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LIPSCHITZ GEOMETRY OF CURVES AND SURFACES DEFINABLE IN O-MINIMAL STRUCTURES

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Abstract. The paper is devoted to the generalization of the theory of Hoelder Complexes, i.e., Lipschitz classification of germs of semialgebraic surfaces, for the definable surfaces in o-minimal structures. The theory is based on the Rosenlicht valuations on the corresponding Hardy fields. We obtain a complete answer for the case of polynomially bounded o-minimal structures and for the case of isolated singularities for general o-minimal structures.

1. Introduction

Metric geometry of singular spaces can be divided into two wide parts. The first one works with invariants under isometries. Investigations of Bröcker and Bernig (see [6], [7]) and others are devoted to the different notions of curvature on singular definable spaces.

Another part of this direction is related to bi-Lipschitz invariants. Bi-Lipschitz equivalence classes of singular spaces are wider than the classes of isometric singular spaces. That is why one can expect to have a complete solution of the problem of Lipschitz classification. This problem appears naturally in Singularity Theory and in classical Differential Geometry. Mostowski [13] studied a question of "tameness" of this problem for complex algebraic sets. He proved that, for any finite dimensional analytic family, the set of equivalence classes (according to a bi-Lipschitz equivalence) is finite. Later, this result was generalized by Parusinski to semialgebraic and subanalytic sets [15]. However, these finiteness results are "existence theorems" and do not give any key to resolve a classification problem. Recently, Valette [16] extended these results to polynomially bounded o-minimal structures.

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For germs of semialgebraic and subanalytic curves (one-dimensional semialgebraic and subanalytic sets), the problem of bi-Lipschitz classification was completely solved in [3]. The main result of [3] is that a Lipschitz equivalence class of a curve is totally determined by orders of contact of all pairs of branches. The paper [1] is devoted to a bi-Lipschitz classification of 2-dimensional semialgebraic or subanalytic sets. The problem is studied with respect to an intrinsic (inner) metric. The inner distance between two points on a semialgebraic set is defined as a minimal length of a rectifiable curve on the set connecting these points. This viewpoint is more usual in differential geometry than in classical singularity theory. The paper [1] gives a complete bi-Lipschitz invariant—so-called Hölder Complex. Hölder Complex is a canonical local triangulation equipped with some rational numbers associated to each 2-dimensional simplex. These numbers characterize the orders of contact of one-dimensional faces of these simplices near a singular point. In particular, it is proved that if x_0 is an isolated singular point of a 2-dimensional semialgebraic set X with a connected link then the germ of X at x_0 is bi-Lipschitz equivalent to a β-horn (a revolution surface of the graph of the function x^β). This result was recently rediscovered by Grieser [10]. In [4], the number β is computed for quasihomogeneous and semiquasihomogeneous singularities.

All results described above are devoted to a special class of singular spaces: semialgebraic and subanalytic sets. The following question looks natural: What happens in a more general situation? In this paper, we consider a more general class of singular spaces: definable sets in o-minimal structures. Topological properties of these sets are similar to a semialgebraic case. What one can say about metric properties? Note, that the results of Mostowski are not true for general o-minimal structures. To see it, consider the following set: $X = X_1 \cup X_2$ where $X_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1 > 0, x_2 > 0, x_3 = x_1^{x_2}\}, X_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1 > 0, x_2 > 0, x_3 = 0\}$. The family of sections of this set by planes $x_2 = const$ has infinitely many equivalence classes according to a bi-Lipschitz equivalence.

The main goal of the present paper is to show that a question of Lipschitz classification also makes sense in o-minimal case (even in the case when the set of equivalence classes is not countable).

In [1] and [3] (see also [2]), orders of contact of semialgebraic (subanalytic) arcs were measured by some rational numbers. These numbers are first exponents of Puiseux decomposition of corresponding distance functions. Note, that these Puiseux exponents can be considered as elements of a value group of a canonical valuation on a Hardy field of germs of semialgebraic functions. In a general case, one can also consider a Hardy field of definable functions and take a value group of the corresponding Rosenlicht valuation (see [14]). Actually, this idea does not work directly. If an o-minimal structure is not polynomially bounded, then the canonical valuation does not create a bi-Lipschitz invariant (see Section 3). That is why we define a notion of *quasivaluation.*

Let A be an o-minimal structure. Let K_A be a Hardy field of germs of definable in A functions $\phi\colon (0,\varepsilon)\to\mathbb{R}$. Let $G_A\subset K_A$ be a group of local homeomorphisms near 0. Let $Lip_A\subset G_A$ be a subgroup of bi-Lipschitz homeomorphisms.

PROPOSITION (Corollary 3.7). *Lip_A is a normal subgroup of G_A if and only if A is polynomially bounded.*

We denote by $\widetilde{H}$ the set of left cosets in G_A with respect to Lip_A. Observe that $\widetilde{H}$ is an ordered set. Let G_A^+ be a subset of G_A of the functions $\phi : (0,\varepsilon)\to\mathbb{R}$ such that $\lim_{t\to 0}\phi'(t)\neq\infty$. A canonical projection $P: G_A^+\to\widetilde{H}$ is called *quasi-valuation.* The term "quasi-valuation" has the following motivation. Let $v: K_A\to H$ be a Rosenlicht valuation on K_A. Then for any pair $\psi_1,\psi_2\in G_A^+$, we have the following: if $v(\psi_1)=v(\psi_2)$, then $P(\psi_1)=P(\psi_2)$. Moreover, if A is polynomially bounded then cosets are totally determined by the valuation v.

Section 4 is devoted to a relation between quasi-valuations and Lipschitz geometry of some special definable sets. Namely, we consider ϕ-semicusps $C_\phi=\{(x_1,x_2)\in\mathbb{R}^2|x_1\geq 0, x_2=0\}\cup\{(x_1,x_2)\in\mathbb{R}^2|x_1\geq 0, x_2=\phi(x_1)\}$ and ϕ-triangles $T_\phi=\{(x_1,x_2)\in\mathbb{R}^2|x_1\geq 0, 0\leq x_2\leq\phi(x_1)\}$ with $\phi\in G_A^+$. We show that two semicusps C_{ϕ_1} and C_{ϕ_2} are bi-Lipschitz equivalent (with respect to the Euclidean metric in $\mathbb{R}^2$) if and only if $P(\phi_1)=P(\phi_2)$. The same result is also true for T_{ϕ_1} and T_{ϕ_2}. Note, that the sets T_{ϕ_1} and T_{ϕ_2} are normally embedded, i.e., the intrinsic and the Euclidean metrics are Lipschitz equivalent (see [5]). The results of this section are important for further investigations.

We generalize, in Section 5, the results of [3] for definable curves. We associate to a germ of a definable curve two combinatorial objects: Valuation Semicomplex and Quasi-valuation Semicomplex. In order to construct these complexes, we take all pairs of the branches of a given curve. Let (γ_i,γ_j) be a pair of branches. We can suppose that γ_i and γ_j are parameterized by a distance to a singular point. Set $\phi_{ij}=\|\gamma_i-\gamma_j\|$. Taking $\alpha_{ij}=v(\phi_{ij})$ we obtain a valuation semicomplex and taking $\tilde{\alpha}_{ij}=P(\phi_{ij})$ we obtain a quasi-valuation semicomplex. We prove that a quasi-valuation semicomplex is a bi-Lipschitz invariant and an isomorphism of valuation semicomplexes is a criterion of a bi-Lipschitz equivalence. In this section, the bi-Lipschitz equivalence is considered with respect to the Euclidean metric. If all the distance functions ϕ_{ij} are nonflat, in particular, if the o-minimal structure A is polynomially bounded, then two semicomplexes (valuation and quasi-valuation) are isomorphic and each of them gives a complete bi-Lipschitz invariant.

Sections 6, 7, 8, and 9 are devoted to the investigation of intrinsic Lipschitz geometry of definable surfaces. In Section 8, we study isolated singularities. The main result of this section is so-called Horn theorem.

Let $\phi \in G_A^+$ be a germ of a definable in A function. A ϕ-horn W_ϕ is a set defined as follows:

$$W_\phi = \left\{(x_1, x_2, y) \in \mathbb{R}^3 | y \geq 0, \sqrt{x_1^2 + x_2^2} = \phi(y)\right\}.$$

We prove that any germ of a definable in A surface with isolated singular point with a connected link is bi-Lipschitz equivalent with respect to the intrinsic metric to a ϕ-horn, for some $\phi \in G_A^+$. Moreover, ϕ_1-horn and ϕ_2-horn are bi-Lipschitz equivalent if and only if $P(\phi_1) = P(\phi_2)$. This result generalizes a horn theorem from [1]. Grieser [9] obtained a related result investigating a problem of classification of riemannian metrics with isolated singularities up to quasi-isometry.

Sections 6, 7, and 9 are devoted to nonisolated singularities. We introduce the notion of Quasi-valuation Complex. It is a generalization, for the o-minimal case, of Hölder Complexes developed in [1]. Quasi-valuation Complex can be defined as a finite graph Γ with a function $\beta : E_\Gamma \to \widetilde{H}$ where E_Γ is the set of edges of Γ and $\widetilde{H}$ is an ordered set related to the notion of quasi-valuation. This graph Γ carries a topological information about a singular point. The function β is responsible for a metric information: "intrinsic orders of contact" of one-dimensional faces of simplices of a triangulation near a singular point. We show that a Canonical quasi-valuation Complex is a bi-Lipschitz invariant and a complete bi-Lipschitz invariant in a nonflat case.

2. Basic notations

2.1. Hardy field of definable functions. Let A be an o-minimal structure. Consider the set of all germs of definable in A functions $\phi : (0, \varepsilon) \to \mathbb{R}$. The usual operations of addition and multiplication of functions provide a structure of a field on this set. We denote this field by K_A. Clearly, K_A is totally ordered, and for each $\phi \in K_A$, we have: $\phi' \in K_A$. Thus, K_A is a Hardy field.

Let H be an ordered group (called *a value group*). A valuation $v : K_A - \{0\} \to H$ is called *a Rosenlicht valuation* if $v(f) > v(g)$ when $\lim_{t \to 0} \frac{f(t)}{g(t)} = 0$. The Rosenlicht valuation is canonical in the following sense. Let $v_1 : K_A - \{0\} \to H_1$ be another Rosenlicht valuation. Then there exists an order-preserving embedding $j : H \to H_1$ such that $v_1(\phi) = j(v(\phi))$. We will use the notation

$$\phi_1 \backsim \phi_2; \quad \text{if } v(\phi_1) = v(\phi_2).$$

2.2. Inner (intrinsic) metric on definable sets. Let $X \subset \mathbb{R}^n$ be a connected definable in A set. We define an intrinsic distance $d_\ell(x_1, x_2)$ between two points $x_1, x_2 \in X$ in the following way. Set $d_\ell(x_1, x_2) = \inf \ell(\gamma)$ where $\gamma : [0, 1] \to X$ be a rectifiable curve such that $\gamma(0) = x_1, \gamma(1) = x_2$ and $\ell(\gamma)$ be a length of the curve γ. By Definable Triangulation Theorem (see [17]), the distance d_ℓ is well defined.

2.3. Bi-Lipschitz equivalence. Let X_1, X_2 be two metric spaces. A map $F: X_1 \to X_2$ is called *bi-Lipschitz* if F and F^{-1} are Lipschitz homeomorphisms. A definable in A subset $X \subset \mathbb{R}^n$ can be considered as a metric space. There are two natural metrics on X: Euclidean metric $d(x_1, x_2) = \|x_1 - x_2\|$ and an intrinsic metric d_ℓ defined above. Two connected definable sets X_1, X_2 are called *bi-Lipschitz equivalent (isomorphic) with respect to the Euclidean metric* if there exists a bi-Lipschitz (with respect to the Euclidean metric) map $F: X_1 \to X_2$. The sets X_1, X_2 are called *bi-Lipschitz equivalent with respect to the inner (intrinsic) metric* if there exists a map $F: X_1 \to X_2$ bi-Lipschitz with respect to the inner metric. The sets X_1, X_2 are called *bi-Lipschitz equivalent (with respect to the Euclidean or to the inner metric) in* A if a bi-Lipschitz map F is definable in A.

3. Germs of definable homeomorphisms

Let A be an o-minimal structure over $\mathbb{R}$ and let K_A be a Hardy field of germs at $0 \in \mathbb{R}$ of definable in A functions $\phi: (0, \varepsilon) \to \mathbb{R}$. Let $G_A \subset K_A$ be the subset of K_A defined as follows:

$$G_A = \left\{ \phi \in K_A, \phi > 0 \text{ and } \lim_{t \to 0} \phi(t) = 0 \right\}.$$

Each function ϕ from G_A can be extended to $[0, \varepsilon)$ putting $\phi(0) = 0$. The elements of G_A are germs of definable homeomorphisms $\phi: [0, \varepsilon_1) \to [0, \varepsilon_2)$. They form a group where the group operation is a composition. Let Lip_A be the subgroup of G_A of the germs of bi-Lipschitz homeomorphisms. Observe that Lip_A can be defined as follows: $Lip_A = \{\phi \in G_A, v(\phi) = v(Id)\}$, where $v: K_A \to H$ is a Rosenlicht valuation and H is a value group of v.

Definition 1. Two homeomorphisms ϕ_1 and ϕ_2 of G_A are called *R-Lipschitz equivalent* if $\phi_1^{-1}\phi_2 \in Lip_A$. If $\phi_1\phi_2^{-1} \in Lip_A$, the homeomorphisms ϕ_1 and ϕ_2 are called *L-Lipschitz equivalent*. Homeomorphisms ϕ_1 and ϕ_2 are called *RL-Lipschitz equivalent* if $\phi_1 = l_1\phi_2 l_2^{-1}$, for some bi-Lipschitz homeomorphisms $l_1, l_2 \in Lip_A$.

Let $G_A^+ \subset G_A$ be a subset defined as follows: $G_A^+ = \{\phi \in G_A, v(\phi) \geq v(Id)\}$. Let $G_A^- = \{\phi \in G_A, v(\phi) \leq v(Id)\}$. Clearly, $G_A = G_A^- \cup G_A^+$ and $Lip_A = G_A^- \cap G_A^+$.

Proposition 3.1.

(1) *If $\phi \in G_A^+$, then $\phi^{-1} \in G_A^-$.*
(2) *If ϕ_1 and ϕ_2 are L-Lipschitz equivalent or R-Lipschitz equivalent and $\phi_1 \in G_A^+$, then $\phi_2 \in G_A^+$.*
(3) *Homeomorphisms ϕ_1 and ϕ_2 are R-Lipschitz equivalent if and only if ϕ_1^{-1} and ϕ_2^{-1} are L-Lipschitz equivalent.*

The proof is straightforward.

Proposition 3.2. *Let A be a polynomially bounded o-minimal structure. Let $\phi_1, \phi_2 \in G_A$. Then the following assertions are equivalent.*

(1) *ϕ_1, ϕ_2 are R-Lipschitz equivalent.*
(2) *ϕ_1, ϕ_2 are L-Lipschitz equivalent.*
(3) *$v(\phi_1) = v(\phi_2)$.*

Proof. Let $v(\phi_1) = v(\phi_2) = \alpha$. Then by the results of van den Dries and Miller [18], $\phi_1 \backsim t^\alpha$, $\phi_2 \backsim t^\alpha$, ${\phi_1}^{-1} \backsim t^{\frac{1}{\alpha}}$ and ${\phi_2}^{-1} \backsim t^{\frac{1}{\alpha}}$. Thus,

$$ {\phi_1}^{-1}\phi_2 \backsim t \quad \text{and} \quad {\phi_2}^{-1}\phi_1 \backsim t. $$

We obtained that $\phi_1^{-1}\phi_2 \in Lip_A$ and $\phi_1\phi_2^{-1} \in Lip_A$.

Assume that $v(\phi_1) = \alpha_1, v(\phi_2) = \alpha_2$ and $\alpha_1 \neq \alpha_2$. Hence,

$$ {\phi_1}^{-1}\phi_2 \backsim t^{\frac{\alpha_2}{\alpha_1}} \quad \text{and} \quad {\phi_2}^{-1}\phi_1 \backsim t^{\frac{\alpha_1}{\alpha_2}}. $$

It means that $\phi_1^{-1}\phi_2 \notin Lip_A$ and $\phi_1\phi_2^{-1} \notin Lip_A$. □

Proposition 3.3. *Let A be an o-minimal structure (not necessary polynomially bounded). Let $\phi_1, \phi_2 \in G_A^+$. Then ϕ_1 and ϕ_2 are L-Lipschitz equivalent if and only if $v(\phi_1) = v(\phi_2)$.*

Proof. Suppose that $v(\phi_1) = v(\phi_2)$. Then $\phi_1\phi_2^{-1} \backsim t$. Let ϕ_1 be L-Lipschitz equivalent to ϕ_2. Thus, there exist two positive constants $\widetilde{K}_1$ and $\widetilde{K}_2$ such that

$$ \widetilde{K}_1 < (\phi_1(\phi_2^{-1}(s)))' < \widetilde{K}_2. $$

Hence,

$$ \widetilde{K}_1 < \frac{\phi_1'(\phi_2^{-1}(s))}{\phi_2'(\phi_2^{-1}(s))} < \widetilde{K}_2. $$

Thus, $\frac{\phi_1'}{\phi_2'}$ is bounded away from 0 and infinity. By L'hospital rule, it is also true for $\frac{\phi_1}{\phi_2}$. It means that $v(\phi_1) = v(\phi_2)$. □

Proposition 3.4. *Let $\phi_1, \phi_2 \in G_A^+$ and let $v(\phi_1) = v(\phi_2)$. Then ϕ_1 and ϕ_2 are R-Lipschitz equivalent.*

To prove this proposition we need the following lemma.

Lemma 3.5. *For all $\phi \in G_A^+$ and for all $K > 0$, the germs ϕ and $K\phi$ are R-Lipschitz equivalent.*

Proof. The property is clear if $v(\phi) = v(Id)$. Suppose that $v(\phi) > v(Id)$. Consider the case $K > 1$. Since ϕ' is a monotone function and $\phi'(0) = 0$, we obtain

$$ \big(\phi(Kx) - K\phi(x)\big)' = K\big(\phi'(Kx) - \phi'(x)\big) > 0, $$

for $x > 0$. Thus,

$$ \phi(x) \leq K\phi(x) \leq \phi(Kx). $$

Since ϕ is a monotone function, we obtain

$$\phi^{-1}(\phi(x)) \le \phi^{-1}(K\phi(x)) \le \phi^{-1}(\phi(Kx)).$$

Finally,

$$x \le \phi^{-1}(K\phi(x)) \le Kx.$$

Hence, $\phi^{-1}(K\phi(x)) \in Lip_A$.

The same arguments give the proof for $K < 1$. $\square$

Proof of Proposition 3.4. Since $v(\phi_1) = v(\phi_2)$, there exist two constants $K_1, K_2 > 0$ such that

$$K_1\phi_2(t) \le \phi_1(t) \le K_2\phi_2(t).$$

Since ϕ_2^{-1} is a monotone function, we obtain

$$\phi_2^{-1}(K_1\phi_2(t)) \le \phi_2^{-1}(\phi_1(t)) \le \phi_2^{-1}(K_2\phi_2(t)).$$

Therefore, $\phi_2^{-1}\phi_1 \in Lip_A$ because, by Lemma 3.5, $\phi_2^{-1}(K_1\phi_2) \in Lip_A$ and $\phi_2^{-1}(K_2\phi_2) \in Lip_A$. $\square$

REMARK. The inverse statement for R-Lipschitz equivalence is wrong. Let A be an exponential o-minimal structure. Then, by [12], the function e^x is definable in A. Thus, $\phi_1 = e^{-\frac{1}{x}}$ and $\phi_2 = e^{-\frac{1}{2x}}$ belong to G_A^+. Clearly, ϕ_1 and ϕ_2 are R-Lipschitz equivalent but $v(\phi_1) \ne v(\phi_2)$.

COROLLARY 3.6. *If $\phi_1, \phi_2 \in G_A^+$ and ϕ_1, ϕ_2 are RL-Lipschitz equivalent then they are R-Lipschitz equivalent.*

Proof. Since ϕ_1 and ϕ_2 are RL-Lipschitz equivalent, there exists $\ell \in Lip_A$ such that ϕ_1 and $\phi_2\ell$ are L-Lipschitz equivalent. Then by Proposition 3.3, $v(\phi_1) = v(\phi_2\ell)$, and by Proposition 3.4, ϕ_1 and $\phi_2\ell$ are R-Lipschitz equivalent. Hence, ϕ_1 and ϕ_2 are R-Lipschitz equivalent. $\square$

COROLLARY 3.7. *Lip_A is a normal subgroup in G_A if and only if A is a polynomially bounded o-minimal structure.*

Proof. If A is a polynomially bounded then the equivalence classes by R-Lipschitz equivalence and by L-Lipschitz equivalence are the same. Thus, Lip_A be normal. Let Lip_A is normal. Let A be an exponential structure. Then there exist two functions ϕ_1 and ϕ_2 (see the Remark) such that they are R-Lipschitz equivalent but not L-Lipschitz equivalent. $\square$

PROPOSITION 3.8. *Let $\phi_1, \phi_2, \phi_3 \in G_A^+$ be the germs of definable functions such that, for $t \ne 0, \phi_1(t) < \phi_2(t) < \phi_3(t)$. Let ϕ_1 and ϕ_3 be R-Lipschitz equivalent. Then ϕ_1 and ϕ_2 are R-Lipschitz equivalent.*

Proof. Since ϕ_1 and ϕ_2 are germs of definable homeomorphisms, we have that $\phi_2 = \phi_1\tilde{\ell}, \phi_3 = \phi_1\ell$, for some definable homeomorphisms ℓ and $\tilde{\ell}$. Since ϕ_1 and ϕ_3 are R-Lipschitz equivalent, $\ell \in Lip_A$. Clearly, $t < \tilde{\ell}(t) < \ell(t)$. Thus, $\tilde{\ell} \in Lip_A$. $\square$

Let $\widetilde{H}$ be the set of left co-sets of G_A^+ with respect to Lip_A. Let $P: G_A^+ \to \widetilde{H}$ be the canonical projection. We can define a natural order in $\widetilde{H}$ in the following way. Let $h_1, h_2 \in \widetilde{H}, h_1 \neq h_2$. Set $h_2 > h_1$ if there exist $\phi_1 \in h_1$ and $\phi_2 \in h_2$ such that, for $t \neq 0, \phi_1(t) > \phi_2(t)$. By Proposition 3.8, this order is well defined and $\widetilde{H}$ is totally ordered. Let $H^+ = v(G_A^+)$.

THEOREM 3.9. *There exists a map* $\bar{v}: H^+ \to \widetilde{H}$ *such that*

(1) *For each* $\eta_1, \eta_2 \in H^+$ *such that* $\eta_1 > \eta_2$, *one has*: $\bar{v}(\eta_1) \geq \bar{v}(\eta_2)$.
(2) *The diagram*

$$\begin{array}{ccc} G_A^+ & \xrightarrow{v} & H^+ \\ P \searrow & & \swarrow \bar{v} \\ & \widetilde{H} & \end{array}$$

is commutative.

Proof. Let $\eta \in H^+$, let $\phi \in v^{-1}(\eta)$. Define $\bar{v}(\eta) = P(\phi)$. By Proposition 3.4, the map $\bar{v}$ is well defined. By the definition of the order in $\widetilde{H}$, the map $\bar{v}$ satisfies the condition (1). □

The map $P: G_A^+ \to \widetilde{H}$ is called *Quasi-valuation map*. If A is polynomially bounded then $\widetilde{H}$ can be identified with H^+ and P is the restriction of a valuation v to G_A^+.

COROLLARY 3.10. *Let* $\phi_1, \phi_2 \in G_A^+$. *Then* $P(\phi_1 + \phi_2) = \min\{P(\phi_1), P(\phi_2)\}$.

Proof. Since $\phi_1, \phi_2 \in G_A^+$, then $v(\phi_1 + \phi_2) = \min\{v(\phi_1), v(\phi_2)\}$. By Theorem 3.9, $P(\phi_1 + \phi_2) = \bar{v}(v(\phi_1 + \phi_2)) = \bar{v}(\min\{v(\phi_1), v(\phi_2)\}) = \min\{P(\phi_1), P(\phi_2)\}$. □

A function $\phi \in G_A^+$ is called *flat* if $\frac{d^n \phi}{(dt)^n}|_{t=0} = 0$, for all $n \geq 1, n \in \mathbb{Z}$. A function ϕ is called *nonflat* if it is not flat. The following result shows that for nonflat functions, a quasi-valuation is equivalent to a valuation.

THEOREM 3.11. *Let* $\phi_1 \in G_A^+$ *be a nonflat function. Let* $\phi_2 \in G_A^+$ *be another function such that* $P(\phi_1) = P(\phi_2)$. *Then* $v(\phi_1) = v(\phi_2)$.

To prove the theorem we need the following

LEMMA 3.12. *Let* $\phi \in G_A^+$ *be a nonflat function. Then for every* $r > 0$, *we have* $v(\phi(t)) = v(\phi(rt))$.

Proof. Without loss of generality, we can suppose that $r > 1$. Let k be a number such that, for $i < k$, we have $\frac{d^i \phi}{(dt)^i}|_{t=0} = 0$ and $\frac{d^k \phi}{(dt)^k}|_{t=0} \neq 0$. We have two possibilities: $\frac{d^k \phi}{(dt)^k}|_{t=0} = \infty$ or $\frac{d^k \phi}{(dt)^k}|_{t=0} = M$ where $M \neq \infty$. If

$\frac{d^k\phi}{(dt)^k}|_{t=0} = M$ using L'Hospital rule we obtain

$$\lim_{t\to 0}\frac{\phi(t)}{\phi(rt)} = \frac{1}{r^k}\lim_{t\to 0}\frac{\phi^{(k)}(t)}{\phi^{(k)}(rt)} = \frac{1}{r^k}.$$

Thus, $v(\phi(t)) = v(\phi(rt))$.

Consider the second case: $\frac{d^k\phi}{(dt)^k}|_{t=0} = \infty$. We have:

$$\lim_{t\to 0}\frac{\phi(t)}{\phi(rt)} = \frac{1}{r^{k-1}}\lim_{t\to 0}\frac{\phi^{(k-1)}(t)}{\phi^{(k-1)}(rt)}.$$

Since $\phi^{(k-1)}(0) = 0$ and $\phi^{(k-1)}(t)$ is a monotone function we obtain that

$$\lim_{t\to 0}\frac{\phi(t)}{\phi(rt)} \le \frac{1}{r^{k-1}}.$$

On the other hand, since $\lim_{t\to 0}\phi^{(k)}(t) = \infty$ and $\phi^{(k)}(t)$ is a monotone function we obtain that

$$\lim_{t\to 0}\frac{\phi(t)}{\phi(rt)} = \frac{1}{r^k}\lim_{t\to 0}\frac{\phi^{(k)}(t)}{\phi^{(k)}(rt)} \ge \frac{1}{r^k}.$$

Hence, $v(\phi(t)) = v(\phi(rt))$. The lemma is proved. □

Proof of Theorem 3.11. Let $P(\phi_1) = P(\phi_2)$. It means that $\phi_2(t) = \phi_1(l(t))$ where $l \in Lip_A$. Thus, there exists a couple of constants $r_1, r_2 > 0$ such that $r_1 t < l(t) < r_2 t$. Since ϕ_1 is an increasing function we have

$$\phi_1(r_1 t) \le \phi_2(t) \le \phi_1(r_2 t).$$

Since ϕ_1 is a nonflat function, by Lemma 3.12, $v(\phi_1(r_1 t)) = v(\phi_1(r_2 t)) = v(\phi_1(t))$. Hence, $v(\phi_1) = v(\phi_2)$. □

PROPOSITION 3.13. *Let ϕ is a flat function. Then for all $r > 0, r \ne 1$, we have $v(\phi(rt)) \ne v(\phi(t))$.*

Proof. Suppose that $r < 1$. We have

$$\lim_{t\to 0}\frac{\phi(rt)}{\phi(t)} = r^k\lim_{t\to 0}\frac{\phi^{(k)}(rt)}{\phi^{(k)}(t)}.$$

Since $\phi^{(k)}(rt) < \phi^{(k)}(t)$, we obtain

$$\lim_{t\to 0}\frac{\phi(rt)}{\phi(t)} \le r^k.$$

This inequality is true for all k. Thus, $v(\phi(rt)) > v(\phi(t))$. □

4. Germs of definable sets in $\mathbb{R}^2$

Here (in this section), a bi-Lipschitz equivalence is considered with respect to the Euclidean metric.

Let $\phi \in G_A^+$ be a germ of a definable homeomorphism. A *ϕ-semicusp C_ϕ* is a germ at $(0,0)$ of a 1-dimensional subset in $\mathbb{R}^2$ defined as follows:

$$C_\phi = \{(x_1,x_2)|x_1 \geq 0, x_2 = 0\} \cup \{(x_1,x_2)|x_1 \geq 0, x_2 = \phi(x_1)\}.$$

THEOREM 4.1. *Two semicusps are bi-Lipschitz equivalent in A if and only if the germs of corresponding definable homeomorphisms ϕ_1 and ϕ_2 are R-Lipschitz equivalent.*

To prove this theorem we need two lemmas.

LEMMA 4.2. *Let $\phi_1,\ \phi_2 \in G_A^+$. Let C_{ϕ_1} and C_{ϕ_2} be bi-Lipschitz equivalent in A. Suppose that $\phi_1, \phi_2 \notin Lip_A$. Let $F : C_{\phi_1} \to C_{\phi_2}$ be a definable bi-Lipschitz map. Then $F((0,0)) = (0,0)$.*

Proof. Suppose that $F((0,0)) = a \in C_{\phi_2}, a \neq (0,0)$. Consider two points $x = (t,0)$ and $y = (t,\phi_1(t))$ sufficiently close to $(0,0)$. Since C_{ϕ_2} is a 1-dimensional smooth manifold near the point a there exist two constants K_1 and K_2 such that

$$K_1 t \leq d(F(x),F(y)) \leq K_2 t.$$

But $d(x,y) = \phi_1(t)$. Thus, $\phi_1(t) \in Lip_A$. This is a contradiction. □

LEMMA 4.3. *Let $F : C_{\phi_1} \to C_{\phi_2}$ be a definable bi-Lipschitz map. Then there exists another definable bi-Lipschitz map $\widetilde{F} : C_{\phi_1} \to C_{\phi_2}$ such that the image of the graph of ϕ_1 by $\widetilde{F}$ belongs to the graph of ϕ_2 and the image of the positive x_1-axis by $\widetilde{F}$ belongs to itself.*

Proof. Let C_ϕ be a ϕ-semicusp. Let $F_\phi : C_\phi \to C_\phi$ be the map defined as follows: $F_\phi(x_1,x_2) = (x_1, -(x_2 - \phi(x_1)))$. F_ϕ is a definable bi-Lipschitz map because $\phi \in G_A^+$. Clearly, the image of the graph of ϕ by F_ϕ belongs to the positive x_1-axis and the image of the positive x_1-axis by F_ϕ belongs to the graph of ϕ. Let $F : C_{\phi_1} \to C_{\phi_2}$ be a bi-Lipschitz map such that the conclusion of the lemma do not hold. By the preceding lemma, $F(0,0) = (0,0)$. Set $\widetilde{F} = F_{\phi_2} \circ F$. Then $\widetilde{F}$ is a bi-Lipschitz map satisfying the conclusion of the lemma. □

Proof of Theorem 4.1. Let C_{ϕ_1} and C_{ϕ_2} be bi-Lipschitz equivalent in A. Note that $\phi \in Lip_A$ if and only if C_ϕ is a Lipschitz submanifold. Thus, if C_{ϕ_1} and C_{ϕ_2} are bi-Lipschitz equivalent then $\phi_1, \phi_2 \in Lip_A$ or $\phi_1, \phi_2 \notin Lip_A$.

Assume that $\phi_1, \phi_2 \notin Lip_A$. Let $\widetilde{F} : C_{\phi_1} \to C_{\phi_2}$ be a bi-Lipschitz map satisfying the conclusion of Lemma 4.3. Then for $(t,\phi_1(t)) \in \gamma^1_{\phi_1}$, we have:

$\widetilde{F}(t,\phi_1(t)) = (h(t),\phi_2(h(t)))$, for some definable bi-Lipschitz map $h : [0,\varepsilon) \to [0,\infty)$. Let us prove that the map $F : C_{\phi_1} \to C_{\phi_2}$ constructed as follows:

$$F(t,\phi_1(t)) = \widetilde{F}(t,\phi_1(t)), \qquad F(t,0) = (h(t),0) \tag{1}$$

is also a definable bi-Lipschitz map. Let $x = (t,0)$ and let $y = (t,\phi_1(t))$ be two points on C_{ϕ_1}. Then $F(x) = (h(t),0), F(y) = (h(t),\phi_2(h(t)))$ and $\widetilde{F}(x) = (\tilde{h}(t),0)$, for some definable bi-Lipschitz map $\tilde{h}$. Since $\widetilde{F}$ is a bi-Lipschitz map, there exists $K_1 > 0$ such that

$$\begin{aligned} d(y,x) &\geq K_1 d(\widetilde{F}(y),\widetilde{F}(x)) = K_1 d((h(t),\phi_2(h(t))),(\tilde{h}(t),0)) \\ &\geq K_1 d((h(t),\phi_2(h(t))),(h(t),0)) = K_1 d(F(y),F(x)). \end{aligned}$$

On the other hand, there exists $K_2 > 0$ such that

$$\begin{aligned} d(F(y),F(x)) &\geq K_2 d(\widetilde{F}^{-1}(F(y)),\widetilde{F}^{-1}(F(x))) = K_2 d(y,\widetilde{F}^{-1}(F(x))) \\ &= K_2 d((t,\phi_1(t)),(\tilde{h}^{-1}(h(t)),0)) \geq K_2\phi_1(t) = K_2 d(y,x). \end{aligned}$$

This proves that F is bi-Lipschitz. The map $F : C_{\phi_1} \to C_{\phi_2}$ can be presented in the following form: $F(x_1,x_2) = (h(x_1),l(x_2))$. Thus, $\phi_2 = l\phi_1 h^{-1}$. Since F is a bi-Lipschitz map, and since the maps h and l are definable in A, we obtain that the maps h and l are bi-Lipschitz. It means that ϕ_1 and ϕ_2 are RL-Lipschitz equivalent. By the results of Section 3 (Corollary 3.6), they are R-Lipschitz equivalent.

If $\phi_1,\phi_2 \in Lip_A$, then ϕ_1 and ϕ_2 are R-Lipschitz equivalent because Lip_A is a group.

Let ϕ_1 and ϕ_2 be R-Lipschitz equivalent: $\phi_1 = \phi_2\ell$. Let us define a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ in the following way:

$$F(x_1,x_2) = \begin{cases} (x_1,x_2), & \text{if } x_1 < 0, \\ (l(x_1),x_2), & \text{if } x_1 \geq 0. \end{cases} \tag{2}$$

F is a bi-Lipschitz map because $l \in Lip_A$. Clearly, $F(C_{\phi_1}) = C_{\phi_2}$. The theorem is proved. ☐

COROLLARY 4.4. *Two semicusps C_{ϕ_1} and C_{ϕ_2} are bi-Lipschitz equivalent in A if and only if $P(\phi_1) = P(\phi_2)$ where $P : G_A^+ \to \widetilde{H}$ is the quasi-valuation.*

DEFINITION 2. Let $\phi \in G_A^+$ and let $T_\phi = \{x = (x_1,x_2) | x_1 \geq 0, 0 \leq x_2 \leq \phi(x_1)\}$. A germ of the set T_ϕ is called *ϕ-triangle*. In other words, T_ϕ is a set bounded by C_ϕ.

PROPOSITION 4.5. *T_{ϕ_1} and T_{ϕ_2} are bi-Lipschitz equivalent in A (with respect to both inner and Euclidean metrics) if and only if $P(\phi_1) = P(\phi_2)$.*

Proof. If T_{ϕ_1} is bi-Lipschitz equivalent to T_{ϕ_2} then C_{ϕ_1} is bi-Lipschitz equivalent to C_{ϕ_2}. Thus, $P(\phi_1) = P(\phi_2)$ by Corollary 4.4.

Let $P(\phi_1) = P(\phi_2)$. Then ϕ_1 and ϕ_2 are R-Lipschitz equivalent. It means that there exists $l \in Lip_A$ such that $\phi_1(t) = \phi_2(l(t))$. Consider the map F defined in the end of the proof of Theorem 4.1. Clearly, F is a definable bi-Lipschitz map and $F(C_{\phi_1}) = C_{\phi_2}$. Observe that $T_{\phi_1} = \bigcup_{0\leq\alpha\leq 1} C_{\alpha\phi_1}$ and $T_{\phi_2} = \bigcup_{0\leq\alpha\leq 1} C_{\alpha\phi_2}$. For each $\alpha \in [0,1]$, we have that $\alpha\phi_1(t) = \alpha\phi_2(l(t))$. Hence, $F(T_{\phi_1}) = T_{\phi_2}$. $\square$

Proposition 4.6. *Let $\Psi : T_{\phi_1} \to T_{\phi_2}$ be a Lipschitz homeomorphism with $\Psi(0,0) = (0,0)$ and let there exists a positive constant C such that, for each $x \in T_{\phi_1}$, we have*:

$$Cd(x,(0,0)) \leq d(\Psi(x),(0,0)).$$

Then $P(\phi_2) \geq P(\phi_1)$.

Proof. By the assumption of the proposition, we obtain that

$$\phi_2\left(\frac{C}{2}t\right) \leq d(\Psi(t,0),\Psi(t,\phi_1(t))).$$

Since Ψ is a Lipschitz map, we have:

$$d(\Psi(t,0),\Psi(t,\phi_1(t))) \leq Kd((t,0),(t,\phi_1(t))) = K\phi_1(t),$$

for some constant K. Finally, we obtain: $\phi_2(\frac{C}{2}t) \leq K\phi_1(t)$. Since $P(\phi(t)) = P(\phi(\frac{C}{2}t))$, and by Lemma 3.5, $P(\phi) = P(K\phi)$, for all $\phi \in G_A^+$, we have: $P(\phi_2) \geq P(\phi_1)$. The proposition is proved. $\square$

Proposition 4.7. *Let $\phi \in G_A^+$ be a flat function. Let $\Psi : C_{\phi(t)} \to C_{\phi(kt)}$ be a map such that one of the following conditions holds*:

(1) *There exists $s < 1$ such that $d(\Psi(t,0),(0,0)) < \frac{s}{k}t$, $t > 0$.*
(2) *There exists $s > 1$ such that $d(\Psi(t,0),(0,0)) > \frac{s}{k}t$, $t > 0$.*

Then the map Ψ is not bi-Lipschitz.

Proof. Consider the first case: $d(\Psi(t,0),(0,0)) < \frac{s}{k}t$, for some $s < 1$. Suppose that the map Ψ is bi-Lipschitz. Using the same arguments as in the proof of Theorem 4.1, one can construct a bi-Lipschitz map $\widetilde{\Psi} : C_{\phi(t)} \to C_{\phi(kt)}$ such that $\widetilde{\Psi}(t,0) = (\psi(t),0)$ and $\widetilde{\Psi}(t,\phi(t)) = (\psi(t),\phi(k\psi(t)))$. Since $\psi(t) < \frac{s}{k}t$, we obtain that $\phi(k\psi(t)) < \phi(st)$. By Proposition 3.13, one has $v(\phi(t)) < v(\phi(st))$. It means that the map $\widetilde{\Psi}$ cannot be bi-Lipschitz.

If Ψ satisfies the condition (2) the proof is similar to the first case. $\square$

Proposition 4.8.

(1) *Two semicusps C_{ϕ_1} and C_{ϕ_2} are bi-Lipschitz equivalent if and only if they are bi-Lipschitz equivalent in A.*
(2) *Definable in A triangles T_{ϕ_1} and T_{ϕ_2} are bi-Lipschitz equivalent if and only if they are bi-Lipschitz equivalent in A.*

Proof. We are going to prove the first part. The second part can be proved in the same way. Note, that it is enough to prove that if there exists a bi-Lipschitz map $\Phi : C_{\phi_1} \to C_{\phi_2}$ then $P(\phi_1) = P(\phi_2)$. Consider a pair of points $(t,0)$ and $(t,\phi_1(t))$. Let $\Phi((t,0)) = (t_1,0)$ and $\Phi((t,\phi_1(t))) = (t_2,\phi_2(t_2))$. Let $\rho(t) = \min\{t_1,t_2\}$. Since Φ is a bi-Lipschitz map, there exists a constant $K_1 > 0$ such that $\rho(t) > K_1 t$. By the same reason, there exists a constant $K_2 > 0$ such that $d(\Phi(t,0),\Phi(t,\phi_1(t))) \le K_2\phi_1(t)$. Finally, we obtain that $K_2\phi_1(t) \ge \phi_2(K_1 t)$. Hence, $P(\phi_1) \le P(\phi_2)$.

Considering the map Φ^{-1} we obtain that $P(\phi_2) \le P(\phi_1)$. □

5. Germs of definable curves in $\mathbb{R}^n$

As in Section 4, we consider here a bi-Lipschitz equivalence with respect to the Euclidean metric.

We call a definable in A set of dimension 1 *a definable in A curve*. Let $X \subset \mathbb{R}^n$ be a definable in A curve and let $x_0 \in X$. By [17], there exists a neighborhood U_{x_0} of x_0 in $\mathbb{R}^n$ such that $X \cap U_{x_0} = \bigcup_{i=1}^k X_i$ satisfies the following conditions:

(1) For all i, X_i is definable in A.
(2) There exists a definable in A homeomorphism $h_i : [0,\varepsilon) \to X_i$ such that $h_i(0) = x_0$.
(3) For $i \neq j, X_i \cap X_j = x_0$.
(4) There exists a number r_0, such that for all $0 < r \le r_0$, we have: $\#(X_i \cap S^{n-1}_{x_0,r}) = 1$ (here $S^{n-1}_{x_0,r}$ is the sphere centered at x_0 of radius r).

The subsets X_i are called *the branches of X at x_0*.

Let X be a germ at x_0 of a definable curve with two branches X_1 and X_2. Let $x_i(t)$ be a point on X_i (where $i = 1,2$) such that $\|x_i(t) - x_0\| = t$. We define a *test function* τ_X as follows: $\tau_X(t) = \|x_1(t) - x_2(t)\|$. Observe that $\tau_X \in G_A^+$.

Theorem 5.1. *Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be two definable in A curves. Let us suppose that X has two branches at $x_0 \in X$ and Y has two branches at $y_0 \in Y$. Then the germs of X at x_0 and of Y at y_0 are bi-Lipschitz equivalent with respect to the Euclidean metric if and only if $P(\tau_X) = P(\tau_Y)$ where P is the quasi-valuation in G_A^+.*

In order to prove the theorem, we need some preliminary results. Observe that a ϕ-semicusp can be considered as a set described above, i.e., a definable curve with exactly two branches. Let τ_{C_ϕ} be the test function for C_ϕ.

Lemma 5.2. *Let v be a Rosenlicht valuation in K_A and let C_ϕ be a ϕ-semicusp. Then $v(\tau_{C_\phi}) = v(\phi)$.*

Proof. Let $v(\phi) > v(Id)$. Suppose that $v(\phi) < v(\tau_{C_\phi})$. Consider the triangle with vertices $A(t), B(t)$ and $C(t)$ where $A(t) = (t,0), B(t) = (t,\phi(t))$ and

$C(t)$ is the intersection of the graph of ϕ with a circle centered at $(0,0)$ of radius t. Since $\phi'(t)$ tends to 0 when t tends to 0, the angle at the vertex $B(t)$ has to tend to $\pi/2$. On the other hand, $\|B(t)-C(t)\| \ll \|A(t)-B(t)\|$.

Suppose that $v(\tau_{C_\phi}) < v(\phi)$. Consider again the triangle $A(t), B(t), C(t)$ defined above. Since $\|A(t)-C(t)\| \gg \|A(t)-B(t)\|$, the angle at the vertex $A(t)$ tends to $\frac{\pi}{2}$ when t tends to 0. It means that the angle at the vertex $A(t)$ in the triangle $(0,0), A(t), C(t)$ must tend to 0. But it is impossible because $\|A(t)\| = \|C(t)\|$.

If $v(\phi) = v(Id)$, the statement is trivial. □

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two definable in A curves. Assume that X have exactly two branches at $x_0 \in X$ and let Y have exactly two branches at $y_0 \in Y$. We define $x_i(t) \in X_i$ and $y_i(t) \in Y_i$ (here $i = 1,2$ and X_i, Y_i are corresponding branches of X and Y) as above: $\|x_i(t) - x_0\| = t$ and $\|y_i(t) - y_0\| = t$. Observe that for a sufficiently small t, these points are well defined. Let $\Phi : (X, x_0) \to (Y, y_0)$ be the map defined as follows: $\Phi(x_i(t)) = y_i(t)(i = 1,2)$.

Lemma 5.3. *Assume that $v(\tau_X) = v(\tau_Y)$. Then Φ is a definable bi-Lipschitz map.*

Proof. Without loss of generality, we can suppose that $X \cap Y = \emptyset$. Let us define a function $r(x)$ in the following way:

$$r(x) = \begin{cases} \|x - x_0\|, & \text{if } x \in X, \\ \|x - y_0\|, & \text{if } x \in Y. \end{cases}$$

Since $r(x)$ is a definable function, we conclude that Φ is a definable map. Let x_1, x_2 be two points sufficiently close to x_0 such that $x_1 \in X_1$ and $x_2 \in X_2$. Suppose that $r(x_1) \le r(x_2)$. Let $x_3 \in X_2$ be the point such that $r(x_3) = r(x_1)$. Since X and Y are definable sets, the branches are sufficiently close to their tangent vectors at x_0 and y_0 (see [18]).

We can suppose that the angles at the vertex x_3 of the triangle (x_1, x_2, x_3) and at the vertex $\Phi(x_3)$ of the triangle $(\Phi(x_1), \Phi(x_2), \Phi(x_3))$ tend to some values $\theta_1 > 0$ and $\theta_2 > 0$. These values θ_1 and θ_2 depend only on angles between the tangent vectors of branches. (For example, if X_1 and X_2 have the same tangent vector then $\theta_1 = \pi/2$.) Thus, there exist two positive constants K_1 and K_2 such that

$$K_1 \max\{|r(x_1) - r(x_2)|, \tau_X(r(x_1))\} \le \|x_1 - x_2\|,$$
$$K_2 \max\{|r(x_1) - r(x_2)|, \tau_X(r(x_1))\} \ge \|x_1 - x_2\|$$

and

$$K_1 \max\{|r(\Phi(x_1)) - r(\Phi(x_2))|, \tau_Y(r(\Phi(x_1)))\} \le \|\Phi(x_1) - \Phi(x_2)\|,$$
$$K_2 \max\{|r(\Phi(x_1)) - r(\Phi(x_2))|, \tau_Y(r(\Phi(x_1)))\} \ge \|\Phi(x_1) - \Phi(x_2)\|.$$

Since $v(\tau_X) = v(\tau_Y)$, there exist two positive constants M_1 and M_2 such that $M_1\tau_X(r(x_1)) \leq \tau_Y(r(x_1)) \leq M_2\tau_X(r(x_1))$. By the definition of the map Φ, we have $r(x_i) = r(\Phi(x_i))$ $(i = 1,2,3)$. Thus, the above inequalities imply that Φ is a bi-Lipschitz map. □

Proof of Theorem 5.1. By Lemma 5.2 and by Lemma 5.3, a set X satisfying the conditions of the theorem is bi-Lipschitz equivalent in A to τ_X-semicusp C_{τ_X}. By Corollary 4.4, two semicusps C_{τ_X} and C_{τ_Y} are bi-Lipschitz equivalent in A if and only if $P(\tau_X) = P(\tau_Y)$. By Theorem 4.8, bi-Lipschitz equivalence of the semicusps is equivalent to bi-Lipschitz equivalence in A. □

Definition 3. A complete finite graph Γ with a function $\alpha : E_\Gamma \to H^+$ (where H^+ is a subset of the value group H defined as follows: $H^+ = v(G_A^+)$) is called *a valuation semicomplex* if α satisfies the “isosceles” condition: for all $a_1, a_2, a_3 \in V_\Gamma$, the following is hold: if $\alpha(a_1, a_2) \leq \alpha(a_2, a_3) \leq \alpha(a_1, a_3)$, then $\alpha(a_1, a_2) = \alpha(a_2, a_3)$.

Definition 4. A complete finite graph $\widetilde{\Gamma}$ with a function $\tilde{\alpha} : E_{\widetilde{\Gamma}} \to \widetilde{H}$ (where $\widetilde{H}$ is an ordered set associated with the quasi-valuation $P : G_A^+ \to \widetilde{H}$) is called *a quasi-valuation semicomplex* if $\tilde{\alpha}$ satisfies the “isosceles” condition.

Let (X, x_0) be the germ at $x_0 \in X$ of a definable in A curve. We associate a valuation semicomplex (Γ, α) to (X, x_0) in the following way. The branches X_i of X correspond to the vertices a_i of Γ. Let $X_{ij} = X_i \cup X_j$. Set $\alpha(a_i, a_j) = v(\tau_{X_{ij}})$. We associate a quasi-valuation semicomplex $(\widetilde{\Gamma}, \tilde{\alpha})$ to (X, x_0) in a similar way: set $\widetilde{\Gamma} = \Gamma$ and $\tilde{\alpha}(a_i, a_j) = P(\tau_{X_{ij}})$.

Proposition 5.4.

(1) (Γ, α) *is a valuation semicomplex.*

(2) $(\widetilde{\Gamma}, \tilde{\alpha})$ *is a quasi-valuation semicomplex.*

Proof.

1. We must prove the isosceles property. Let X_i, X_j, X_k be three branches of X at x_0. Since K_A is a Hardy field, we can suppose that $\tau_{X_{ij}} \leq \tau_{X_{ik}} \leq \tau_{X_{jk}}$. Thus, $v(\tau_{X_{ij}}) \geq v(\tau_{X_{ik}}) \geq v(\tau_{X_{jk}})$. But, $\tau_{X_{jk}} \leq \tau_{X_{ij}} + \tau_{X_{ik}}$. Since v is a Rosenlicht valuation, we obtain: $v(\tau_{X_{jk}}) = \min\{v(\tau_{X_{ij}}), v(\tau_{X_{ik}})\}$.
2. The proof of Assertion 2 is the same. □

Definition 5. Two valuation (quasi-valuation) semicomplexes (Γ, α) and (Γ', α') are called *isomorphic* if there exists an isomorphism $f : \Gamma \to \Gamma'$ such that, for all a_i, a_j, we have: $\alpha(a_i, a_j) = \alpha'(f(a_i), f(a_j))$.

Let (Γ, α) be a valuation semicomplex. We can define a corresponding quasi-valuation semicomplex $(\widetilde{\Gamma}, \tilde{\alpha})$ in the following way. Set $\widetilde{\Gamma} = \Gamma$ and set $\tilde{\alpha}(a_i, a_j) = \bar{v}(\alpha(a_i, a_j))$ where $\bar{v}$ is the map defined in Theorem 3.9. Clearly, $(\widetilde{\Gamma}, \tilde{\alpha})$ is a quasi-valuation semicomplex. The following result shows that a quasi-valuation semicomplex is an invariant under bi-Lipschitz maps and an isomorphism of valuation semicomplexes is a criterion of a bi-Lipschitz equivalence.

THEOREM 5.5.

(1) *If two germs of definable curves* (X, x_0) *and* (Y, y_0) *are bi-Lipschitz equivalent, then the corresponding quasi-valuation semicomplexes are isomorphic.*
(2) *If the valuation semicomplexes associated to the germs of definable curves* (X, x_0) *and* (Y, y_0) *are isomorphic, then these germs are bi-Lipschitz equivalent.*

Proof.

1. If $F : (X, x_0) \to (Y, y_0)$ is a bi-Lipschitz map, then for all pairs of branches X_i, X_j, by Theorem 5.1, we have: $P(\tau_{X_{ij}}) = P(\tau_{F(X_{ij})})$. Hence, the corresponding quasi-valuation semicomplexes are isomorphic.

2. Let (X, x_0) and (Y, y_0) be a pair of germs of definable curves such that the corresponding valuation semicomplexes are isomorphic. Let $x_i(\varepsilon)$ be a point on the branch X_i of X such that $\|x_i(\varepsilon) - x_0\| = \varepsilon$. Let $y_i(\varepsilon)$ be a point on the branch Y_i of Y such that $\|y_i(\varepsilon) - y_0\| = \varepsilon$. Set $\Phi(x_i(\varepsilon)) = y_i(\varepsilon)$. By the same argument, as in the proof of Lemma 5.3, the germ of Φ at x_0 is a germ of a bi-Lipschitz map. □

REMARK. A valuation semicomplex, in general, is not a bi-Lipschitz invariant. To see it consider the semicusps C_{ϕ_1} and C_{ϕ_2} with $\phi_1(t) = e^{-1/t}$ and $\phi_2(t) = e^{-1/2t}$.

The following statement shows that in the nonflat case a quasi-valuation semicomplex is a complete bi-Lipschitz invariant.

COROLLARY 5.6. *Let* (X, x_0) *and* (Y, y_0) *be the germs of definable in* A *curves where* A *is an o-minimal structure, not necessarily polynomially bounded. Suppose that all functions* $\tau_{X_{ij}}$ *and* $\tau_{Y_{ij}}$ *are nonflat at* 0. *Then* (X, x_0) *and* (Y, y_0) *are bi-Lipschitz equivalent if and only if the corresponding valuation* (*quasi-valuation*) *semicomplexes are isomorphic.*

Proof. By Theorem 3.11, in this case valuation semicomplexes are totally determined by quasi-valuation semicomplexes. Thus, the statement follows from Theorem 5.5. □

THEOREM 5.7 (Realization theorem for definable curves). *Let* (Γ, α) *be a valuation semicomplex. Then there exist a definable in* A *curve* $X \subset \mathbb{R}^2$ *and a point* $x_0 \in X$ *such that* (Γ, α) *is a valuation semicomplex associated to* (X, x_0). *The germ* (X, x_0) *is called a* realization of (Γ, α).

Proof. Let V_Γ be a set of vertices of Γ. We use the induction on $\sharp V_\Gamma$ (the number of vertices of Γ). For $\sharp V_\Gamma = 1$, the statement is trivial. Suppose that the statement is proved for all Γ such that $\sharp V_\Gamma \le k$. Moreover, suppose that there exists a realization of (Γ,α) satisfying the following conditions:

(1) $x_0=(0,0)\in\mathbb{R}^2$.
(2) Each branch X_i of X at x_0 is a graph of a definable in A function $\phi_i:[0,\varepsilon)\to\mathbb{R}$ such that $\phi_i(0)=0,\phi_i(t)\ge 0$, for $t\ge 0$, and $\phi_i(t)>\phi_{i-1}(t)$, for $t>0$.

Let $\alpha_0=\max_{1\le i,j\le k+1}\alpha(a_i,a_j)$. We can suppose without loss of generality that $\alpha_0=\alpha(a_k,a_{k+1})$. Let $\widetilde{\Gamma}$ be a graph obtained from Γ by exclusion of the vertex a_{k+1}. Let $\tilde\alpha=\alpha|_{V_{\tilde\Gamma}\times V_{\tilde\Gamma}}$. Let $(\widetilde{X},(0,0))$ be a realization of $(\widetilde{\Gamma},\tilde\alpha)$ satisfying the conditions (1) and (2). Let $\{\widetilde{X}_i\}$ be the branches of $\widetilde{X}$. Let each $\widetilde{X}_i$ be a graph of a definable function $\tilde\phi_i:[0,\varepsilon)\to\mathbb{R}$. Let $\psi\in G_A^+$ be a definable function such that $v(\psi)=\alpha_0$. Set $\phi_i=\tilde\phi_i$, for $i=1,2,\dots,k$ and $\phi_{k+1}=\tilde\phi_k+\psi$. Then by Lemma 5.2 and straightforward calculations, we obtain that $X=\bigcup graph(\phi_i)$ is a realization of (Γ,α). $\square$

Corollary 5.8. *Let $(\widetilde{\Gamma},\tilde\alpha)$ be a quasi-valuation semicomplex. Then there exist a definable in A curve $X\subset\mathbb{R}^2$ and a point $x_0\in X$ such that $(\widetilde{\Gamma},\tilde\alpha)$ is a quasi-valuation semicomplex associated to (X,x_0).*

Proposition 5.9. *Let (X,x_0) and (Y,y_0) be germs of definable in A curves such that they have exactly two branches. Suppose that τ_X and τ_Y are flat functions such that $v(\tau_X)=v(\tau_Y)$. Let $F:(X,x_0)\to(Y,y_0)$ be a map satisfying one of the following conditions*:

(1) *$F(x_0)=y_0$ and there exists $0<S<1$ such that $d(F(x),y_0)<Sd(x,x_0)$, for all $x\in X$.*
(2) *$F(x_0)=y_0$ and there exists $S>1$ such that $d(F(x),y_0)>Sd(x,x_0)$, for all $x\in X$.*

Then the map F is not bi-Lipschitz.

This proposition is a corollary of Theorem 5.1 and Proposition 4.7.

Corollary 5.10. *Let ϕ be a definable in A flat function. Let (X,x_0) and (Y,y_0) be germs of definable in A curves such that they have exactly two branches. Let $v(\tau_X)=v(\phi(t))$ and $v(\tau_Y)=v(\phi(Kt))$, for some $K>0$. Let $F:(X,x_0)\to(Y,y_0)$ with $F(x_0)=y_0$ satisfying one of the following conditions*:

(1) *There exists $0<S<1$ such that $d(F(x),y_0)<\frac{S}{K}d(x,x_0)$, for all $x\in X$.*
(2) *There exists $S>1$ such that $d(F(x),y_0)>\frac{S}{K}d(x,x_0)$, for all $x\in X$.*

Then a map F is not bi-Lipschitz.

The following result shows that the quasi-valuation semicomplex is not a complete bi-Lipschitz invariant.

THEOREM 5.11. *Let A be an o-minimal structure which is not polynomially bounded. Then there exists a pair of germs (X, x_0) and (Y, y_0) of definable in A curves which are not bi-Lipschitz equivalent but the corresponding quasi-valuation semicomplexes are isomorphic.*

Proof. Let ϕ be a definable in A flat function. Let (Γ_1, α_1) be a valuation semicomplex with vertices a_1^1, a_2^1, a_3^1 such that $\alpha_1(a_1^1, a_2^1) = \alpha_1(a_2^1, a_3^1) = \alpha_1(a_1^1, a_3^1) = v(\phi(t))$. Let (Γ_2, α_2) be a valuation semicomplex with vertices a_1^2, a_2^2, a_3^2 defined as follows:

$$\alpha_2(a_1^2, a_2^2) = \alpha_2(a_1^2, a_3^2) = v(\phi(t)), \qquad \alpha_2(a_2^2, a_3^2) = v\big(\phi(t/2)\big).$$

Note, that the quasi-valuation semicomplexes corresponding to (Γ_1, α_1) and to (Γ_2, α_2) are isomorphic because $P(\phi(t)) = P(\phi(t/2))$

Let (X, x_0) be a realization of (Γ_1, α_1) and let (Y, y_0) be a realization of (Γ_2, α_2). Suppose that (X, x_0) and (Y, y_0) are bi-Lipschitz equivalent. Let $F : (X, x_0) \to (Y, y_0)$ be a corresponding bi-Lipschitz map. Let X_1, X_2, X_3 be branches of X and let Y_1, Y_2, Y_3 be corresponding branches of Y. The set $X_1 \cup X_2$ is bi-Lipschitz equivalent to $Y_1 \cup Y_2$. By Proposition 5.9, applied to this pair of germs, we obtain that $d(F(x), y_0) < \frac{3}{2} d(x, x_0)$, for x sufficiently close to x_0. But, by Corollary 5.10 applied to $F : X_2 \cup X_3 \to Y_2 \cup Y_3$, we obtain that $d(F(x), y_0) > \frac{3}{2} d(x, x_0)$. This is a contradiction. □

6. Quasi-valuation complexes

Let A be an o-minimal structure and let $P : G_A^+ \to \widetilde{H}$ be a quasi-valuation in A. Let Γ be a finite graph and let E_Γ be the set of edges of Γ. A pair (Γ, β) (where $\beta : E_\Gamma \to \widetilde{H}$) is called *a quasi-valuation complex.* Two quasi-valuation complexes (Γ, β) and (Γ', β') are called *isomorphic* if there exists an isomorphism $i : \Gamma \to \Gamma'$ such that, for each edge $g \in E_\Gamma$, we have $\beta(g) = \beta'(i(g))$.

A vertex $a \in V_\Gamma$ is called *a smooth* vertex if it is connected with exactly two vertices and by exactly one edge with each of them. A vertex $a \in V_\Gamma$ is called *a loop* vertex if it is connected with only one other vertex by two edges. A graph Γ is called *simplified* if it has no smooth vertices. A quasi-valuation complex (Γ, β) is called *simplified* if Γ is simplified and, for any loop vertex a and for two edges g_1 and g_2 connected to a, we have $\beta(g_1) = \beta(g_2)$.

REMARK. The Hölder complexes considered in [1] give examples of quasi-valuation complexes. In this case, since the semi-algebraic structure is polynomially bounded, the quasi-valuation coincides with the canonical Rosenlicht valuation.

Now, we are going to describe a simplification procedure of the quasi-valuation complexes. This procedure is essentially the same one as in [1] but we are going to present it here in order to make our exposition self contained.

Elimination of a smooth vertex. Let (Γ,β) be a quasi-valuation complex and let $a\in V_\Gamma$ be a smooth vertex. Let g_1 and g_2 be two edges connected to a. Let a_1 and a_2 be two other vertices connected with a. Let us define a quasi-valuation complex (Γ',β') in the following way. Let us cut the union of g_1, g_2 and a from Γ and connect the vertices a_1 and a_2 by a new edge g'. Set $\beta'(g')=\min(\beta(g_1),\beta(g_2))$. For other edges $g\in E_{\Gamma'}$, we put $\beta'(g)$ the same as it was in (Γ,β).

Correction near a loop vertex. Let (Γ,β) be a quasi-valuation complex and let $a\in V_\Gamma$ be a loop vertex. Let g_1 and g_2 be two edges connected to a. We define a quasi-valuation complex (Γ',β') in the following way. Set $\Gamma'=\Gamma$. Set $\beta'(g)=\beta(g)$, for all edges $g\neq g_1$ and $g\neq g_2$. Set $\beta'(g_1)=\beta'(g_2)=\min(\beta(g_1),\beta(g_2))$.

A simplified quasi-valuation complex (Γ',β') is called *a simplification of* (Γ,β) if it can be obtained from (Γ,β) by a finite sequence of operations described above.

THEOREM 6.1 ([1]). *For any quasi-valuation complex* (Γ,β), *there exists a simplification. Two simplifications of the same quasi-valuation complex are isomorphic.*

7. Quasi-valuation complexes and definable surfaces

In this section, we study a bi-Lipschitz equivalence with respect to the inner metric. The word "bi-Lipschitz" means bi-Lipschitz with respect to this metric.

Let A be an o-minimal structure. Let (Γ,β) be a quasi-valuation complex.

DEFINITION 6. A germ at a point x_0 of a definable in A surface X is called *a Geometric Quasi-valuation Complex associated to* (Γ,β) if:

1. For some small ε, $X\cap B_{x_0,\varepsilon}$ is homeomorphic to $C\Gamma$ (here $B_{x_0,\varepsilon}$ is a ball centered at x_0 of radius ε and $C\Gamma$ is a cone over Γ).
2. Let $\Phi: C\Gamma\to X\cap B_{x_0,\varepsilon}$ be a homeomorphism and let $\Phi(a_0)=x_0$ (here a_0 is a vertex of $C\Gamma$). Let $Cg\subset C\Gamma$ be the subcone of $C\Gamma$ corresponding to the edge g. Then there exist a function $\psi\in G_A^+$ and a definable in A bi-Lipschitz map $\Psi:\Phi(Cg)\to T_\psi$ such that $P(\psi)=\beta(g)$ and $\Psi(x_0)=(0,0)$.

Let $X\subset\mathbb{R}^n$ be a definable in A closed surface and let $x_0\in X$. Let $\{X_i\}$ be a definable triangulation and simultaneously a pancake decomposition of X. Let S be a standard simplicial complex corresponding to the triangulation $\{X_i\}$. Let $\theta: S\to X$ be a definable triangulation map. Let $\tilde{x}_0=\theta^{-1}(x_0)$. Let $\widetilde{S}$ be a star of the vertex $\tilde{x}_0$, i.e., $\widetilde{S}$ contains the simplices of S such that $\tilde{x}_0$ is a vertex of these simplices. A quasi-valuation complex (Γ,β) corresponding to the germ (X,x_0) can be constructed in the following way. Let Γ be a graph-link of $\tilde{x}_0$ in S, i.e., the vertices of Γ are one-dimensional faces of $\widetilde{S}$

and the edges of Γ are two-dimensional faces of $\widetilde{S}$. Moreover, two vertices are connected by an edge if and only if the corresponding one-dimensional faces belong to the boundary of the corresponding two-dimensional face. Clearly, $\widetilde{S}$ is homeomorphic to the cone over Γ. Let g be an edge of Γ. Let a_1 and a_2 be two vertices connected to g. Let s_1 and s_2 be the one-dimensional faces corresponding to a_1 and a_2. Let $\gamma_1 = \theta(s_1)$ and $\gamma_2 = \theta(s_2)$. Clearly, $x_0 = \gamma_1 \cap \gamma_2$. Let τ be the test function defined, in Section 5, for the pair of curves γ_1 and γ_2. Set $\beta(g) = P(\tau)$.

Proposition 7.1. *The germ of X at x_0 is a Geometric quasi-valuation Complex associated to* (Γ, β).

Proof. Condition 1 of the definition of the Geometric quasi-valuation Complex is satisfied because $\{X_i\}$ is a triangulation of X.

Let X_j be a simplex of a triangulation $\{X_i\}$ of X such that x_0 is a vertex of this simplex. If $\dim X_j = 1$, then it corresponds to an isolated vertex in Γ. If $\dim X_j = 2$, then the germ of X_j at x_0 is bi-Lipschitz equivalent to a germ at $(0,0)$ of some 2-dimensional definable set $Y \subset \mathbb{R}^2$, because X_j is a pancake (see [11]). The set Y can be obtained from X_j using a projection to some 2-dimensional subspace of $\mathbb{R}^n$ (see [1] for a complete description of this procedure in a semialgebraic case). Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two boundary curves of Y. Then we can choose a coordinate system in $\mathbb{R}^2$ such that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are graphs of some definable in A functions: $\tilde{\gamma}_1 = (x, \psi_1(x)), \tilde{\gamma}_2 = (x, \psi_2(x))$. Then using the same construction as in Section 4 (Theorem 4.1) we can show that Y is bi-Lipschitz equivalent to a germ at $(0,0)$ of a set T_ψ where $\psi = \psi_1 - \psi_2$. By Proposition 4.5 and Theorem 5.5, $P(\tau) = P(\psi)$. $\square$

In fact, Proposition 7.1 can be reformulated in the following form.

Theorem 7.2. *Let $X \subset \mathbb{R}^n$ be a definable in A closed surface and let $x_0 \in X$. Then there exists a quasi-valuation complex (Γ, β) such that a germ of X at x_0 is a Geometric quasi-valuation Complex associated to (Γ, β).*

Remark. Note that the quasi-valuation complex defined in Theorem 7.2 is not canonical, i.e., depends on the choice of a pancake decomposition. It becomes canonical if we use the simplification procedure.

Theorem 7.3. *Let (Γ, β) be a quasi-valuation complex. Let X be a definable in A set, let $x_0 \in X$ and let the germ of X at x_0 be a Geometric quasi-valuation Complex associated to (Γ, β). Let $(\widetilde{\Gamma}, \tilde{\beta})$ be the simplification of (Γ, β). Then the germ of X at x_0 is a Geometric quasi-valuation Complex associated to $(\widetilde{\Gamma}, \tilde{\beta})$.*

We need some preliminary results.

Lemma 7.4. *Let $Y \subset \mathbb{R}^n$ be definable set such that there exists a definable bi-Lipschitz map $\Psi : Y \to T_\psi$ where $\psi \in G_A^+$ and $\Psi(y_0) = (0,0)$, for some*

$y_0 \in Y$. Let γ_1 and γ_2 be curves defined as follows: $\gamma_1(t) = \Psi^{-1}((t,0))$ and $\gamma_2(t) = \Psi^{-1}((t,\psi(t))$. Then there exist $\phi \in G_A^+$ and a definable bi-Lipschitz map $\widetilde{\Psi} : Y \to T_\phi$ such that $P(\psi) = P(\phi)$ and

$$\widetilde{\Psi}(y) = \begin{cases} (r(y),0), & \text{if } y \in \gamma_2, \qquad (3) \\ (r(y),\phi(r(y))), & \text{if } y \in \gamma_1, \qquad (4) \end{cases}$$

where $r(y) = \|y - y_0\|$.

Proof. Since γ_1 is a definable curve, it has a tangent vector at y_0, and thus, the germ at 0 of the function $\tilde{r}(t) = r(\gamma_1(t))$ belongs to Lip_A.

Let $\Psi_1 : [0,\infty) \times \mathbb{R} \to [0,\infty) \times \mathbb{R}$ be a map defined as follows: $\Psi_1(x_1,x_2) = (\tilde{r}(x_1),x_2)$. Set $\widetilde{\Psi}_1 = \Psi_1 \circ \Psi$. Thus, for $y \in \gamma_1$, we obtain $\widetilde{\Psi}_1(y) = (r(y),0)$. By the construction, $\widetilde{\Psi}_1$ is a definable bi-Lipschitz map. The image of T_ψ by the map Ψ_1 is a set bounded by the straight line $x_2 = 0$ and a graph of some function $\phi \in G_A^+$. By the definition of the map $\widetilde{\Psi}_1$, the germs of ϕ and ψ are R-Lipschitz equivalent and, thus, $P(\phi) = P(\psi)$. Let $\theta : [0,\infty) \times \mathbb{R} \to [0,\infty) \times \mathbb{R}$ be the map defined as follows:

$$\theta(x_1,x_2) = (x_1,-x_2), \qquad \theta_1(x_1,x_2) = \big(x_1, x_2 + \phi(x_1)\big).$$

We define $\widetilde{\Psi}_2 : Y \to T_\phi$ in the following way:

$$\widetilde{\Psi}_2 = \theta_1 \circ \theta \circ \widetilde{\Psi}_1.$$

Clearly, $\widetilde{\Psi}_2$ is a definable bi-Lipschitz map. For $y \in \gamma_1$, we obtain that

$$\widetilde{\Psi}_2(y) = (r(y),\phi(r(y))). \tag{5}$$

Thus, the condition (4) is satisfied and now we are going to correct the map $\widetilde{\Psi}_2$ in order to obtain the condition (3).

Let $R(x) = r(\widetilde{\Psi}_2^{-1}(x,0))$. We will show that

$$R(x) = x + \phi_1(x),$$

for some $\phi_1 \in G_A^+$ such that $P(\phi_1(x)) \geq P(\phi(x))$.

Since $\widetilde{\Psi}_2$ is a bi-Lipschitz map, there exists a number $K > 0$ such that

$$K\phi(x) = K\|(x,0) - (x,\phi(x))\| \geq d_\ell(\widetilde{\Psi}_2^{-1}(x,0), \widetilde{\Psi}_2^{-1}(x,\phi(x))).$$

On the other hand,

$$\begin{aligned} d_\ell(\widetilde{\Psi}_2^{-1}(x,0), \widetilde{\Psi}_2^{-1}(x,\phi(x))) &\geq \|\widetilde{\Psi}_2^{-1}(x,0) - \widetilde{\Psi}_2^{-1}(x,\phi(x))\| \\ &\geq |\|\widetilde{\Psi}_2^{-1}(x,0) - y_0\| - \|\widetilde{\Psi}_2^{-1}(x,\phi(x)) - y_0\||. \end{aligned}$$

Using (5), we obtain

$$\|\widetilde{\Psi}_2^{-1}(x,\phi(x)) - y_0\| = x$$

and by the definition of $R(x)$, we have:

$$\|\widetilde{\Psi}_2^{-1}(x,0) - y_0\| = R(x).$$

Thus,

$$K\phi(x) \geq R(x) - x = \phi_1(x) \quad \text{and} \quad v(\phi_1(x)) \geq v(\phi(x)).$$

By the results of Section 3, we obtain

$$P(\phi_1(x)) \geq P(\phi(x)).$$

Suppose now that $v(\phi(x)) > v(Id)$. We define a map $\Psi_3 : T_\phi \to T_\phi$ in the following way:

$$\begin{aligned}(6)\qquad &\Psi_3(x_1, x_2)\\ &= \begin{cases} \big(R(x_1)\big(1 - \frac{x_2}{\phi(x_1)}\big) + x_1\big(\frac{x_2}{\phi(x_1)}\big), x_2\big), & \text{if } (x_1, x_2) \neq (0,0), \\ (0,0), & \text{if } (x_1, x_2) = (0,0). \end{cases}\end{aligned}$$

Let us show that Ψ_3 is a bi-Lipschitz map. Computing the derivatives we obtain

$$\frac{\partial \Psi_3^1}{\partial x_1} = 1 - \frac{d\phi_1}{dx_1}\left(\frac{x_2}{\phi(x_1)}\right) + \frac{d\phi}{dx_1}\left(\frac{x_2\phi_1(x_1)}{\phi(x_1)^2}\right) + \frac{d\phi_1}{dx_1}.$$

Since $v(\phi) > v(Id)$, then $\frac{d\phi}{dx_1}$ and $\frac{d\phi_1}{dx_1}$ tend to zero when x_1 tends to zero. Thus, for small $\varepsilon > 0$, there exists $\delta > 0$ such that if $(x_1, x_2) \in T_\phi \cap B_{(0,0),\delta}$ then $\frac{\partial \Psi_3^1}{\partial x_1} \in (1 - \varepsilon, 1 + \varepsilon)$. Computing $\frac{\partial \Psi_3^1}{\partial x_2}$, we obtain

$$\frac{\partial \Psi_3^1}{\partial x_2}(x_1, x_2) = -\frac{\phi_1(x_1)}{\phi(x_1)}.$$

Since $v(\phi_1) \geq v(\phi)$, we have that $\frac{\partial \Psi_3^1}{\partial x_2}$ is bounded.

Finally, $\|D\Psi_3\|$ is bounded away from 0 and infinity, Ψ_3 is homeomorphism near $(0,0)$, and thus, the germ of Φ_3 at $(0,0)$ is a germ of a bi-Lipschitz map.

Let $v(\phi) = v(Id)$. We can suppose that $\phi(x) < \frac{x}{3}$ and $\frac{d\phi}{dx} < \frac{1}{3}$, for x sufficiently close to 0. Otherwise, we apply a corresponding linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. Then the map $\Psi_3 : T_\phi \to T_\phi$ defined in the same way as above is bi-Lipschitz by the same arguments as in the first part of the proof. Note, that all the points belonging to the curve $(x_1, \phi(x_1))$ are the fixed points of Ψ_3.

Let us define $\widetilde{\Psi} : Y \to T_\phi$ as follows:

$$\widetilde{\Psi} = \Psi_3 \circ \widetilde{\Psi}_2.$$

It easy to see that for $y \in \gamma_1$, we obtain $\widetilde{\Psi}(y) = (r(y), \phi(r(y)))$, and for $y \in \gamma_2$, we have $\widetilde{\Psi}(y) = (r(y), 0)$. The lemma is proved. $\square$

Proof of Theorem 7.3. We can suppose that a simplification $(\widetilde{\Gamma}, \tilde{\beta})$ is obtained from (Γ, β) by using a single operation: *an elimination of a smooth vertex* or *a correction near a loop vertex.*

Consider the first case. Let a be a smooth vertex and let g_1 and g_2 be two edges connected to the vertex a. Let $\Phi : C\Gamma \to X \cap B_{x_0,\varepsilon}$ be a homeomorphism from the definition of a Geometric quasi-valuation Complex associated to (Γ, β) (see Definition 6). Using Lemma 7.4, we can construct definable bi-Lipschitz maps $\Psi_{g_1} : \Phi(Cg_1) \to T_{\phi_1}$ and $\Psi_{g_2} : \Phi(Cg_2) \to T_{\phi_2}$ such that $P(\phi_1) = \beta(g_1)$ and $P(\phi_2) = \beta(g_2)$. Let $\tilde{\theta} : T_{\phi_2} \to \mathbb{R}^2$ be a map defined as follows:

$$\tilde{\theta}(x_1, x_2) = \big(x_1, x_2 + \phi_1(x_1)\big).$$

Let us define a map $\Psi_g : \Phi(Cg_1) \cup \Phi(Cg_2) \to T_{\phi_1+\phi_2}$ in the following way:

$$\Psi_g(x) = \begin{cases} \Psi_{g_1}(x), & \text{if } x \in \Phi(Cg_1), \\ \tilde{\theta}(\Psi_{g_2}(x)), & \text{if } x \in \Phi(Cg_2). \end{cases}$$

This map is definable in A, continuous on $\Phi(Cg_1) \cup \Phi(Cg_2)$ (by Lemma 7.4) and bi-Lipschitz on $\Phi(Cg_1)$ and on $\Phi(Cg_2)$. Hence, it is bi-Lipschitz on $\Phi(Cg) = \Phi(Cg_1) \cup \Phi(Cg_2)$ with respect to the intrinsic metric. Since P is a quasi-valuation, we obtain that $P(\phi_1 + \phi_2) = \min(P(\phi_1), P(\phi_2))$. It means that for some $\varepsilon > 0$, the set $X \cap B_{x_0,\varepsilon}$ is a Geometric quasi-valuation Complex associated to $(\widetilde{\Gamma}, \tilde{\beta})$.

Consider the second case when $(\widetilde{\Gamma}, \tilde{\beta})$ can be obtained from (Γ, β) using *correction near a loop vertex*. Observe that a set T_ϕ can be considered as a union of two sets T_1 and T_2 such that they are bi-Lipschitz equivalent to T_ϕ. Namely, $T_1 = T_{\frac{\phi}{2}}$ and $T_2 = \{(x_1, x_2) | \frac{\phi(x_1)}{2} \leq x_2 \leq \phi(x_1)\}$. Let a be a loop vertex of (Γ, β). Let g_1 and g_2 be the edges connecting a with another vertex b. Suppose that $\beta(g_2) > \beta(g_1)$. Let $\Psi_{g_1} : \Phi(Cg_1) \to T_{\phi_1}$ and $\Psi_{g_2} : \Phi(Cg_2) \to T_{\phi_2}$ be maps constructed in Lemma 7.4. The set T_{ϕ_1} can be divided into the sets T_1 and T_2 such that T_1 and T_2 are bi-Lipschitz equivalent to T_{ϕ_1}. Now, we can construct a quasi-valuation Complex (Γ', β') such that Γ' is obtained from Γ by adding an additional vertex a' on the edge g_1. The edge g_1 is decomposed into new edges g'_1 and g'_2 connecting a' with a and b correspondingly. Set $\beta(g'_1) = \beta(g'_2) = \beta(g_1)$. Clearly, $X \cap B_{x_0,\varepsilon}$ is a Geometric quasi-valuation Complex associated to (Γ', β'). The set $\Phi(Cg'_1)$ is defined as $\Psi_{g_1}^{-1}(T_2)$. Then a is a smooth vertex of (Γ', β'), and the first part of proof can be applied to this case. Clearly, $(\widetilde{\Gamma}, \tilde{\beta})$ is a simplification of (Γ', β'). We obtained that $X \cap B_{x_0,\varepsilon}$ is a Geometric quasi-valuation Complex associated to $(\widetilde{\Gamma}, \tilde{\beta})$. □

8. Horns. Isolated singularities

Let A be an o-minimal structure and let $\phi \in G_A^+$ be a germ of a definable in A function. A set $W_\phi \subset \mathbb{R}^3$ defined as follows:

$$W_\phi = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3, x_3 \geq 0, \sqrt{x_1^2 + x_2^2} = \phi(x_3)\right\}$$

is called ϕ-*horn*.

W_ϕ can be obtained as a "surface of revolution" of the graph of ϕ. It is easy to see that W_ϕ is normally embedded in $\mathbb{R}^3$. Each point $x=(x_1,x_2,x_3)$ belonging to W_ϕ has natural "polar" coordinates: $\rho(x)=x_3$ and the angle coordinate $\eta(x)$ defined as follows: $\cos\eta(x)=\frac{x_1}{\phi(x_3)}, \sin\eta(x)=\frac{x_2}{\phi(x_3)}$.

THEOREM 8.1. *W_{ϕ_1} and W_{ϕ_2} are bi-Lipschitz equivalent if and only if $P(\phi_1)=P(\phi_2)$.*

REMARK. Here, a bi-Lipschitz equivalence can be considered with respect to the inner or to the Euclidean metric. The both notions are the same because W_ϕ is normally embedded in $\mathbb{R}^3$.

Proof of Theorem 8.1. If $P(\phi_1)=P(\phi_2)$, then by Corollary 4.4, the semi-cusps C_{ϕ_1} and C_{ϕ_2} are bi-Lipschitz equivalent. Hence, one can extend a bi-Lipschitz map to surfaces obtained by revolution of C_{ϕ_1} and C_{ϕ_2}.

Let $F: W_{\phi_1}\to W_{\phi_2}$ be a bi-Lipschitz map. Let S_ε be the set of points $x\in W_{\phi_1}$ such that $\rho(x)=\varepsilon$. Let $a(\varepsilon)$ and $\tilde{a}(\varepsilon)$ be two points on $F(S_\varepsilon)$, such that for each other pair $y,\tilde{y}\in F(S_\varepsilon)$, we have: $\|y-\tilde{y}\|\le\|a(\varepsilon)-\tilde{a}(\varepsilon)\|$. Since F is a bi-Lipschitz map, there exists a positive constant C_1 such that

$$C_1\|x-\tilde{x}\|\ge\|a(\varepsilon)-\tilde{a}(\varepsilon)\|$$

where $F(x)=a(\varepsilon)$ and $F(\tilde{x})=\tilde{a}(\varepsilon)$. Since $x,\tilde{x}\in S_\varepsilon$, we obtain that $\|x-\tilde{x}\|\le 2\phi_1(\varepsilon)$ and, hence,

$$2C_1\phi_1(\varepsilon)\ge\|a(\varepsilon)-\tilde{a}(\varepsilon)\|.$$

Let $b(\varepsilon)\in F(S_\varepsilon)$ be a point such that $\rho(b(\varepsilon))=\min_{y\in F(S_\varepsilon)}\rho(y)$. Let $\tilde{b}(\varepsilon)$ be a point on $F(S_\varepsilon)$ such that $\eta(\tilde{b}(\varepsilon))=\eta(b(\varepsilon))+\pi$, i.e. $\tilde{b}(\varepsilon)$ is an opposite to $b(\varepsilon)$ point on $F(S_\varepsilon)$. By the definition of $a(\varepsilon)$ and $\tilde{a}(\varepsilon)$, we obtain:

$$\|a(\varepsilon)-\tilde{a}(\varepsilon)\|\ge\|b(\varepsilon)-\tilde{b}(\varepsilon)\|.$$

But

$$\|b(\varepsilon)-\tilde{b}(\varepsilon)\|\ge 2\phi_2(\rho(b(\varepsilon))).$$

Since F is a bi-Lipschitz map, there exists $C_2>0$ such that

$$\rho(b(\varepsilon))\ge C_2\varepsilon.$$

Using the above inequalities, we obtain:

$$C_1\phi_1(\varepsilon)\ge\phi_2(C_2\varepsilon).$$

Hence, $P(\phi_1)\le P(\phi_2)$. Considering the map F^{-1}, we conclude that $P(\phi_1)=P(\phi_2)$. □

PROPOSITION 8.2. *Let X be a definable in A set and let $x_0\in X$. Suppose that X is a union of two definable subsets X_1 and X_2 such that $X_1\cap X_2=\gamma_1\cup\gamma_2$ where γ_1 and γ_2 are two definable in A curves and $\gamma_1\cap\gamma_2=x_0$. Let X_1 and X_2 be bi-Lipschitz equivalent in A to T_{ψ_1} and to T_{ψ_2} correspondingly* *with respect to the inner metric and let the image of* x_0 *by corresponding bi-Lipschitz maps is a point* $(0,0) \in \mathbb{R}^2$. *Suppose that* $P(\psi_1) \leq P(\psi_2)$.

Then the germ of X *at* x_0 *is bi-Lipschitz equivalent in* A *with respect to the inner metric to the germ of the horn* W_{ψ_1} *at* $(0,0,0) \in \mathbb{R}^3$.

Proof. Let $\Psi_1 : X_1 \to T_{\psi_1}$ be a definable bi-Lipschitz map. The set T_{ψ_1} can be decomposed into union of the sets T_1 and T_2 such that $T_1 = T_{\frac{\psi_1}{2}}$ and T_2 is a closure of $T_{\psi_1} - T_1$. Note, that T_1 and T_2 are bi-Lipschitz equivalent in A to T_{ψ_1}. By Theorem 7.3, we obtain that $X_2 \cup \Psi_1^{-1}(T_2)$ is bi-Lipschitz equivalent in A to T_{ψ_1}.

Let $W_1 \subset W_{\psi_1}$ be a subset of W_{ψ_1} defined as follows: $W_1 = \{x \in W_{\psi_1}, x_1 \geq 0\}$. Let $W_2 = \{x \in W_{\psi_1}, x_1 \leq 0\}$. Clearly, $W_{\psi_1} = W_1 \cup W_2$ and both W_1 and W_2 are bi-Lipschitz equivalent to T_{ψ_1}. By the same arguments as in the proof of Theorem 7.3, there exists a bi-Lipschitz (with respect to the inner metric) map $\Phi_1 : X_2 \cup \Psi_1^{-1}(T_2) \to W_1$ such that $\|x - x_0\| = \|\Phi_1(x)\|$, for x belonging to the boundary of $X_2 \cup \Psi_1^{-1}(T_2)$. Using the same procedure, we can construct a bi-Lipschitz (with respect to the inner metric) map $\Phi_2 : \Psi_1^{-1}(T_1) \to W_2$ such that $\|x - x_0\| = \|\Phi_2(x)\|$, for x belonging to the boundary of $\Psi_1^{-1}(T_1)$.

Let $\Phi : X \to W_{\psi_1}$ be a map defined as follows:

$$\Phi(x) = \begin{cases} \Phi_1(x), & \text{if } x \in X_2 \cup \Psi_1^{-1}(T_2), \\ \Phi_2(x), & \text{if } x \in \Psi_1^{-1}(T_1). \end{cases}$$

By construction, Φ is a bi-Lipschitz map. $\square$

THEOREM 8.3 (Horn theorem). *Let* $X \subset \mathbb{R}^n$ *be a definable set. Let* $x_0 \in X$ *be an isolated singular point such that the link of* X *at* x_0 *is connected. Then there exists a definable in* A *function* $\psi \in G_A^+$ *such that the germ of* X *at* x_0 *is bi-Lipschitz equivalent in* A *with respect to the inner metric to the germ of* W_ψ *at* $(0,0,0) \in \mathbb{R}^3$.

Proof. Let (Γ, β) be a quasi-valuation complex corresponding to (X, x_0). Clearly, that a simplification $(\widetilde{\Gamma}, \tilde{\beta})$ of (Γ, β) must have the following form: the graph $\widetilde{\Gamma}$ contains only two vertices a_1 and a_2 connected by two edges g_1 and g_2 and $\tilde{\beta}(g_1) = \tilde{\beta}(g_2)$. By Theorem 7.3, (X, x_0) is a Geometric quasi-valuation Complex associated to $(\widetilde{\Gamma}, \tilde{\beta})$. Let ψ be a function such that $P(\psi) = \beta(g_1) - \beta(g_2)$. Then by Proposition 8.2, the germ of X at x_0 is bi-Lipschitz equivalent to the germ W_ψ at $(0,0,0) \in \mathbb{R}^3$. $\square$

The main result of this section is the following.

THEOREM 8.4 (Classification theorem for definable surfaces with isolated singularities). *Let* X *be a definable in* A *surface and let* $x_0 \in X$ *be an isolated singular point. Then*:

1. *There exists a finite family of definable in A functions $\psi_1 < \psi_2 < \cdots < \psi_k$ such that the germ of X at x_0 is bi-Lipschitz equivalent in A with respect to the inner metric to the germ of $W_{\psi_1} \cup W_{\psi_2} \cup \cdots \cup W_{\psi_k}$ at $(0,0,0) \in \mathbb{R}^3$.*
2. *The sets $W_{\psi_1} \cup W_{\psi_2} \cup \cdots \cup W_{\psi_k}$ and $W_{\tilde{\psi}_1} \cup W_{\tilde{\psi}_2} \cup \cdots \cup W_{\tilde{\psi}_k}$ are bi-Lipschitz equivalent with respect to the inner metric if and only if $P(\psi_i) = P(\tilde{\psi}_i)$, for $i = 1, \ldots, k$.*

Remark. The functions $\psi_1, \psi_2, \ldots, \psi_k$ defined in the theorem are not unique, but the collection of values $P(\psi_1), P(\psi_2), \ldots, P(\psi_k)$ is unique and gives a complete bi-Lipschitz invariant for this type of singularities.

Proof of Theorem 8.4. 1. Using general properties of length-spaces (see [8]), one can observe that if Y and Z be length-spaces such that $Y = Y_1 \cup Y_2$, $Z = Z_1 \cup Z_2, \#\{Y_1 \cap Y_2\} = 1, \#\{Z_1 \cap Z_2\} = 1, Y_1$ is bi-Lipschitz equivalent with respect to the inner metric to Z_1 and Y_2 is bi-Lipschitz equivalent with respect to the inner metric to Z_2 then Y is bi-Lipschitz equivalent to Z.

Let X be a definable surface and let $x_0 \in X$ be a singular point. Then X can be decomposed into a union of surfaces $X_1, X_2, \ldots, X_k$ such that $\bigcap_{i=1}^k X_i = x_0$ and the link of each X_i at x_0 is connected. By Theorem 8.3, each X_i is bi-Lipschitz equivalent to W_{ψ_i}, for some $\psi_i \in G_A^+$. By the observation from the beginning of the proof, we obtain the part 1.

The part 2 is a direct corollary of Theorem 8.1. □

Corollary 8.5. *Let (X, x_0) and (X', x_0') be germs of definable surfaces with isolated singular points $x_0 \in X$ and $x_0' \in X'$. Suppose that there exists a bi-Lipschitz (with respect to the inner metric) map $F : (X, x_0) \to (X', x_0')$. Then there exists a definable in A bi-Lipschitz (with respect to the inner metric) map $G : (X, x_0) \to (X', x_0')$.*

Observe that Theorem 8.1 is proved for any bi-Lipschitz map and all the maps constructed in the proofs of Theorem 8.3 and Theorem 8.4 are definable in A.

9. Canonical quasi-valuation Complex

Definition 7. Let A be an o-minimal structure and let $X \subset \mathbb{R}^n$ be a definable in A surface such that $x_0 \in X$. Let (Γ, β) be a quasi-valuation Complex such that (X, x_0) is a Geometric quasi-valuation Complex associated to (Γ, β). Let $(\widetilde{\Gamma}, \tilde{\beta})$ be a simplification of (Γ, β). Then $(\widetilde{\Gamma}, \tilde{\beta})$ is called *a Canonical quasi-valuation Complex of X at x_0*.

Theorem 9.1. *Let X, X' be definable in A surfaces such that $x_0 \in X$, $x_0' \in X'$. Then:*

1. *A Canonical quasi-valuation Complex of X at x_0 is well defined up to an isomorphism.*

2. *If germ of X' at x'_0 is bi-Lipschitz equivalent with respect to the inner metric to the germ of X at x_0, then their Canonical quasi-valuation Complexes are isomorphic.*

Proof. Note that the Statement 1 is a direct corollary of the Statement 2 because the identity map is bi-Lipschitz. Let (Γ,β) be a Canonical quasi-valuation Complex of X at x_0 and let (Γ',β') be a Canonical quasi-valuation Complex of X' at x'_0. Let $\Phi : C\Gamma \to X \cap B_{x_0,\varepsilon}$ and $\Phi' : C\Gamma' \to X' \cap B_{x'_0,\varepsilon'}$ be the corresponding homeomorphisms. Let $F : (X,x_0) \to (X',x'_0)$ be a bi-Lipschitz map. Since Γ and Γ' do not have smooth vertices, F induces an isomorphism i between Γ and Γ'. Let g be an edge of Γ connecting vertices a_1 and a_2. Suppose that a_1 and a_2 are not loop vertices. Let $g' = i(g)$. Let $\Psi_g : \Phi(Cg) \to T_\phi$ and $\Psi'_{g'} : \Phi'(Cg') \to T_{\phi'}$ be the corresponding bi-Lipschitz maps described in the definition of Geometric quasi-valuation Complex (see Section 7). By the construction, the map $\Psi'_{g'} \circ F \circ \Psi_g^{-1} : T_\phi \to T_{\phi'}$ is a bi-Lipschitz map. Thus, by Proposition 4.5 and Theorem 4.8, we obtain $P(\phi) = P(\phi')$.

Let $a \in V_\Gamma$ be a loop vertex. Let g_1 and g_2 be edges connecting a to another vertex a_1. Let Γ_1 be a subgraph of Γ such that $V_{\Gamma_1} = \{a,a_1\}$ and $E_{\Gamma_1} = \{g_1,g_2\}$. By Proposition 8.2, we have that $\Phi(C\Gamma_1)$ is bi-Lipschitz equivalent to ϕ-horn W_ϕ where $\phi \in \beta(g_1)$. By Theorem 8.1, we obtain that $F(\Phi(C\Gamma_1))$ is bi-Lipschitz equivalent to $W_{\phi'}$ and $P(\phi) = P(\phi')$. Since (Γ',β') is a simplified quasi-valuation complex, $i(\Gamma_1)$ is a graph Γ'_1 (a subgraph of Γ') with vertices a', a'_1 and the edges g'_1, g'_2. By Theorem 8.1, we have: $\beta(g'_1) = \beta(g'_2) = \beta(g_1) = \beta(g_2)$. The theorem is proved. $\square$

The following result shows that Canonical quasi-valuation Complex is not a complete bi-Lipschitz invariant, for o-minimal structures not polynomially bounded.

Theorem 9.2. *Let A be an o-minimal structure which is not polynomially bounded. Then there exists a pair of germs of definable in A surfaces $(X,x_0) \subset \mathbb{R}^3$ and $(Y,y_0) \subset \mathbb{R}^3$ such that the corresponding Canonical quasi-valuation Complexes are isomorphic but the germs (X,x_0) and (Y,y_0) are not bi-Lipschitz equivalent.*

Proof. Let ϕ be a definable in A flat function. Let (Γ,β) be the following quasi-valuation complex:

$$V_\Gamma = \{a_0,a_1,a_2,a_3\}, \qquad E_\Gamma = \{(a_0,a_1),(a_0,a_2),(a_0,a_3)\}$$

and $\beta(a_0,a_1) = \beta(a_0,a_2) = \beta(a_0,a_3) = P(\phi)$.

Let (X,x_0) be a realization of (Γ,β) constructed as follows. Let V_1,V_2,V_3 be three planes in $\mathbb{R}^3$ such that $V_1 \cap V_2 \cap V_3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_1 \in \mathbb{R}, x_2 = x_3 = 0\}$. Let T_1,T_2,T_3 be three copies of T_ϕ on these planes such that $T_1 \cap T_2 \cap T_3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_1 \geq 0, x_2 = x_3 = 0\}$. Set $X = T_1 \cup T_2 \cup T_3$.

Clearly, $(X,(0,0,0))$ is a realization of (Γ,β). Let $T_3' \subset V_3$ be a copy of $T_{\phi(\frac{t}{2})}$ on the plane V_3. Set $Y = T_1 \cup T_2 \cup T_3'$. Clearly, $(Y,(0,0,0))$ is another realization of (Γ,β). By the same arguments as in the proof of Theorem 5.11, the sets $(Y,(0,0,0))$ and $(X,(0,0,0))$ are not bi-Lipschitz equivalent. □

Definition 8. A germ (X,x_0) of a definable surface is called *totally nonflat* if the Canonical quasi-valuation Complex (Γ,β) satisfies the following condition: for every edge $g \in E_\Gamma$, there exists a nonflat germ $\psi \in G_A^+$ such that $P(\psi) = \beta(g)$.

Remark. If A is polynomially bounded, then every germ of every definable in A surface is totally nonflat.

Theorem 9.3. *Let A be an o-minimal structure. Let (X,x_0) and (Y,y_0) be two germs of definable in A totally nonflat surfaces. Then the germs (X,x_0) and (Y,y_0) are bi-Lipschitz equivalent with respect to the inner metric if and only if the corresponding Canonical quasi-valuation Complexes are isomorphic.*

Proof. Let (Γ_1,β_1) and (Γ_2,β_2) be Canonical quasi-valuation Complexes associated to (X,x_0) and (Y,y_0), correspondingly. Let $\{X_j\}$ and $\{Y_j\}$ be triangulations of X and Y corresponding to (Γ_1,β_1) and (Γ_2,β_2). Let $\bar{X}$ be a simplex of the triangulation $\{X_j\}$ or of the triangulation $\{Y_j\}$. Let γ_1 and γ_2 be boundary curves of $\bar{X}$. Then by the same arguments as in the proof of Lemma 7.4, we obtain that there exists a definable bi-Lipschitz (with respect to the inner metric) map $\widetilde{\Psi}_{\bar{X}} : \bar{X} \to \widetilde{T}_\phi$ (here $\widetilde{T}_\phi = \{(x_1,x_2) \in \mathbb{R}^2 | x_1 \geq 0, -\phi(x_1) \leq x_2 \leq \phi(x_1)\}$) such that

$$(\star) \qquad \widetilde{\Psi}_{\bar{X}}(x) = \begin{cases} (r_1(x), \phi(r_1(x))), & \text{for } x \in \gamma_1, \\ (r_2(x), -\phi(r_2(x))), & \text{for } x \in \gamma_2, \end{cases}$$

where $r_1(x) = \|x - x_0\|$ or $r_2(x) = \|x - y_0\|$.

Let $i : (\Gamma_1,\beta_1) \to (\Gamma_2,\beta_2)$ be an isomorphism. Let $\bar{Y} \in \{X_j\}$ be a simplex of the triangulation $\{X_j\}$ corresponding to some edge $g \in \Gamma_1$. Let $i(\bar{Y})$ be a simplex of $\{Y_j\}$ corresponding to $i(g)$. Now, we can define a map $\Psi : (X,x_0) \to (Y,y_0)$ independently on each simplex $\bar{Y} \in \{X_j\}$. Let $\Psi = \widetilde{\Psi}_{i(\bar{Y})}^{-1} \circ \widetilde{\Psi}_{\bar{Y}}$. Since ϕ is a nonflat function, we obtain that Ψ is bi-Lipschitz on each simplex $\bar{Y} \in \{X_j\}$. Since Ψ is well defined and continuous on boundary curves, we conclude that Ψ is a definable bi-Lipschitz map on X. □

Hence, the Canonical quasi-valuation complex is a complete bi-Lipschitz invariant for totally nonflat surfaces. In particular, it is a complete bi-Lipschitz invariant for all definable surfaces in polynomially bounded o-minimal structures.

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References

[1] L. Birbrair, *Local bi-Lipschitz classification of 2-dimensional semialgebraic sets*, Houston J. Math. **25** (1999), 453–472. MR 1730886

[2] L. Birbrair and F. Cano, *Characteristic exponents of semialgebraic singularities*, Math. Nachr. **276** (2004), 23–30. MR 2100044

[3] L. Birbrair and A. Fernandes, *Metric theory of semialgebraic curves*, Rev. Mat. Complut. **XIII** (2000), 369–382. MR 1822120

[4] L. Birbrair and A. Fernandes, *Local Lipschitz geometry of real weighted homogeneous surfaces*, Geom. Dedicata **135** (2008), 211–217. MR 2413338

[5] L. Birbrair and T. Mostowski, *Normal embedding of semialgebraic sets*, Michigan Math. J. **47** (2000), 125–132. MR 1755260

[6] A. Bernig and L. Bröcker, *Lipschitz–Killing invariants*, Math. Nachr. **245** (2002), 5–25. MR 1936341

[7] L. Bröcker, M. Kuppe and W. Scheufler, *Inner metric properties of 2-dimensional semi-algebraic sets*, Rev. Mat. Compl. **10** (1997), 51–78. MR 1485291

[8] D. Burago, Yu. Burago and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, Amer. Math. Soc., Providence, RI, 2001, 415 pp. MR 1835418

[9] D. Grieser, *Quasiisometry of singular metrics*, Houston J. Meth. **28** (2002), 741–752. MR 1953683

[10] D. Grieser, *Local geometry of singular real analytic surfaces*, Trans. Amer. Math. Soc. **355** (2003), 1559–1577. MR 1946405

[11] K. Kurdyka, *On a subanalytic stratification satisfying a Whitney property with exponent 1*, Real algebraic geometry, proceedings, Rennes (1991) (M. Coste, ed.), Lecture Notes of Mathematics, vol. 1524, 1992, pp. 316–323. MR 1226263

[12] Ch. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), 257–259. MR 1195484

[13] T. Mostowski, *Lipschitz equisingularity*, Dissertationes Math. **CCXLIII** (1985). MR 0808226

[14] M. Rosenlicht, *Hardy fields*, J. Math. Anal. Appl. **93** (1983), 297–311. MR 0700146

[15] A. Parusinski, *Lipschitz stratification of subanalytic sets*, Ann. Sci. Ec. Norm. Super. (4) **27** (1994), 661–696. MR 1307677

[16] G. Valette, *Lipschitz triangulations*, Illinois J. Math. **49** (2005), 953–979 (electronic). MR 2210270

[17] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge Univ. Press, Cambridge, 1998, 180 pp. MR 1633348

[18] L. van den Dries and Ch. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540. MR 1404337

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