by applying a variational formula for Green's function due to Heins [2], which is closely related to Hadamard variation.

2. Relation between the Hadamard and Loewner variations

2.1. Outward variations: the case of smoothly bounded initial domain. A conformal map $f : \mathbb{D} \to \mathbb{C}$ defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1|\}$ is called a normalized Riemann map if $f(0) = 0$ and $f'(0) > 0$. Let $f_t : \mathbb{D} \to \mathbb{C}$, $t \in [0,T]$, be normalized Riemann maps such that $f_s \prec f_t$ for $0 \le s \le t \le T$ (that is, $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$). Then

$$f_t(z) = \alpha(t) z + \cdots,$$

where $\alpha : [0,T] \to (0,\infty)$ is a monotonically increasing function. The subordination chain f_t is called a normalized subordination chain [6] or a Loewner chain [7] if

$$\alpha(t) = e^t.$$

Let f_t be a Loewner chain defined on the interval $[0,T]$ for some T.

Definition 1. We say that

$$F : [a,b] \times [0,L] \to \mathbb{C} \tag{1}$$

is a "C^m injective homotopy of closed curves" if F is injective on $[a,b] \times [0,L)$, $F(t,0) = F(t,L)$ for all $t \in [a,b]$, and F has a C^m extension to an open set containing $[a,b] \times \mathbb{R}$ which is L periodic in the second variable. We say that a subordination chain f_t defined on the interval $[a,b]$ is C^m on $[a,b]$ if f_t has

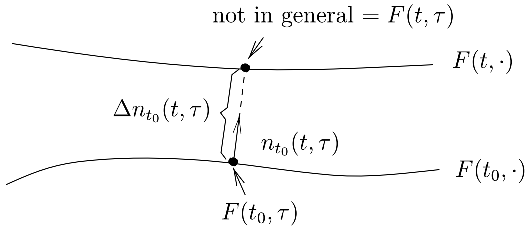

FIGURE 1. Definition 2.

a continuous injective extension to $\overline{\mathbb{D}}$ for each $t \in [a,b]$ and the corresponding injective homotopy

$$F : [a,b] \times [0,2\pi] \to \mathbb{C}, \\ (t,\tau) \mapsto f_t(e^{i\tau})$$

is C^m.

REMARK 1. The "closed curves" in the above terminology are of course the curves $\tau \mapsto F(t,\tau)$ for fixed t.

Up to first order, any sufficiently smooth homotopy behaves like a Hadamard variation. To make this precise, we need to make some definitions. Consulting Figure 1 may be helpful.

DEFINITION 2. Let $F : [a,b] \times [0,L]$ be a C^2 injective homotopy of closed curves. Let $n_t(\tau)$ denote the unit outward normal to $F(t,\cdot)$ at τ. For sufficiently small $t - t_0$, let $\Delta n_{t_0}(t,\tau)$ be the distance from $F(t_0,\tau)$ to the curve $F(t,\cdot)$ along the normal $n_{t_0}(\tau)$. Define

$$\nu_{t_0}(\tau) = \frac{d}{dt}\bigg|_{t=t_0} \Delta n_{t_0}(t,\tau).$$

It is intuitively clear that for small enough $t - t_0$, $\Delta n_{t_0}(t,\tau)$ is well defined, and hence ν_{t_0} is well defined. Proofs can be found in [9].

REMARK 2. It will sometimes be convenient to write $\nu_{t_0}(u)$ for $\nu_{t_0}(\tau)$ where u is the complex variable $u = F(t_0,\tau)$ parameterizing the boundary of $f_{t_0}(\mathbb{D})$. Similarly, $n(u)$ or n_u will denote the unit outward normal at u, etc.

Let g_t denote Green's function of the domain $f_t(\mathbb{D})$. One would expect that the first-order variation of g_{t_0} should behave as though the homotopy were in fact a variation along the normal lines by the amount $(t-t_0)\nu_{t_0}(\tau)$ at each point $F(t_0,\tau)$. (This is because the variation in the direction tangent to the boundary does not change the domain up to first order). More precisely, we have that [9, Theorem 1]

$$g_t(z,w) - g_{t_0}(z,w) = \frac{1}{2\pi}(t-t_0)\int_{\partial D_t} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,w)\nu_{t_0}(u)\,ds_u + O(|t-t_0|^2), \tag{2}$$

where for convenience we denote $\nu_{t_0}(u) = \nu_{t_0}(\tau)$ for $u = F(t_0,\tau)$ along the boundary of D_{t_0}, s_u is arc length in the u variable, and n_u denotes the unit outward normal at u. The remainder term is understood to be $O(|t-t_0|^2)$ uniformly on compact subsets of D_{t_0} in both z and w. Furthermore, the remainder term is harmonic. Differentiating

$$\dot{g}_t(z,w) = \frac{1}{2\pi}\int_{\partial D_t} \frac{\partial g_t}{\partial n_u}(u,z)\frac{\partial g_t}{\partial n_u}(u,w)\nu_t(u)\,ds_u. \tag{3}$$

Let $\mathcal{P}$ denote the set of holomorphic functions p defined on $\mathbb{D}$ satisfying $p(0)=1$ and $\mathrm{Re}(p)>0$. We now give an expression for ν_t in terms of the generator $p_t \in \mathcal{P}$ appearing in the Loewner equation.

Theorem 1. *Let f_t be a C^2 Loewner chain on $[a,b]$, and let p_t be the infinitesimal generator in the Loewner partial equation*

$$\dot{f}_t(z) = zp_t(z)f_t'(z).$$

For $t_0 \in [a,b)$, if s denotes arc length along the boundary of $f_{t_0}(\mathbb{D})$, then for the homotopy $F(t,\tau) = f_t(e^{i\tau})$ we have

$$\nu_{t_0}(u) = -\,\mathrm{Re}\left(\frac{1}{i}\frac{f_{t_0}^{-1}(u)}{f_{t_0}^{-1\prime}(u)} p_{t_0}\circ f_{t_0}^{-1}(u)\frac{d\bar{u}}{ds}\right).$$

Proof. Fix $u \in \partial f_{t_0}(\mathbb{D})$ and let $z = f_{t_0}^{-1}(u)$. Define $x(t) = u + \Delta n_{t_0}(u,t) \times n_{t_0}(u)$. We claim that

$$\lim_{t\to t_0} \mathrm{Re}\left(\frac{f_t(z)-x(t)}{t-t_0}\overline{n_{t_0}(u)}\right) = 0;$$

that is

$$\lim_{t\to t_0}\frac{f_t(z)-x(t)}{t-t_0} \tag{4}$$

is in the direction of the tangent to ∂D_{t_0} at $u(s)$. To see this, by Definition 2,

$$\begin{aligned}\lim_{t\to t_0}\frac{x(t)-f_{t_0}(z)}{t-t_0} &= \lim_{t\to t_0}\mathrm{Re}\left(\frac{\Delta n_{t_0}(z,t)}{t-t_0}n_{t_0}(u)\right)\\ &= \nu_{t_0}(u)n_{t_0}(u).\end{aligned}$$

Since,

$$\lim_{t\to t_0}\frac{f_t(z)-f_{t_0}(z)}{t-t_0} = \dot{f}_{t_0}(z)$$

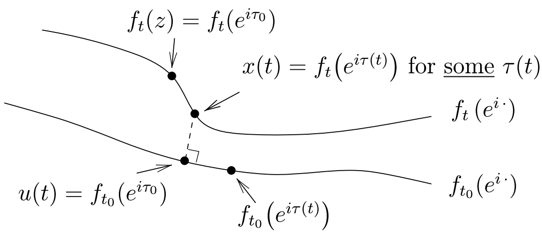

FIGURE 2. Proof of Theorem 1.

it follows that the limit (4) exists. Next, we set $z = e^{i\tau_0}$, and observe that $x(t) = f_t(e^{i\tau(t)})$ for some $\tau(t)$ (see Figure 2). Since the homotopy is C^2, it follows that

$$\lim_{t\to t_0} \frac{f_t(e^{i\tau_0}) - f_{t_0}(e^{i\tau_0})}{t-t_0} = \lim_{t\to t_0} \frac{f_t(e^{i\tau(t)}) - f_{t_0}(e^{i\tau(t)})}{t-t_0} = \dot{f}_{t_0}(e^{i\tau_0}).$$

Thus by the existence of the limit (4), we may rearrange the terms above to get

$$\lim_{t\to t_0} \frac{f_t(z) - x(t)}{t-t_0} = \lim_{t\to t_0} \frac{f_{t_0}(e^{i\tau_0}) - f_{t_0}(e^{i\tau(t)})}{t-t_0}$$

which is clearly in the direction of the tangent to $\partial f_{t_0}(\mathbb{D})$. This proves the claim.

Thus,

$$\begin{aligned}\operatorname{Re}(\dot{f}_{t_0}(z)\overline{n_{t_0}(u)}) &= \lim_{t\to t_0} \operatorname{Re}\left(\frac{f_t(z) - f_{t_0}(z)}{t-t_0}\overline{n_{t_0}(u)}\right)\\ &= \operatorname{Re}\left(\left(\frac{f_t(z) - x(t)}{t-t_0} + \frac{x(t) - f_{t_0}(z)}{t-t_0}\right)\overline{n_{t_0}(u)}\right)\\ &= \nu_{t_0}(u).\end{aligned}$$

The lemma now follows from the observation that the outward unit normal is given by

$$n_{t_0}(u) = \frac{1}{i}\frac{du}{ds}. \qquad \square$$

Thus, we have the following extension of the Hadamard variational formula.

COROLLARY 1. *Let f_t, p_t, etc. be as in Theorem 1. Let $j_t(z) = zp_t(z)$. We have*

$$\dot{g}_t(z,w) = \operatorname{Re}\left(\frac{2}{\pi i}\int_{\partial D_t} \frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w) j_t \circ f_t^{-1}(u) f_t' \circ f_t^{-1}(u)\, du\right).$$

Proof. Since g is constant along the boundary,

$$\operatorname{Im}\left(\frac{1}{i}\frac{\partial g_t}{\partial u}\frac{du}{ds}\right) = 0$$

so

$$\frac{\partial g_t}{\partial n} = \frac{2}{i}\frac{\partial g_t}{\partial u}\frac{du}{ds}.$$

Thus,

$$\frac{\partial g_t}{\partial n_u}(u,z)\frac{\partial g_t}{\partial n_u}(u,w) = -4\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)\left(\frac{du}{ds}\right)^2$$

and this expression is real. The claim follows from (2) and the fact that $|du/ds| = 1$. □

2.2. Extensions of the Hadamard variational formula. In this section, we clarify to what extent the Loewner equation for the Riemann map provides an extension of the Hadamard variational formula for Green's function.

We first note that one can easily derive a variational formula for Green's function from the Loewner partial differential equation.

THEOREM 2. *Let f_t be a Loewner chain on $[0,T]$, and $p_t \in \mathcal{P}$ be the corresponding generator (measurable in t) in the Loewner partial differential equation $\dot{f}_t(z) = zp_t(z)f_t'(z)$. If g_t is Green's function of $f_t(\mathbb{D})$, then for almost all $t \in [0,T]$*

$$\dot{g}_t(z,w) = -2\,\mathrm{Re}\left(\frac{\partial g_t}{\partial z}(z,w)j_t \circ f_t^{-1}(z)f_t' \circ f_t^{-1}(z) + \frac{\partial g_t}{\partial w}(z,w)j_t \circ f_t^{-1}(w)f_t' \circ f_t^{-1}(w)\right). \tag{5}$$

Proof. Green's function in terms of f_t is

$$g_t(z,w) = -\log\left|\frac{f_t^{-1}(z) - f_t^{-1}(w)}{1 - \overline{f_t^{-1}(w)}f_t^{-1}(z)}\right|. \tag{6}$$

Differentiate and apply the Loewner equation. □

Evaluating the integral in Corollary 1 results in the formula above. It is natural to ask whether Theorem 2 can be given in a form closer to the Hadamard variational formula with a suitable interpretation of the integral. This is easily done as follows.

THEOREM 3. *If f_t, p_t and g_t satisfy the hypotheses of Theorem 2 then*

$$\dot{g}_t(z,w) = \lim_{r\to\infty} \mathrm{Re}\left(\frac{2}{\pi i}\int_{\gamma_r} \frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)j_t \circ f_t^{-1}(u)f_t' \circ f_t^{-1}(u)\,du\right),$$

where γ_r is the hyperbolic circle of radius r centred on 0 in $f_t(\mathbb{D})$.

Proof. Since $\partial g/\partial u$ is holomorphic in u with a simple pole at z (resp. w), for all r large enough the above integral can be evaluated and equals the expression for $\dot{g}_t$ in Theorem 2. □

It is clear that Theorem 1 also holds for solutions to the inwardly directed Loewner partial differential equation

$$\dot{f}_t = -zp_t(z)f_t'(z), \tag{7}$$

so long as $F(t,\tau) = f_t(e^{i\tau})$ is a C^2 injective homotopy. One simply changes the sign of the formula:

$$\nu_{t_0}(u) = \operatorname{Re}\left(\frac{1}{i}\frac{f_{t_0}^{-1}(u)}{f_{t_0}^{-1\prime}(u)}p_{t_0}\circ f_{t_0}^{-1}(u)\frac{d\bar{u}}{ds}\right).$$

Equation (7) was considered by Friedland and Schiffer [3] and is sometimes called the time-reversed Loewner equation or the Friedland–Schiffer equation. They established the existence of solutions for any $p_t \in \operatorname{ext}\mathcal{P}$ measurable in t where

$$\operatorname{ext}\mathcal{P} = \left\{\frac{1+\kappa z}{1-\kappa z}\;\middle|\;|\kappa| = 1\right\}$$

and for any holomorphic initial function $f_0(z)$ on the disc. The solutions are of the form $f_t(z) = f_0(g_t(z))$ where $g_t(z)$ is a bounded univalent function on $\mathbb{D}$ satisfying $g_0(z) = z$ and $g_t'(0) = e^{-t}$. In particular, if f_0 is univalent, then f_t can be thought of as an inwardly directed Loewner chain with initial function f_0. However, their existence proof does not rely in any way on the fact that p is of the above form, and holds for any $p_t \in \mathcal{P}$ which is measurable in t. A proof can also be found in [8].

Theorems 2 and 3 thus clearly hold with a change of sign. That is, let f_t be a solution to equation (7) on the interval $[0,T]$. For almost all $t \in [0,T]$, we have that

$$\begin{aligned}\dot{g}_t(z,w) &= -\lim_{r\to\infty}\operatorname{Re}\left(\frac{2}{\pi i}\int_{\gamma_r}\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)j_t\circ f_t^{-1}(u)f_t'\circ f_t^{-1}(u)\,du\right)\\ &= 2\operatorname{Re}\bigg(\frac{\partial g_t}{\partial z}(z,w)j_t\circ f_t^{-1}(z)f_t'\circ f_t^{-1}(z)\\ &\qquad + \frac{\partial g_t}{\partial w}(z,w)j_t\circ f_t^{-1}(w)f_t'\circ f_t^{-1}(w)\bigg),\end{aligned} \tag{8}$$

where γ_r is the hyperbolic circle of radius r centred on 0 in $D_t = f_t(\mathbb{D})$ and $j_t(z) = zp_t(z)$.

REMARK 3. For sufficiently smooth solutions to the time-reversed Loewner equation, Corollary 1 holds with the opposite sign.

However, given the existence of solutions to the time-reversed Loewner equation (7) for arbitrary measurable $p_t \in \mathcal{P}$, equation (8) can be stated in a stronger form using the Herglotz representation of p_t. Furthermore, the quantity $\partial g/\partial n$ can be defined in a natural way for any simply-connected domain by making use of the conformal invariance of Green's function. The idea is to "parameterize" the boundary by hyperbolic angle, and write $\partial g/\partial n$ in terms of the Poisson kernel on $\mathbb{D}$.

Let $f:\mathbb{D}\to D$ be a normalized Riemann map. Then the hyperbolic circle γ_r of radius r centred at 0 is given by $\theta\mapsto f(Re^{i\theta})$ for some $R>0$, and θ can be interpreted as the hyperbolic angle between the geodesics $f(s)$, $s\in[0,1)$ and $f(se^{i\theta})$, $s\in[0,1)$. Green's function is constant on γ_r so for $u=f(Re^{i\theta})$ and any $z\in D$

$$\operatorname{Im}\left(\frac{1}{i}\frac{\partial g}{\partial u}(u,z)\frac{du}{d\theta}\right)=0$$

so setting $\zeta=Re^{i\theta}$

$$\begin{aligned}\frac{\partial g}{\partial n}(\zeta,z)\frac{ds}{d\theta}&=\frac{2}{i}\frac{\partial g}{\partial u}(\zeta,z)\frac{du}{d\theta}=2\zeta\frac{\partial g}{\partial u}(\zeta,f^{-1}(z))f'(\zeta)\\&=2\zeta\frac{\partial g_{\mathbb{D}}}{\partial\zeta}(\zeta,f^{-1}(z)),\end{aligned}\tag{9}$$

where $g_{\mathbb{D}}$ is Green's function of $\mathbb{D}$. A computation shows that

$$\lim_{R\nearrow 1}2\zeta\frac{\partial g_{\mathbb{D}}}{\partial\zeta}(Re^{i\theta},f^{-1}(z))=-\operatorname{Re}\left(\frac{e^{i\theta}+f^{-1}(z)}{e^{i\theta}-f^{-1}(z)}\right).\tag{10}$$

THEOREM 4. *Let D_0 be a simply connected domain containing* 0, *and let μ_t be an increasing function of bounded variation on $[0,2\pi)$ measurable in t on $[0,T]$, such that $d\mu_t$ has total measure one. Then there exists a family of simply connected domains D_t such that $D_t\subset D_s$ for all $s<t$ whose Green's functions g_t satisfy*

$$\dot g_t(z,w)=-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left[\frac{e^{i\theta}+f_t^{-1}(z)}{e^{i\theta}-f_t^{-1}(z)}\right]\operatorname{Re}\left[\frac{e^{i\theta}+f_t^{-1}(w)}{e^{i\theta}-f_t^{-1}(w)}\right]d\mu_t(\theta)$$

for almost all t in $[0,T]$.

Proof. Let p_t be the normalized function of positive real part associated with the measure $d\mu_t$, let f_t be the corresponding solution of the Friedland–Schiffer equation (7) and let $D_t=f_t(\mathbb{D})$. Let γ_r be the hyperbolic circle of radius r centred at 0 in D_t; so γ_r is the image of the Euclidean circle of radius R under f_t for some R. By equation (8), for all r large enough, we have for $u=f_t(\zeta)$

$$\begin{aligned}\dot g_t(z,w)&=-\operatorname{Re}\left(\frac{2}{\pi i}\int_{\gamma_r}\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)j_t\circ f_t^{-1}(u)f_t'\circ f_t^{-1}(u)\,du\right)\\&=-\operatorname{Re}\left(\frac{1}{2\pi i}\int_{f_t^{-1}\circ\gamma_r}\left(\zeta\frac{\partial g}{\partial u}(f_t(\zeta),z)f_t'(\zeta)\right)\right.\\&\qquad\left.\times\left(\zeta\frac{\partial g}{\partial u}(f_t(\zeta),w)f_t'(\zeta)\right)p_t(\zeta)\frac{d\zeta}{\zeta}\right).\end{aligned}$$

By equation (9), the quantities in brackets are real and

$$\dot{g}_t(z,w) = -\frac{1}{2\pi}\int_0^{2\pi}\left(\mathrm{Re}^{i\theta}\,\frac{\partial g}{\partial u}(f_t(Re^{i\theta}),z)f_t'(Re^{i\theta})\right) \cdot \left(Re^{i\theta}\frac{\partial g}{\partial u}(f_t(Re^{i\theta}),w)f_t'(Re^{i\theta})\right)\mathrm{Re}\,p_t(Re^{i\theta})\,d\theta.$$

Now, we can choose a sequence $R_n \nearrow 1$ for which the measures $\mathrm{Re}\,p_t(R_n e^{i\cdot})$ converge in the weak*-topology to the probability measure μ_t on $[0,2\pi]$ associated with p_t via Herglotz formula, and the claim follows from equation (10). □

Theorem 4 is a natural extension of the Hadamard formula (3). In the case that D_t is smoothly bounded and p_t is smooth up to the boundary, it follows from Theorem 1 that if ds denotes infinitesimal arc length then (with ζ, u, etc. as above)

$$\nu(u)\frac{d\theta}{ds} = -\,\mathrm{Re}\,p_t(\zeta).$$

Thus, by equation (10)

$$\begin{aligned}\dot{g}_t(z,w) &= -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left[\frac{e^{i\theta}+f_t^{-1}(z)}{e^{i\theta}-f_t^{-1}(z)}\right]\mathrm{Re}\left[\frac{e^{i\theta}+f_t^{-1}(w)}{e^{i\theta}-f_t^{-1}(w)}\right]d\mu_t(\theta)\\ &= \frac{1}{2\pi}\int_{\partial D_t}\left(\frac{\partial g}{\partial n_u}(u,z)\frac{ds}{d\theta}\right)\left(\frac{\partial g}{\partial n_u}(u,w)\frac{ds}{d\theta}\right)\nu(u)\frac{d\theta}{ds}\,d\theta\end{aligned}$$

which agrees with equation (2).

Remark 4. Theorem 3 can also be written in terms of the Herglotz representation of p_t. However, given an arbitrary increasing μ_t of bounded variation and unit total measure there need not be a solution to the Loewner equation (see Example 3 ahead).

3. An application

3.1. Existence of solutions to the Loewner partial differential equation with prescribed initial generator. As an application of Theorem 1, we establish the existence of solutions to the Loewner equation with sufficiently smooth initial infinitesimal generators $p_0 \in \mathcal{P}$.

Theorem 5. *Let $f_0 : \mathbb{D} \to D_0$ be a one-to-one and onto holomorphic mapping such that $f_0(0) = 0 \in D_0$. Assume that $f_0 \in C^3(\overline{\mathbb{D}})$, and that the boundary of D_0 is a simple curve. For any $p \in \mathcal{P} \cap C^2(\overline{\mathbb{D}})$, there exists a Loewner chain f_t defined on an interval $[0,T]$ satisfying the Loewner partial differential equation*

$$\dot{f}_t = zp_t(z)f_t'(z)$$

such that $p_0 = p$.

The proof requires the intuitive geometric fact that for any smooth simple closed curve there exists an interval on which a normal variation is injective.

LEMMA 1. *Let $\gamma : [0,L] \to \mathbb{C}$ be a C^2 simple closed curve with outward normal $n(t)$. Let K be the maximum of the curvature of γ. Let $d(t_1,t_2) = |\gamma(t_1) - \gamma(t_2)|$, and let M be the minimum of d on*

$$\{(t_1,t_2) \mid \pi/(5K) \leq |t_1 - t_2| \leq L/2\}.$$

Finally let $R = \min\{(\sqrt{2}K)^{-1}, M/2\}$. Then the map $(t,r) \mapsto \gamma(t) + rn(t)$ is injective on $[0,L] \times (-R,R)$.

Proof. By [9, Lemma 2], $(t,r) \mapsto \gamma(t) + rn(t)$ is one-to-one on $[\alpha,\beta] \times (-1/(\sqrt{2}K), 1/(\sqrt{2}K))$ whenever $|\beta - \alpha| < \pi/(4K)$. Now assume that there exist t_1 and t_2 such that $|t_1 - t_2| \leq L/2$ and $\gamma(t_1) + r_1 n(t_1) = \gamma(t_2) + r_2 n(t_2) = w$ for $|r_i| < R$, $i = 1,2$. It follows that $|t_1 - t_2| > \pi/(5K)$. On the other hand, we must also have that

$$\begin{aligned} |\gamma(t_1) - \gamma(t_2)| &\leq |\gamma(t_1) - w| + |\gamma(t_2) - w| \\ &= |r_1| + |r_2| < 2R < M \end{aligned}$$

which is a contradiction. □

Proof of Theorem 5. Let $u(s)$ parameterize ∂D_0 by arc length. Let $\nu(u(s))$ be defined by

$$\nu(u(s)) = -\operatorname{Re}\left(f^{-1}(u(s)) p \circ f^{-1}(u(s)) f' \circ f^{-1}(u(s)) \frac{1}{i} \frac{d\bar{u}}{ds} \right).$$

By setting $u(s(t)) = f(e^{it})$, it is easily computed that $\nu(u(s)) > 0$ for all s.

Consider the curve $s \mapsto u(s) + \nu(u(s))r$. Since p is in $C^2(\overline{\mathbb{D}})$ and $f' \in C^2(\overline{\mathbb{D}})$, $\nu(u(s))$ is C^2 and in particular uniformly bounded on $[0,L]$. Thus, since ∂D_0 is C^2, by Lemma 1 the homotopy $(s,r) \mapsto u(s) + \nu(u(s))r$ is injective on $[0,L] \times [0,T']$ for some T', and furthermore since $\nu(u(s))$ is C^2 the homotopy is C^2. In particular, for each fixed r the resulting curve bounds a simply connected domain D_r. Let $\nu_r(s)$ be as in Definition 2. It follows from Theorem 1 that $\nu_0(s) = \nu(u(s))$. Note that this is not true for other values of r.

Let $\hat{f}_r : \mathbb{D} \to D_r$ be the conformal mapping such that $\hat{f}_r(0) = 0$ and $\hat{f}_r'(0) > 0$. We claim that the conformal radius $\log|\hat{f}_r'(0)|$ is a C^1 function of r. To see this, by equation (2) we have

$$\frac{d}{dr} g_r(z,w) = \frac{1}{2\pi} \int_{\partial D_r} \frac{\partial g_r}{\partial n_u}(u,z) \frac{\partial g_r}{\partial n_u}(u,w) \nu_r(u)\, ds_u,$$

where ν_r is C^1 in r and $\partial g_r / \partial n_u$ is C^2 in r on ∂D_r. Thus,

$$\frac{d}{dr} \log|\hat{f}_r'(0)| = \frac{d}{dr} \lim_{z \to 0} \big(g_r(z,0) + \log|z| \big) = \lim_{z \to 0} \frac{d}{dr} \big(g_r(z,0) + \log|z| \big)$$

is C^1. Furthermore, since $\nu_0(s) = \nu(u(s))$ one has that the derivative of the conformal radius is 1 at $r = 0$. To see this, we proceed as in the proof of Corollary 1:

$$\begin{aligned}
&\frac{d}{dr}\bigg|_{r=0} \log|\hat{f}_r'(0)| \\
&= \lim_{z\to 0} \frac{d}{dr}\bigg|_{r=0} g_r(z,0) \\
&= \lim_{z\to 0} \operatorname{Re}\left(\frac{2}{\pi i}\int_{\partial D_0} \frac{\partial g_0}{\partial u}(u,0)\frac{\partial g_0}{\partial u}(u,z) f^{-1}(u) p\circ f^{-1}(u) f'\circ f^{-1}(u)\, du\right) \\
&= \lim_{z\to 0} \operatorname{Re}(p\circ f^{-1}(z)) = 1.
\end{aligned}$$

Now, choose a reparameterization of the subordination chain $f_t = \hat{f}_{r(t)}$ so that $f_t'(0) = e^t$. By the above computation, $dr/dt = 1$. Thus,

$$\begin{aligned}
&\frac{d}{dt} g_{r(t)}(z,0)\bigg|_{t=0} \\
&= \frac{d}{dr}\bigg|_{r=0} g_r(z,0) \\
&= \operatorname{Re}\left(\frac{2}{\pi i}\int_{\partial D_0} \frac{\partial g_0}{\partial u}(u,0)\frac{\partial g_0}{\partial u}(u,z) f^{-1}(u) p\circ f^{-1}(u) f'\circ f^{-1}(u)\, du\right) \\
&= \operatorname{Re} p\circ f^{-1}(z).
\end{aligned}$$

For simplicity, we will denote $g_t = g_{r(t)}$; thus, $\dot{g}_0(z,0) = \operatorname{Re} p\circ f^{-1}(z)$.

To complete the proof, let p_t be the infinitesimal generator in the Loewner equation for f_t. We want to show that $p_0 = p$. Let

$$h_t(z) = -\log \frac{f_t^{-1}(z)}{z}$$

denote the unique choice of analytic completion of $g_t(z,0) + \log|z|$ satisfying $\operatorname{Im} h_t(0) = 0$. By the Loewner equation,

$$\dot{h}_0 = -\frac{1}{f_0^{-1}} \frac{d}{dt}\bigg|_{t=0} f_0^{-1} = p_0 \circ f_0^{-1}$$

is a holomorphic function, whose real part is

$$\operatorname{Re}(\dot{h}_0) = \dot{g}_0 = \operatorname{Re}(p\circ f_0^{-1}).$$

Since $p(0) = p_0(0) = 1$, it follows that $p_0 = p$. □

Theorem 5 also shows that one can arbitrarily prescribe the endpoint and initial generator in the ordinary Loewner equation, so long as these are sufficiently smooth.

COROLLARY 2. *Let f be any univalent function on $\mathbb{D}$ satisfying $f(0)=0$ and $f'(0)=1$ such that the boundary of $f(\mathbb{D})$ is C^3. Let $p\in\mathcal{P}\cap C^2(\overline{\mathbb{D}})$. There exists a solution to the Loewner ordinary differential equation*

$$\dot{w}_t=-w_t\cdot p_t\circ w_t$$

with $w_0(0)=0$, $w_t'(0)=e^{-t}$, and $p_0=p$, such that

$$\lim_{t\to\infty} e^t w_t(z)=f(z).$$

Proof. By Theorem 5, there exists a Loewner chain $\tilde{f}_t$ defined on $[0,T]$ with initial generator $p_0=p$. By reparameterizing (and thus possibly changing T) one can ensure that $\tilde{f}_t'(0)=e^t$. Let $\hat{f}$ be any normalized Loewner chain on $[T,\infty)$ starting at $\tilde{f}_T$. Defining $f_t=\tilde{f}_t$ for $t\in[0,T]$ and $f_t=\hat{f}_t$ for $t\in(T,\infty)$, we have constructed a normalized Loewner chain on $[0,\infty)$ satisfying the Loewner equation with initial generator $p_0=p$. Thus $w_t=f_t^{-1}\circ f$ has the desired properties. □

3.2. Some examples. It is unclear to what extent the assumptions of Theorem 5 can be weakened. The following examples put some limits on this.

For some choices of initial functions f_0, there are $p\in\mathcal{P}$ for which there does not exist a subordination chain on any interval $[0,T]$ so that the initial generator p_0 in the Loewner equation is equal to p.

EXAMPLE 1. Let $k(z)=z/(1-z)^2$, $k_t(z)=e^t z/(1-z)^2$ for some $b>1$ and $f_0(z)=k_t^{-1}\circ k(z)$. For some interval $I=(-1,x_0]$ on the real axis, f_0 maps $\mathbb{D}$ onto $\mathbb{D}\backslash I$. Furthermore f_0 extends continuously to $\mathbb{D}$, and maps some point $z_0\in\partial\mathbb{D}$ onto x_0. Assume that $p\in\mathcal{P}\cap C^2(\overline{\mathbb{D}})$, and $p\neq 0$ on $f_0^{-1}(J)$ for some open interval $J\subset(-1,x_0)$. It is clear that there is no subordination chain starting at f_0 with initial generator p.

It is easy to see that one could find a similar example for which $f_0(\partial\mathbb{D})$ is smooth.

If the boundary of $f_0(\mathbb{D})$ is not smooth, $f^{-1}\cdot p\circ f^{-1}\cdot f'\circ f^{-1}$ need not be continuous even if $p\in\mathcal{P}\cap C^2(\overline{\mathbb{D}})$.

EXAMPLE 2. Set $w_0=-(1+i)/2$ and let $f(z)=\sqrt{z+i}+w_0$ where the branch of square root is chosen so that $\mathbb{D}$ is contained in its domain and $\sqrt{i}=(1+i)/\sqrt{2}$. Thus f has a continuous extension mapping $-i$ to a corner of interior angle $\pi/2$ located at w_0.

It is easily computed that

$$f^{-1}(w)=(w-w_0)^2-i \quad\text{and}\quad f'\circ f^{-1}(w)=\frac{1}{2(w-w_0)}.$$

Setting $p(z)=1+z$, we have

$$f^{-1}(w)\cdot p\circ f^{-1}(w)\cdot f'\circ f^{-1}(w)=-\frac{1+i}{2(w-w_0)}+\frac{1-2i}{2}(w-w_0)+\frac{1}{2}(w-w_0)^3.$$

One would expect that for $p \in \text{ext}(\mathcal{P})$ and $f_0(z) = z$, the singularity of p on the boundary would prevent the existence of a subordination chain such that f_t is differentiable in t at $t = 0$ and $p_0 = p$. We have been unable to demonstrate this. However, the following example shows that there is a choice of p_t for $t \in [0, T]$ with p_t in $C^2(\overline{\mathbb{D}})$ on $(0, T]$ and $p_0 \in \text{ext}\,\mathcal{P}$, for which there is no solution to the Loewner partial differential equation on any interval $[0, T']$ with generator p_t and initial point $f_0(z) = z$.

Example 3. Let

$$p_t(z) = \frac{1 + e^{-3t}z}{1 - e^{-3t}z}.$$

Then the (normalized) solution f_t to the Loewner equation

$$\frac{\partial f_t}{\partial t} = z \frac{\partial f_t}{\partial z} p_t$$

with $f_0(z) = z$ is

$$f_t(z) = 1 - \sqrt{1 - 2e^t z + e^{-2t}z^2}.$$

This function is not analytic in $\mathbb{D}$ for any $t > 0$.

It should be noted that by a result of Becker [1] every solution f_t to the Loewner partial differential equation which is analytic in the disk $|z| < r(t)$ such that $e^t r(t) \to +\infty$ as $t \to +\infty$ is actually analytic in the whole unit disk (see [1, Satz 2]). Thus, if a solution f_t to the Loewner equation does not live on all of $\mathbb{D}$, its domain of definition has to shrink sufficiently fast. This makes it difficult to construct such solutions. Example 3 also shows that the assumption $e^t r(t) \to \infty$ in Becker's result is sharp in a sense.

4. A Loewner equation for general subordination chains

In [2], Heins gave an interesting derivation of the Loewner equation. His approach was to first prove that Green's function satisfies a kind of Loewner equation directly, and then use this to derive the Loewner equation for the mapping function. He considered only the special case of Loewner chains of maps onto the disc minus an arc joining the boundary.

In this section, we will show that his approach extends to arbitrary Loewner chains. In fact, this allows the removal of any assumption on the continuity of $f_t'(0)$. We will also show that Heins' formula agrees with Theorem 2 and thus with the Hadamard variational formula.

Recall that a subordination chain is called normalized if $f_t(0) = 0$ and $f_t'(0) = e^t$. It is shown in [6], [7] that a *normalized* subordination chain f_t is differentiable (a.e.) w.r.t. t and that the evolution of f_t can be described with the help of a differential equation (the Loewner equation). The differentiability is based on the fact that every *normalized* subordination chain satisfies a Lipschitz condition w.r.t. t locally uniformly in $\mathbb{D}$.

REMARK 5. If $f_t(z) = \alpha(t)z + \cdots$ is a subordination chain such that $\alpha : [0, T] \to (0, \infty)$ is continuous, then the substitution $t^* = \log \alpha(t)$ introduces a new parameter and (w.r.t. to the new parameter) yields a normalized subordination chain f_{t^*}, see [6], [7]. Thus, if $\alpha(t)$ is continuous, one can *select* a parameterization which ensures differentiability (a.e.). This is akin to Hilbert's fifth problem, concerning the introduction of differentiable coordinates in continuous groups.

The following theorem shows that in fact *every* subordination chain is differentiable (a.e.). The proof is based on an idea of Heins [2]. The differentiability and the associated Loewner-type equation come ultimately from the monotonicity of Green's function g_t of $f_t(\mathbb{D})$.

THEOREM 6. *Let $f_t : \mathbb{D} \to \mathbb{C}$, $t \in [0, T]$, be normalized Riemann maps such that $f_s \prec f_t$ for $0 \le s \le t \le T$. Then there exists a function $p_t(z)$ analytic in $|z| < 1$ and measurable in $t \in [0, T]$ satisfying*

$$\operatorname{Re} p_t(z) \ge 0, \quad z \in \mathbb{D}, t \in [0, T]$$

and a set $N \subset [0, T]$ of measure zero such that

$$\dot{f}_t(z) = z p_t(z) f_t'(z), \quad t \in [0, T] \backslash N, z \in \mathbb{D}.$$

The map $t \mapsto f_t(\mathbb{D})$ is continuous on $[0, T] \backslash N$ in the sense of kernel convergence.

Proof. (a) Let $A_t := f_t(\mathbb{D})$ and let $g_t(w)$ denote Green's function of A_t with pole at $w = 0$. Note that there exists an open disk K around $w = 0$, which is compactly contained in A_0, and thus in every A_t. By subordination, $A_s \subset A_t$ for $0 \le s \le t \le T$, so $t \mapsto g_t(w)$ is monotonically increasing for every fixed $w \in K$. If $E := \{w_m\}$, $w_1 := 0$, is a dense countable subset of K, then there exists for every nonnegative integer m a nullset $N_m \subseteq [0, 1]$ such that $t \mapsto g_t(w_m)$ is differentiable on $[0, T] \backslash N_m$ with derivative ≥ 0. Thus, for $N := \bigcup_{m \ge 1} N_m$, the derivative of $g_t(w)$ w.r.t. t exists on $[0, T] \backslash N$ for every $w \in E$.

Then for each $t_0 \in [0, T] \backslash N$ the limit

$$h_{t_0}(w) := \lim_{t \to t_0} \frac{g_t(w) - g_{t_0}(w)}{t - t_0} \tag{11}$$

exists locally uniformly in K and h_{t_0} is a nonnegative and harmonic function in K. This follows from the facts that the difference quotients on the right side are nonnegative harmonic functions in K and thus from a normal family, and the limit (11) exists on a dense subset of K.

In particular, $g_t(w) + \log |w| \to g_{t_0}(w) + \log |w|$ locally uniformly in K as $t \to t_0$ for every $t_0 \notin N$.

(b) We now show $A_t \to A_{t_0}$ as $t \to t_0$ for any $t_0 \notin N$ in the sense of kernel convergence.

The idea is simply this. We know that $t \mapsto g_t(w)$ is continuous at every $t_0 \notin N$ locally uniformly w.r.t. $w \in K$. Using the relation of Green's function g_t to the conformal map f_t, this implies $f_t \to f_{t_0}$ locally uniformly first in a neighborhood of $z=0$ and then, by normality, in the whole of $\mathbb{D}$, so $A_t \to A_{t_0}$ as $t \to t_0 \notin N$.

Fix $t_0 \notin N$ and let G_t denote the holomorphic function in A_t with $\operatorname{Re} G_t(w) = g_t(w) + \log|w|$ and $\operatorname{Im} G_t(0) = 0$, that is, by the Schwarz integral formula and shrinking K a little,

$$G_t(w) = \frac{1}{2\pi i}\int_{\partial K} \frac{\zeta + w}{\zeta - w}\big(g_t(\zeta) + \log|\zeta|\big)\frac{d\zeta}{\zeta}, \quad w \in K. \tag{12}$$

In particular, $G_t \to G_{t_0}$ uniformly in K as $t \to t_0$ and $\dot{G}_{t_0}(w)$ exists uniformly in $w \in K$. Using the relation between Green's function g_t and the conformal map f_t,

$$f_t^{-1}(w) = we^{-G_t(w)},$$

we see that $f_t^{-1} \to f_{t_0}^{-1}$ uniformly in K as $t \to t_0$. By the Koebe one-quarter theorem, there is a disk $D \subseteq \mathbb{D}$ such that $D \subset f_t^{-1}(K)$ for all $t \in [0,T]$. It follows that $f_t \to f_{t_0}$ locally uniformly in D as $t \to t_0$. Since $\{f_t : t \in [0,T]\}$ is a normal family, we deduce that $f_t \to f_{t_0}$ locally uniformly in $\mathbb{D}$ as $t \to t_0$, so $f_t(\mathbb{D}) \to f_{t_0}(\mathbb{D})$ as $t \to t_0$ in the sense of kernel convergence.

(c) From (a) and (b), we deduce that for any $t_0 \in [0,T]\backslash N$ the limit (11) exists locally uniformly in A_{t_0} and h_{t_0} is harmonic and nonnegative in A_{t_0}. Hence, the function G_{t_0} is analytic in A_{t_0}, $\dot{G}_{t_0}(w)$ exists locally uniformly for $w \in A_{t_0}$ and

$$\operatorname{Re} \dot{G}_{t_0}(w) = h_{t_0}(w), \quad w \in A_{t_0}.$$

If H_{t_0} denotes the analytic function in A_{t_0} with $H_{t_0}(0) = \dot{G}_{t_0}(0)$ and $\operatorname{Re} H_{t_0}(w) = h_{t_0}(w) \geq 0$, we therefore get

$$\dot{G}_{t_0}(w) = H_{t_0}(w), \quad w \in A_{t_0}.$$

Since $f_t^{-1}(w) = we^{-G_t(w)}$, we arrive at

$$\frac{d}{dt}(f_t^{-1})\bigg|_{t=t_0}(w) = -H_{t_0}(w) f_{t_0}^{-1}(w), \quad w \in A_{t_0}.$$

Again, the derivative w.r.t. t at $t=t_0$ on the left side exists locally uniformly for $w \in A_{t_0}$. This also implies that $(f_t^{-1})'(w)$ is differentiable w.r.t. t at $t=t_0$ locally uniformly for $w \in A_{t_0}$. By the Bürmann–Lagrange formula,

$$f_t(z) = \frac{1}{2\pi i}\int_\gamma \frac{\zeta (f_t^{-1})'(\zeta)}{f_t^{-1}(\zeta) - z} d\zeta, \quad |z| < r, 0 < r < 1, \tag{13}$$

where γ is a smooth Jordan curve in A_t which contains $f_t(|\eta| = r)$ in its interior. Thus for fixed $0 < r < 1$, since $f_t(\mathbb{D}) \to f_{t_0}(\mathbb{D})$ as $t \to t_0$, there is a smooth Jordan curve $\gamma \subset A_{t_0}$ which contains $f_t(|\eta|=r)$ in its interior for all t sufficiently close to t_0. Hence, (13) implies that the limit

$$\dot{f}_{t_0}(z) = \lim_{t\to t_0} \frac{f_t(z)-f_{t_0}(z)}{t-t_0}$$

exists locally uniformly in $\mathbb{D}$. From $f_t^{-1}(f_t(z))=z$, we therefore get

$$\dot{f}_{t_0}(z) = z f_{t_0}'(z) H_{t_0}(f_{t_0}(z)), \quad z\in\mathbb{D}.$$

If we define

$$p_t(z) := H_t(f_t(z)),$$

then $p_t(z)$ is analytic in $|z|<1$ with nonnegative real part and measurable in $[0,T]$, and we arrive at the Loewner differential equation for f_t. □

Remark 6. This proof does not rely on the second coefficient estimate, distortion theorem or growth theorem for univalent functions. It only relies on the Koebe 1/4 theorem, whose proof also does not require them (see for example [4]).

We conclude with a few observations. First, Theorem 6 generalizes Theorem 2 to the case of an *arbitrary* subordination chain:

Remark 7. Let D_t be any sequence of domains parameterized by $t\in[0,T]$ such that $D_t\subset D_s$ whenever $t<s$. Let $g_t(z,w)$ be Green's function for D_t. There exists a function $p_t\in\mathcal{P}$ which is measurable in t and a set N of measure zero such that

$$\begin{aligned}\dot{g}_t(z,w) = -2\,\mathrm{Re}\bigg(&\frac{\partial g_t}{\partial z}(z,w) j_t\circ f_t^{-1}(z) f_t'\circ f_t^{-1}(z)\\ &+\frac{\partial g_t}{\partial w}(z,w) j_t\circ f_t^{-1}(w) f_t'\circ f_t^{-1}(w)\bigg),\end{aligned}$$

where $j_t(z)=zp_t(z)$, for any $t\in[0,T]\backslash N$. Furthermore, $g_t(z,w)+\log|z-w|\to g_{t_0}(z,w)+\log|z-w|$ locally uniformly for all $t_0\in[0,T]\backslash N$.

Proof. Differentiate equation (6) using Theorem 6. □

Remark 8. Remark 7 (and thus Theorem 2) agree with Heins' Loewner equation for Green's function. To see this, set $w=0$ in the above formula. In that case, by equation (6)

$$\frac{\partial g}{\partial z}(z,0) = -\frac{{f_t^{-1}}'(z)}{2f_t^{-1}(z)}.$$

Thus since $j_t(0)=0$ and $f_t(0)=0$, we have

$$\dot{g}_t(z,0) = -\,\mathrm{Re}(p_t\circ f_t^{-1}(z)).$$

This is Heins' formula (see equations (2) and (3) in [2]).

REMARK 9. Theorem 6 shows that the assumption that $d\mu_t$ have unit total measure can be removed from Theorem 4.

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