Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 551–565
S 0019-2082

# ARTINIAN-FINITARY GROUPS OVER COMMUTATIVE RINGS

B. A. F. WEHRFRITZ

*In Memory of Reinhold Baer (1902–1979)*

Abstract. Let $M$ be a module over the commutative ring $R$. We consider the group $G$ of all automorphisms $g$ of $M$ for which $M(g-1)$ is $R$-Artinian. We show that $G$ has a locally residually nilpotent normal subgroup modulo which $G$ is a subdirect product of finitary linear groups over field images of $R$. This can be used to study certain subgroups of $G$. For example, if $H$ is a locally finite subgroup of $G$, then $H$ is isomorphic to a finitary linear group of characteristic zero if $R$ is an algebra over the rationals and $H/O_p(H)$ is isomorphic to a finitary linear group of characteristic the prime $p$ if R has characteristic a power of $p$. It also gives information about $\mathrm{Aut}_R\, M$ if $M$ itself is $R$-Artinian.

## 1. Introduction

Throughout this paper $M$ denotes a module over the (almost always) commutative ring $R$. The finitary automorphism group $F\,\mathrm{Aut}_R\, M$ of $M$ over $R$ is the subgroup of the group $\mathrm{Aut}_R\, M$ of $R$-automorphisms $g$ of $M$ such that $M(g-1)$ is Noetherian (as $R$-module). This includes all the finitary general linear groups $\mathrm{FGL}(V)$ of vector spaces $V$ over fields and is studied in this generality in [12] and [13].

In [14] we considered a wide range of variations of the notion of a finitary group of automorphisms. Here we concentrate on the Artinian analogue of finitary groups, just over commutative rings. Thus we are primarily concerned with the subgroup

$$F_1\,\mathrm{Aut}_R\, M = \{g \in \mathrm{Aut}_R\, M : M(g-1) \text{ is Artinian}\}$$

of $\mathrm{Aut}_R\, M$. (The subscript 1 here is part of a more systematic notation and refers to the fact that a module is Artinian if and only if it has Krull dimension less than 1. An alternative notation for $F\,\mathrm{Aut}_R\, M$ would be $F^1\,\mathrm{Aut}_R\, M$, since a module is Noetherian if and only if it has Krull codimension less than 1; see

Received September 25, 2001.
2000 *Mathematics Subject Classification.* 20H99, 20F28, 20K30.

[14].) Some information about the groups $F_1 \operatorname{Aut}_R M$, even for modules $M$ over potentially non-commutative rings, is given in [14]; see especially [14, 1.2] and [14, 1.3]. Here, by restricting our ground rings $R$ to being commutative we can derive stronger conclusions, of which the most obvious is the replacement of various cartesian products by direct products.

THEOREM 1. *Let $M$ be a module over the commutative ring $R$. Then the group $G = F_1 \operatorname{Aut}_R M$ contains a locally residually nilpotent normal subgroup $N$ such that $G/N$ embeds into a direct product of finitary linear groups over (commutative) fields.*

More can be said about the normal subgroup $N$ of Theorem 1. For example, it satisfies the natural analogue in this context of unipotence in linear groups; see below, especially Section 4. Not surprisingly, one can strengthen the conclusions of the theorem for locally finite groups.

COROLLARY. *Let $G$ be a locally finite subgroup of $F_1 \operatorname{Aut}_R M$.*

(a) *The group $G$ has a locally nilpotent normal subgroup modulo which $G$ is a subdirect product of irreducible finitary linear groups.*
(b) *If $R$ is an algebra over the rationals $\mathbb{Q}$, then $G$ is isomorphic to a finitary linear group over the complex numbers $\mathbb{C}$.*
(c) *If $R$ has characteristic a power of the prime $p$, then $G/O_p(G)$ is isomorphic to a finitary linear group of characteristic $p$ (and $O_p(G)$ is locally nilpotent).*

We give an example in Section 5 that shows that we cannot improve Theorem 1 by always choosing $N$ to be $\langle 1 \rangle$. In fact, we give examples of $R$ and $M$ as above such that $F_1 \operatorname{Aut}_R M$ cannot be embedded into any cartesian product of finitary linear groups.

If $M$ is Artinian, then $F_1 \operatorname{Aut}_R M = \operatorname{Aut}_R M$ and stronger conclusions can be drawn. Recall that a group is said to be *quasi-linear* if it can be embedded into a direct product of a finite number of linear groups of finite degree; equivalently, if it is isomorphic to a subgroup of some $\mathrm{GL}(n, J)$ for some integer $n$ and some (cartesian) product $J$ of a finite number of fields. Such groups arise naturally in many places (e.g., see [7, Chapter 13], [8] and [9, §6]).

THEOREM 2. *Let $M$ be an Artinian module over the commutative ring $R$. Then the group $G = \operatorname{Aut}_R M$ contains a locally residually nilpotent normal subgroup $N$ such that $G/N$ is quasi-linear.*

COROLLARY. *Let $G$ be a locally finite subgroup of $\operatorname{Aut}_R M$, where $M$ is Artinian.*

(a) *$G$ is locally-nilpotent by quasi-linear.*

(b) *If $R$ is an algebra over the rationals $\mathbb{Q}$, then $G$ is abelian by finite and isomorphic to a linear group of finite degree over the complex numbers $\mathbb{C}$.*
(c) *If $R$ has characteristic a power of the prime $p$, then $G/O_p(G)$ is isomorphic to a linear group of finite degree over the algebraic closure of the field of $p$ elements (and again $O_p(G)$ is locally nilpotent).*

If the ring $R$ is itself Artinian, the Noetherian and the Artinian versions of finitary coincide. More generally, if $M$ is a left module over the left Artinian ring $R$, then $M$ is Noetherian if and only if $M$ is Artinian (e.g., see [5, 3.25]), so $F_1 \operatorname{Aut}_R M = F \operatorname{Aut}_R M$. If instead the ring $R$ is Noetherian, our conclusions are somewhat weaker.

Theorem 3. *Let $M$ be a module over the commutative Noetherian ring $R$.*
(a) *The group $G = F_1 \operatorname{Aut}_R M$ contains an abelian normal subgroup $A$ such that $G/A$ is isomorphic to a finitary group of automorphisms of some module over some commutative ring.*
(b) *If also $M$ is Artinian, then $\operatorname{Aut}_R M$ is quasi-linear.*

We can use the finitary or the quasi-linear cases and Theorem 3 above simply to read off results. For example, Theorem 3(a) above and [13, Theorem 1] immediately yield the following.

Corollary. *Let $M$ be a module over the commutative Noetherian ring $R$.*
(a) *A locally soluble subgroup of $F_1 \operatorname{Aut}_R M$ is hyperabelian, is abelian by (locally-nilpotent by abelian by locally-finite) and has a local system of soluble normal subgroups.*
(b) *Let $G$ be any subgroup of $F_1 \operatorname{Aut}_R M$. Then $G$ has a unique maximal locally soluble, normal subgroup, $S$ say, $S$ contains every ascendant (in particular every normal) locally soluble subgroup of $G$ and $S$ has a local system of soluble normal subgroups of $G$.*

In connection with Theorem 3(b), note that if $M$ is a Noetherian module over the commutative ring $R$, then $\operatorname{Aut}_R M$ is quasi-linear (see [8] or [9, 6.1]), if $M$ is Artinian over the commutative ring $R$, then $\operatorname{Aut}_R M$ is not too far from being quasi-linear (by Theorem 2) and if $M$ is Artinian over a commutative Noetherian ring, then $\operatorname{Aut}_R M$ is again quasi-linear by Theorem 3. In Section 5 we give an example of a module $M$ over a commutative Noetherian ring $R$ such that neither $F_1 \operatorname{Aut}_R M$ nor $F \operatorname{Aut}_R M$ is quasi-linear (or even embeddable into a cartesian product of finitary linear groups).

By an old theorem of Mal'cev [7, 4.2] finitely generated linear groups are residually finite. Our final theorem is a generalization of this.

Theorem 4. *Let $M$ be a module over the commutative ring $R$. Then both $F \operatorname{Aut}_R M$ and $F_1 \operatorname{Aut}_R M$ are locally residually finite.*

We show (by examples) that this is a particular phenomenon for $F \operatorname{Aut}_R M$ and $F_1 \operatorname{Aut}_R M$. It does not, for example, extend to $F_\infty \operatorname{Aut}_R M$, even if $R$ is Noetherian.

## 2. General commutative rings

**2.1**. *Let $R$ be a commutative ring and let $\{1\} = U_0 < U_1 < \cdots < U_i < \cdots \leq \bigcup_{i \geq 0} U_i = M$ be a composition series for the $R$-module $M$. If $U_1$ is essential in $M$, then each $U_{i+1}/U_i$ is isomorphic to $U_1$.*

*Proof.* Now $U_{i+1}/U_i \cong R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ of $R$. Suppose $\mathfrak{m}_0 = \mathfrak{m}_1 = \cdots = \mathfrak{m}_{i-1} \neq \mathfrak{m}_i$. Then $\mathfrak{m}_0^i \mathfrak{m}_i U_{i+1} = \{0\}$. If $\mathfrak{m}_0^i U_{i+1} = \{0\}$, then $\mathfrak{m}_0$ kills $U_{i+1}/U_i$ and $\mathfrak{m}_0 \leq \mathfrak{m}_i$. This is false, since $\mathfrak{m}_0 \neq \mathfrak{m}_i$ is maximal. Hence $\mathfrak{m}_0^i U_{i+1} \neq \{0\}$. But $U_1$ is irreducible and essential in $M$, so $U_1 \leq \mathfrak{m}_0^i U_{i+1}$. Hence $\mathfrak{m}_i U_1 \leq \mathfrak{m}_0^i \mathfrak{m}_i U_{i+1} = \{0\}$ and $\mathfrak{m}_i \leq \mathfrak{m}_0$, again a contradiction. □

**2.2**. *Let $R$ be a commutative ring and $U \cong R/\mathfrak{m}$ an irreducible essential submodule of the $R$-module $M$. If either $M$ is Artinian or $R$ is Noetherian or $M$ is Noetherian then*

$$M = \bigcup_{1 \leq j < \infty} \operatorname{ann}_M(\mathfrak{m}^j),$$

*where $\operatorname{ann}_M(A) = \{x \in M : Ax = \{0\}\}$ for $A$ any subset of $R$.*

*Proof.* Suppose $M$ is Artinian and let $x \in M \setminus \{0\}$. Then $Rx$ is Artinian, so $R/\operatorname{ann}_R(x)$ is Artinian. By Hopkin's theorem (e.g., [1, 5.4.8]) it is also Noetherian, so $Rx$ has a composition series of finite length, $r$ say. By 2.1 we have $\mathfrak{m}^r x = \{0\}$ and the claim in this case follows.

If $R$ is Noetherian, consider a finitely generated submodule $W$ of $M$ containing $U$. Then $W$ is Noetherian and by the Artin-Rees lemma (e.g., [3, 11.C]) we have $\mathfrak{m}^r W = \{0\}$ for some integer $r$. Again the claim follows. If $M$ is Noetherian, then so is $R/\operatorname{ann}_R(M)$ and the previous case applies. □

**2.3**. *Let $R$ be a commutative ring, $\mathfrak{m}$ a maximal ideal of $R$ and $M$ an $R$-module. Set $A_j = \operatorname{ann}_M(\mathfrak{m}^j)$ for each $j \geq 0$ and let $G$ denote the centralizer in $\operatorname{Aut}_R M$ of $A_1$. Then $[A_{j+1}, G] \leq A_j$ for each $j \geq 0$.*

*Proof.* Let $j \geq 1$ and suppose $[A_j, G] \leq A_{j-1}$, a statement that is certainly true if $j = 1$. If $g \in G$, then $\mathfrak{m} A_{j+1}(g-1) \leq A_j(g-1) \leq A_{j-1}$. Thus $\mathfrak{m}^j A_{j+1}(g-1) \leq \mathfrak{m}^{j-1} A_{j-1} = \{0\}$ and hence $A_{j+1}(g-1) \leq A_j$. The claim follows. □

Denote the socle of a module $M$ by $\operatorname{soc} M$.

**2.4**. *Let $M$ be a module over the ring $R$ and let $U$ be an irreducible submodule of $M$. Suppose $V$ is a submodule of $M$ that is maximal subject* *to being an essential extension of* $U$ *and suppose* $\operatorname{soc} M = U \oplus W$. *Then* $\operatorname{soc}(M/V) = (V \oplus W)/V \cong W$.

*Proof.* Clearly $V \cap \operatorname{soc} M = U$ and $V \cap W = \{0\}$. Also $(V \oplus W)/V$ does lie in $\operatorname{soc}(M/V)$. If these are not equal there is some $X \leq M$ with $X \not\leq V \oplus W$, $V \leq X$, $(V \oplus W) \cap X = V$ and $X/V$ irreducible. By the maximality of $V$, the submodule $U$ is not essential in $X$, so there exists $Y \leq X$ with $Y \neq \{0\} = U \cap Y = V \cap Y$, the final equality being since $U$ is essential in $V$. Then $U \oplus Y \leq X$ and $W \cap X \leq V \cap W = \{0\}$. Hence $Y \oplus \operatorname{soc} M = Y \oplus U \oplus W \leq M$. Further $U$ is essential in $V$, so $Y \not\leq V$, $Y \oplus V = X$ and $Y \cong X/V$ is irreducible. This contradicts the definition of $\operatorname{soc} M$ and 2.4 follows. □

**2.5**. Corollary. *Let* $M$ *be an Artinian module over the ring* $R$. *Then* $M$ *has a series of submodules of finite length with uniform factors.*

*Proof.* Let $\operatorname{soc} M = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, where each $U_i$ is irreducible. By Zorn's lemma there is a submodule $V_1$ of $M$ that is maximal subject to being an essential extension of $U_1$. By 2.4 we have $\operatorname{soc}(M/V_1) = W_2 \oplus \cdots \oplus W_n$, where $W_i = (V_1 + U_i)/V_1$ is a copy of $U_i$. By induction on $n$ there is a series

$$\{0\} = V_0 < V_1 < V_2 < \cdots < V_n \leq M,$$

where $V_i/V_{i-1}$ is an essential extension of an irreducible submodule isomorphic to $U_i$, and consequently $V_i/V_{i-1}$ is uniform, and $\operatorname{soc}(M/V_n) = \{0\}$. Also $M/V_n$ is Artinian. Therefore $V_n = M$. □

If $\mathfrak{m}$ is a maximal ideal of the commutative ring $R$, call an $R$-module $P$ an $\mathfrak{m}$-*primary* module if each of its elements is killed by some power of $\mathfrak{m}$. It follows that each composition factor of such a $P$ is isomorphic to $R/\mathfrak{m}$ (but not conversely in general).

**2.6**. *Let* $\mathfrak{m}$ *be a maximal ideal of the commutative ring* $R$ *and suppose that* $\mathfrak{m}^r$ *is Artinian and* $\mathfrak{m}$-*primary for some positive integer* $r$. *Then* $\mathfrak{m}$ *is nilpotent.*

*Proof.* Since $\mathfrak{m}^r$ is Artinian, there is some $s \geq r$ with $\mathfrak{m}^s = \mathfrak{m}^{s+1}$. Let $x \in \mathfrak{m}^r$. Since $\mathfrak{m}^r$ is $\mathfrak{m}$-primary, there exists $j$ with $\mathfrak{m}^j x = \{0\}$. Thus $\mathfrak{m}^s x = \{0\}$ and so $\mathfrak{m}^s = \mathfrak{m}^{r+s} = \{0\}$. Thus $\mathfrak{m}$ is nilpotent. □

Let $M$ be a module over the commutative ring $R$. If $M$ contains a non-zero Artinian submodule, it contains an irreducible submodule, $U_0$ say. Then $U_0 \cong R/\mathfrak{m}_0$ for some maximal ideal $\mathfrak{m}_0$ of $R$. Set $M_1 = \bigcup_j \operatorname{ann}(\mathfrak{m}_0^j)$. Repeat with $M/M_1$ in place of $M$ and keep going, transfinitely if necessary. We construct in this way maximal ideals $\mathfrak{m}_\sigma$ of $R$ for all $\sigma < \tau$ and submodules $M_\sigma$ of $M$ for $\sigma \leq \tau$, where $M_{\sigma+1}/M_\sigma = \bigcup_j \operatorname{ann}_{M/M_\sigma}(\mathfrak{m}_\sigma^j)$ and $M/M_\tau$ has no non-zero Artinian submodules. Each $M_\sigma$ is fully invariant. Define $A_{\sigma,j}$ by $A_{\sigma,j}/M_\sigma = \operatorname{ann}_{M/M_\sigma}(\mathfrak{m}_\sigma^j)$, so $A_{\sigma,0} = M_\sigma$. Then

$$\{0\} = A_{0,0} < A_{0,1} \le \cdots \le A_{1,0} < A_{1,1} \le \cdots \le M_\tau \le M$$

is a fully invariant ascending series of $R$-submodules of $M$.

Let $G = F_1 \operatorname{Aut}_R M$ and set $N = \bigcap_{\sigma<\tau} C_G(A_{\sigma,1}/M_\sigma)$. Clearly $G$ centralizes $M/M_\tau$. From 2.3 it follows that $N$ stabilizes the above series. In particular $N$ is locally residually nilpotent by Hall and Hartley's Theorem A2 of [2]. Let $\rho_\sigma : \operatorname{End}_R M \to \operatorname{End}_{k_\sigma}(A_{\sigma,1}/M_\sigma)$ be the natural map, where $k_\sigma$ denotes the field $R/\mathfrak{m}_\sigma$. Then $G\rho_\sigma \le \operatorname{FGL}(V_\sigma)$ by [14, 2.2] for $V_\sigma = A_{\sigma,1}/M_\sigma$ regarded as a $k_\sigma$-space in the obvious way. Thus we obtain an embedding of $G/N$ into $\prod_{\sigma<\tau} \operatorname{FGL}(V_\sigma)$.

Let $g \in G$. We claim that $g\rho_\sigma = 1$ for almost all $\sigma < \rho$. If so we will have that $G/N$ embeds into the direct product $\times_{\sigma<\tau} \operatorname{FGL}(V_\sigma)$. Suppose $M(g-1) \cap M_1 \ne \{0\}$. Then $M(g-1)$ contains a copy $U_0$ of the irreducible $R$-module $R/\mathfrak{m}_0$. Let $W_0 \ge U_0$ be maximal subject to being an essential extension of $U_0$ in $M(g-1)$. Since $M(g-1)$ is Artinian, we have $W_0 \le M_1$ by 2.2. Clearly $M_1/W_0 \le \bigcup_{j\ge1} \operatorname{ann}_{M/W_0}(\mathfrak{m}_0^j)$. In fact we have equality here: for suppose $x \in M$ with $x + W_0$ in the right-hand side. There exists a positive integer $r$ with $\mathfrak{m}_0^r x \le W_0$ and the latter is Artinian and $\mathfrak{m}_0$-primary. By 2.6 applied to $R/\operatorname{ann}_R(x)$, there is a positive integer $s$ with $\mathfrak{m}_0^s \le \operatorname{ann}_R(x)$, that is, with $\mathfrak{m}_0^s x = \{0\}$, so $x \in M_1$. By 2.4 we can apply induction on the composition length $n$ of the socle of $M(g-1)$ to $M/W_0$ to deduce that $g\rho_\sigma = 1$ for all but $n$ of the $\rho_\sigma$.

Suppose $M$ is Artinian. Then $n_\sigma = \dim V_\sigma$ is finite for every $\sigma < \tau$ and $\rho_\sigma$ maps $G$ into $\operatorname{GL}(n_\sigma, k_\sigma)$. Also $\tau$ is at most the composition length of the socle of $M$ (for let $W_0 \le M$ be maximal subject to being an essential extension of $U_0$ and apply induction to $M/W_0$; cf. the previous argument using 2.4 and 2.6). Actually it is easy to see that $\tau$ is the number of non-zero homogeneous components of the socle of $M$. Finally $M/M_\tau$ is Artinian, so $M = M_\tau$. We have now proved the following result.

**2.7**. Theorem. *Let $M$ be a module over the commutative ring $R$ and let $G$ be a subgroup of $F_1 \operatorname{Aut}_R M$. Then there is an exact sequence*

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathop{\times}_{\sigma<\tau} \operatorname{FGL}\big(V_\sigma\big),$$

*where $N$ stabilizes an ascending series in $M$ and in particular is locally residually nilpotent, and $V_\sigma$ is a vector space over some field image $k_\sigma$ of $R$. If $M$ is Artinian as $R$-module we can choose $\tau$ to be finite and choose each $V_\sigma$ to be finite dimensional; in particular $G/N$ is then quasi-linear.*

Of course Theorems 1 and 2 follow at once from 2.7. Unlike the finitary case, that is, unlike the case of $F \operatorname{Aut}_R M$ (see [13, 2.2, 3.2 and 3.7]), the subgroup $N$ constructed in the proof of 2.7 need not be locally nilpotent or even locally soluble, even if $M$ is Artinian. For example, let $R = \mathbb{Z}$, the integers, and let $M$ be the direct sum of two Prüfer $p^\infty$-groups for some prime $p$. Then $F_1 \operatorname{Aut}_R M = \operatorname{Aut}_R M = \mathrm{GL}(2, \mathbb{Z}_p)$ and $N = \{x \in G : x \equiv 1 \bmod p\}$. In this case $N$ contains free subgroups of rank 2. (Here $\mathbb{Z}_p$ denotes the $p$-adic integers.)

If $R$ is a Noetherian commutative ring we can do rather better than 2.7, as we shall see in the next section.

## 3. Noetherian rings

For the moment we consider again Artinian modules over an arbitrary commutative ring. Suppose $\{0\} < U < V$ is a series of $R$-modules with $U$ and $V/U$ irreducible. If $U$ is essential in $V$, then $V/U \cong U$ by 2.1. If not, there is a proper submodule $W$ of $V$ with $U \cap W = \{0\}$. Then $U + W = V$ and $V = U \oplus W$. Hence $\{0\} < W < V$ is a series of $V$ with $W$ isomorphic to $V/U$ and $V/W$ isomorphic to $U$. In this way we can feed non-isomorphic composition factors of a module past each other. Thus a module $M$ of positive finite composition length can be uniquely written as a direct sum $M = \oplus P_i$, where $P_i$ is $\mathfrak{m}_i$-primary and non-zero and the $\mathfrak{m}_i$ are finitely many distinct maximal ideals of $R$. Note that for each $\mathfrak{m}_i$ there is an irreducible submodule of $M$ isomorphic to $R/\mathfrak{m}_i$.

Now assume that $M$ is Artinian and non-zero. Each finitely generated submodule of $M$ has finite composition length, as does the socle of $M$. Thus an elementary localization argument shows that $M = \oplus P_i$ exactly as in the previous case. (More generally, the same conclusion holds if $M$ is just locally Artinian, meaning that each finitely generated submodule of $M$ is Artinian, except that now there may be infinitely many distinct maximal ideals $\mathfrak{m}_i$.) Clearly the $P_i$ are fully invariant. Thus we have the following result.

**3.1**. *Let $M$ be a non-zero Artinian (or just locally Artinian) $R$-module. Then we have ring isomorphisms* $\operatorname{End}_R M \cong \operatorname{End}_R(\oplus P_i) \cong \prod \operatorname{End}_R P_i$ *and group isomorphisms* $\operatorname{Aut}_R M \cong \operatorname{Aut}_R(\oplus P_i) \cong \prod \operatorname{Aut}_R P_i$.

From now on in this section assume that $R$ is also Noetherian. Let $M$ be Artinian and $\mathfrak{m}$-primary, so the socle of $M$ is a direct sum of a finite number, $n$ say, of copies of $U = R/\mathfrak{m}$. Now $M = \bigcup_j \operatorname{ann}_M(\mathfrak{m}^j)$. Hence $M$ is naturally a module over the inverse limit $S$ of the $R/\mathfrak{m}^j$ (taken over $j = 1, 2, \dots$). Then $S$ is a complete local ring and $S$ is Noetherian (by [9, 2.14] for example). Let $E$ denote the injective hull of $U$ over $S$ and set $M^* = \operatorname{Hom}_S(M, E)$. Then $M^*$ is Noetherian [5, 5.19]. Consequently [9, Theorem 6.1] yields that $\operatorname{Aut}_S M^*$ is quasi-linear. Clearly $\operatorname{Aut}_R M = \operatorname{Aut}_S M \to \operatorname{Aut}_S M^*$, the map here, $\sigma$ say, being given by $(\phi\sigma)\eta = \phi\eta$ for $\phi \in \operatorname{Aut}_S M$ and $\eta \in M^*$ (alternatively $\eta(\phi\sigma) = \phi^{-1}\eta$ if you prefer $\operatorname{Aut}_S M$ and $\operatorname{Aut}_S M^*$ to act on the same side).

Suppose $\phi \neq 1$. There exists some $x$ in $M$ with $x\phi \neq x$. By [5, 2.24] there exists $\eta$ in $M^*$ with $(x\phi - x)\eta \neq 0$. Then $(\phi\sigma)\eta = \phi\eta \neq \eta$ and so $\phi\sigma \neq 1$. Therefore $\mathrm{Aut}_R M$ embeds into $\mathrm{Aut}_S M^*$ and consequently it too is quasi-linear.

If $M$ is Artinian, but not necessarily primary, we can write $M = \oplus P_i$ as in 3.1 and apply the above to each $P_i$. Thus again we obtain that $\mathrm{Aut}_R M$ is quasi-linear. We have now proved the following result.

**3.2**. Theorem. *Let $M$ be an Artinian module over the commutative Noetherian ring $R$. Then $\mathrm{Aut}_R M$ is quasi-linear.*

**3.3**. Proposition. *Let $R$ be a complete local commutative Noetherian ring with maximal ideal $\mathfrak{m}$. Let $E$ denote the injective hull of $R/\mathfrak{m}$ over $R$ and for any $R$-module $M$ set $M^* = \mathrm{Hom}_R(M, E)$. Then $F_1 \mathrm{Aut}_R M$ embeds into $F \mathrm{Aut}_R M^*$ and $F \mathrm{Aut}_R M$ embeds into $F_1 \mathrm{Aut}_R M^*$,*

*Proof.* Let $g \in F_1 \mathrm{Aut}_R M$ and set $X = M(g-1)$. Then $X^*$ embeds into $M^*$ via

$$\gamma : \psi \longmapsto (g-1)\psi \quad \text{for } \psi \in X^*.$$

If $\phi \in M^*$, then $(g-1)\phi = (g-1)\phi|_X \in X^*\gamma$. Thus $(g-1)M^* \leq X^*\gamma$. By [5, 5.19] the module $X^*$ is Noetherian, so $(g-1)M^*$ is too. Thus the standard map $(\eta \mapsto (\phi \mapsto \eta\phi))$ of $\mathrm{End}_R M$ to $\mathrm{End}_R M^*$ maps $F_1 \mathrm{Aut}_R M$ homomorphically into $F \mathrm{Aut}_R M^*$.

Let $\eta \in \mathrm{End}_R M$ with $\eta \neq 0$. Pick $x \in M$ with $x\eta \neq 0$. By [5, 2.24] there is some $\phi$ in $M^*$ with $x\eta\phi \neq 0$. Then $\eta\phi \neq 0$ and so $\mathrm{End}_R M$ embeds into $\mathrm{End}_R M^*$. The first claim of the proposition follows. The proof of the second is similar, using [5, 5.18] in place of [5, 5.19]. (For this second part the completeness of $R$ is not required.) □

**3.4**. Theorem. *Let $M$ be a module over the commutative Noetherian ring $R$ and set $G = F_1 \mathrm{Aut}_R M$. Then there is a commutative ring $S$ and an $S$-module $L$ and a homomorphism $\phi$ of $G$ into $F \mathrm{Aut}_S L$ with the kernel of $\phi$ abelian.*

Note that Theorem 3 follows from 3.2 and 3.4. I do not know whether in 3.4 one can choose $S$ to be Noetherian, nor whether one can choose $\phi$ to be an embedding.

*Proof.* Let $N$ be the sum of all the Artinian submodules of $M$. Then $N$ is locally Artinian. If $X \leq G$ is finitely generated, then $[M, X]$ is Artinian (e.g., [14, 2.1]). In particular $[M, G] \leq N$. Thus we have an exact sequence

$$1 \longrightarrow C_G(N) \longrightarrow G \longrightarrow F_1 \mathrm{Aut}_R N,$$

where by stability theory $C_G(N)$ embeds into $\mathrm{Hom}_R(M/N, N)$ and in particular $C_G(N)$ is abelian.

Now $N = \oplus_{\mathfrak{m}} P_{\mathfrak{m}}$, where $\mathfrak{m}$ ranges over the maximal ideals of $R$ and each $P_{\mathfrak{m}}$ is $\mathfrak{m}$-primary; see 3.1 and its proof. Let $g \in F_1 \operatorname{Aut}_R N$ and set $X = N(g-1)$. Then $X$ is Artinian, so $X$ is a direct sum of only finitely many of its primary components and the $\mathfrak{m}$-primary component of $X$ is $X \cap P_{\mathfrak{m}}$. Thus $X \cap P_{\mathfrak{m}} = \{0\}$ for almost all $\mathfrak{m}$ and $g$ induces the identity map on almost all the $P_{\mathfrak{m}}$. Therefore $F_1 \operatorname{Aut}_R N$ embeds into $\times_{\mathfrak{m}} F_1 \operatorname{Aut}_R P_{\mathfrak{m}}$. Let $S_{\mathfrak{m}}$ denote the inverse limit of the $R/\mathfrak{m}^j$ (taken over $j \geq 1$) and let $P_{\mathfrak{m}}^*$ denote the group of $S_{\mathfrak{m}}$-homomorphisms of $P_{\mathfrak{m}}$ into the injective hull of $R/\mathfrak{m}$ over $S_{\mathfrak{m}}$. Then $F_1 \operatorname{Aut}_R P_{\mathfrak{m}}$ embeds into $F \operatorname{Aut}_{s_{\mathfrak{m}}}(P_{\mathfrak{m}})^*$; see 3.3. Thus $F_1 \operatorname{Aut}_R N$ embeds into $\times_{\mathfrak{m}} F \operatorname{Aut}_{s_{\mathfrak{m}}}(P_{\mathfrak{m}})^*$. The latter embeds into $F \operatorname{Aut}_S L$ for $S = \prod_{\mathfrak{m}} S_{\mathfrak{m}}$ and $L = \oplus_{\mathfrak{m}}(P_{\mathfrak{m}})^*$. The theorem is proved. □

The proof of 3.4 above also shows the following (note that $N = M$ here).

**3.5**. *Let $M$ be a locally Artinian module over the commutative Noetherian ring $R$. Then there is a commutative ring $S$ and an $S$-module $L$ such that $F_1 \operatorname{Aut}_R M$ is embeddable into $F \operatorname{Aut}_R M$.*

## 4. Some applications

In this section we assume the notation of 2.7 and its proof. Let $G$ be a subgroup of $F_1 \operatorname{Aut}_R M$. By [14, 4.6] the group $G$ has a unique maximal normal $s$-subgroup $s(G)$, an $s$-subgroup being a subgroup $S$ of $\operatorname{Aut}_R M$ such that each finitely generated subgroup of $S$ stabilizes some ascending series of $R$-submodules of $M$. Now $s(G)$ acts as an $s$-subgroup on each section of $M$ [14, 4.2] and in particular on each $V_\sigma = A_{\sigma,1}/M_\sigma$. Consequently $s(G)\rho_\sigma \leq s(G\rho_\sigma)$ and hence $s(G) \leq \bigcap_{\sigma<\tau}(s(G\rho_\sigma))\rho_\sigma^{-1} \cap G$.

Suppose $G \leq F_1 \operatorname{Aut}_R M$ is such that $G\rho_\sigma$ is an $s$-subgroup of $\mathrm{FGL}(V_\sigma)$ for every $\sigma < \tau$. If $G_1$ is a finitely generated subgroup of $G$ and if $\mathfrak{g}_1$ is the obvious image of the augmentation ideal of the group ring $\mathbb{Z}G_1$ in $\operatorname{End}_R M$, then $V_\sigma \mathfrak{g}_1^r = \{0\}$ for some integer $r = r(\sigma)$. As in the proof of 2.3 we obtain $A_{\sigma,j+1}\mathfrak{g}_1^r \leq A_{\sigma,j}$ for each $j \geq 0$. (Specifically, if $A_{\sigma,j}\mathfrak{g}_1^r \leq A_{\sigma,j-1}$, then $\mathfrak{m}_\sigma A_{\sigma,j+1}\mathfrak{g}_1^r \leq A_{\sigma,j-1}$, so $\mathfrak{m}_\sigma^j(A_{\sigma,j+1}\mathfrak{g}_1^r) \leq M_\sigma$ and so $A_{\sigma,j+1}\mathfrak{g}_1^r \leq A_{\sigma,j}$, as required.) Thus $G_1$ stabilizes an ascending series in $M$ and so $G$ is an $s$-subgroup.

Now let $G$ be any subgroup of $F_1 \operatorname{Aut}_R M$. The previous paragraph shows that $\bigcap_{\sigma<\tau}(s(G\rho_\sigma))\rho_\sigma^{-1} \cap G$ is an $s$-subgroup that is clearly normal in $G$. Therefore we have proved the following (since the second claim follows from the first and the finitary linear case).

**4.1**. *Let $G$ be a subgroup of $F_1 \operatorname{Aut}_R M$ for $M$ a module over the commutative ring $R$. Then $s(G) = \bigcap_{\sigma<\tau}(s(G\rho_\sigma))\rho_\sigma^{-1} \cap G$ and $G/s(G)$ is a subdirect product of irreducible finitary linear groups.*

4.1 looks superficially like a special case of [14, 4.6], but this is not so since the $\rho_\sigma$ of [14, 4.6] are not the same as the $\rho_\sigma$ of 4.1 above. In fact one can choose the former so that the latter form a subset of the former. Continuing with the notation of 2.7, analogous to [14, 4.5 and 4.17] we have the following.

**4.2**. *Let $G \leq F_1 \operatorname{Aut}_R M$, where $M$ is a module over the commutative ring $R$. The following three conditions are equivalent.*

(a) *$G$ is an $s$-subgroup.*
(b) *Each $G\rho_\sigma$ is a stability subgroup of* $\mathrm{FGL}(V_\sigma)$.
(c) *Each $\mathfrak{g}\rho_\sigma$ is locally nilpotent.*

*($\mathfrak{g}$ is the image of the augmentation ideal of the group ring $\mathbb{Z}G$ in $\operatorname{End}_R M$.)*
*If $g \in F_1 \operatorname{Aut}_R M$, the following are equivalent.*

(d) *The element $g$ is a $u$-element (i.e., $\langle g \rangle$ is an $s$-subgroup).*
(e) *Each $g\rho_\sigma$ for $\sigma < \tau$ is unipotent.*

*Proof.* Now (a) implies (c) by [14, 4.1 and 4.2]. Also (b) and (c) are equivalent by the finitary linear case. Further (b) implies that $G = s(G)$ by 4.1, so (a) holds. Therefore (a), (b) and (c) are equivalent. Now set $G = \langle g \rangle$. Then $g$ is a $u$-element if and only if $G$ is an $s$-subgroup and $g\rho_\sigma$ is unipotent if and only if $\mathfrak{g}\rho_\sigma$ is nilpotent. Thus the equivalence of (a) and (c) yields the equivalence of (d) and (e). □

**4.3**. *Let $G \leq F_1 \operatorname{Aut}_R M$, where $M$ is a module over the commutative ring $R$. Then $G$ is a $u$-subgroup (i.e., has all its elements $u$-elements) if and only if $G$ is an $s$-subgroup.*

*Proof.* The result holds classically for linear groups of finite degree and consequently holds (almost immediately) for finitary linear groups. Thus it holds for subgroups of the $\mathrm{FGL}(V_\sigma)$. Therefore 4.3 follows from 4.2. □

Note that a similar argument does not apply to the situation in [14], for there the $V_\sigma$ are only vector spaces over division rings and the conclusion is not known to hold for skew linear groups of finite degree (see [6, §1.3] for a discussion of this). If $R$ is a $\mathbb{Q}$-algebra, then a similar result does hold in general, even for subgroups of $F_\infty \operatorname{Aut}_R M$ (cf. [14, 4.17]). Possibly 4.3 still holds for $R$ commutative and subgroups of $F_\infty \operatorname{Aut}_R M$.

**4.4.** *Proof of the Corollary to Theorem 1.* (a) Set $H = s(G)$ and apply 4.1. By [14, 4.6] the group $H$ is locally residually nilpotent. Since $H$ is also locally finite, so $H$ is locally nilpotent.

(b) By [14, 4.12(a)] we have $H = \langle 1 \rangle$. If $K$ is a locally finite, irreducible finitary linear group of characteristic 0, then $K$ has a faithful finitary linear representation over the complex field (cf. the proof of [11, Corollary 3(a)]).

Part (b) follows since a direct product of finitary linear groups over the same field $k$ is isomorphic to a finitary linear group over $k$.

(c) Here $H = O_p(G)$ by [14, 4.12(b)]. A set of fields of characteristic $p$ can all be embedded into a single field of characteristic $p$. Part (c) follows. □

**4.5.** *Proof of the Corollary to Theorem 2.* (a) By 2.7 we have $G/N$ quasi-linear and $N \leq s(G)$ is locally nilpotent as in 4.4(a).

(b) Here each $\operatorname{char} k_\sigma = 0$ and $N \leq s(G) = \langle 1 \rangle$; see 4.4(b). Thus $G$ is (quasi-)linear of characteristic zero. Therefore $G$ is abelian-by-finite by Schur's theorem [7, 9.4] and $G$ is isomorphic to a linear group of finite degree over the complex numbers.

(c) This follows from the Winter-Zalesskii theorem (see the proof of [7, 9.5] or see [6, 2.3.1]). □

## 5. Examples

Throughout this section $p$ denotes a prime, $P$ a Prüfer $p^\infty$-group, $U = \operatorname{Aut} P$ the group of units of the $p$-adic integers $\mathbb{Z}_p$ and $G = UP$ the split extension of $P$ by $U$.

**5.1**. *If $\phi : G \to \mathrm{GL}(n, F)$ is a homomorphism, for $n$ an integer and $F$ a field, then $P\phi = \langle 1 \rangle$.*

*Proof.* If $P\phi \neq \langle 1 \rangle$, then $P\phi$ is isomorphic to $P$, so $\operatorname{char} F \neq p$ and $(G : C_G(P\phi))$ is finite, the latter by [7, 1.6 and 1.12]. But $(\ker \phi) \cap P$ is finite, so $(G : C_G(P)) = |U|$ is finite, which is false. Therefore $P\phi = \langle 1 \rangle$. □

**5.2**. *The group $G$ is not quasi-linear, does not embed into the automorphism group of a Noetherian module over a commutative ring and does not embed into the automorphism group of an Artinian module over a commutative Noetherian ring.*

*Proof.* Apply 5.1, [9, 6.1] (or [8] if you prefer) and 3.2 above. □

**5.3**. *Let $\phi : G \to \mathrm{FGL}({}_D V)$ be a homomorphism, where $V$ is a left vector space over the division ring $D$. If $P\phi$ is unipotent, then $P\phi = \langle 1 \rangle$.*

*Proof.* Now $U$ contains an element $x$ of infinite order. Then $[P, x] = P$ and $\langle x^G \rangle = \langle x \rangle P$. Assuming $P\phi$ is unipotent, we have $P\phi \leq u(\langle x\phi^{G\phi} \rangle)$. By the proof of [10, 2.3] the group $P\phi$ stabilizes a finite series in $V$, say of length $r$. If $\operatorname{char} D = 0$, then $P\phi$ is torsion-free. If not, then $P\phi$ has finite exponent dividing $(\operatorname{char} D)^{r-1}$. Either way $P\phi = \langle 1 \rangle$. □

**5.4**. *Let $\phi : G \to \mathrm{FGL}({}_F V)$ be a homomorphism, where $V$ is a vector space over the field $F$. Then $P\phi = \langle 1 \rangle$.*

The group $G$ does embed into $\mathrm{GL}(n, D)$ for a suitable positive integer $n$ and division ring $D$, for example of characteristic 0, so we do need to restrict $D$ in 5.4. (For example, $U$ contains a torsion-free (abelian) subgroup $W$ of finite index $p-1$. Let $F$ be the subfield of the complex numbers generated by all $p$-th power roots of unity. There is an obvious action of $U$ and hence $W$ on $F$. Let $D$ be the division ring of quotients of the skew group ring of $W$ over $F$ (see [6, 1.4.3]). Then $G$ embeds into $\mathrm{GL}(2(p-1), D)$.)

*Proof.* If $V$ is FG-irreducible, then $\dim_F V$ is finite, since $G$ is soluble (see [4, Theorem A]). In this case $P\phi = \langle 1 \rangle$ by 5.1. In general this shows that $P\phi \leq u(G\phi)$. Consequently $P\phi = \langle 1 \rangle$ by 5.3. □

**5.5**. *The group $G$ cannot be embedded into any cartesian product of finitary linear groups.*

*Proof.* This follows from 5.4. □

**5.6**. *Set $R = \mathbb{Z}$, the integers, and $M = \mathbb{Z} \oplus P$. Then $G$ embeds into $F_1 \operatorname{Aut}_R M$.*

Of course $R$ here is Noetherian, while $M$ is neither Artinian nor Noetherian.

*Proof.* We have

$$\operatorname{End}_R M = \begin{pmatrix} \operatorname{End}\mathbb{Z} & \operatorname{Hom}(\mathbb{Z}, P) \\ 0 & \operatorname{End} P \end{pmatrix} = \begin{pmatrix} R & P \\ 0 & \mathbb{Z}_p \end{pmatrix}.$$

Set $G_1 = \left(\begin{smallmatrix} 1 & P \\ 0 & U \end{smallmatrix}\right) \leq \operatorname{Aut}_R M$. Clearly $G_1$ and $G$ are isomorphic. If $u \in U$ and if $x \in P$, then

$$M\left(\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} - 1\right) = P(u-1) \leq P \quad \text{and} \quad M\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} - 1\right) = \mathbb{Z}x \leq P.$$

Since $P$ is Artinian we have $G_1 \leq F_1 \operatorname{Aut}_R M$, as required. □

**5.7**. *Set $R = \mathbb{Z}_p$ and $M = \mathbb{Z}_p \oplus P$. Then $G$ can be embedded into both $F \operatorname{Aut}_R M$ and $F_1 \operatorname{Aut}_R M$.*

*Proof.* We have

$$\operatorname{End}_R M = \begin{pmatrix} \mathbb{Z}_p & P \\ 0 & \mathbb{Z}_p \end{pmatrix}.$$

Set $G_2 = \left(\begin{smallmatrix} U & P \\ 0 & 1 \end{smallmatrix}\right) \leq \operatorname{Aut}_R M$. Clearly $G_2$ and $G$ are isomorphic. With $u$ in $U$ and $x$ in $P$ as before, we have

$$M\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} - 1\right) = \mathbb{Z}_p(u-1) \quad \text{and} \quad M\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} - 1\right) = \mathbb{Z}_p x = \langle x \rangle \leq P.$$

Thus $G_2 \leq F \operatorname{Aut}_R M$.

As in the proof of 5.6, set $G_1 = \left(\begin{smallmatrix} 1 & P \\ 0 & U \end{smallmatrix}\right)$, but working now over the current ring $R$. Then as in the proof of 5.6 we have $G \cong G_1 \leq F_1 \operatorname{Aut}_R M$.

Of course here $R$ is a Noetherian complete local ring, while $M$ is neither Artinian nor Noetherian. □

**5.8**. *Set $R = \mathbb{Z}_p \oplus P$, where $PP = \{0\}$. Then $G$ embeds into* $\mathrm{GL}(2, R)$.

*Proof.* $G$ is isomorphic to $\left(\begin{smallmatrix} U & 0 \\ P & 1 \end{smallmatrix}\right) \leq \mathrm{GL}(2, R)$ with $U \leq \mathbb{Z}_p \leq R$ and $P \leq R$ as given. □

## 6. Residual properties

**6.1**. *Let $G$ be a finitely generated subgroup of $F_1 \operatorname{Aut}_R M$, where $M$ is a module over the commutative ring $R$, and let $X$ be a finite subset of $M$. Then*

$$RXG = \sum_{x \in X,\, g \in G} Rxg$$

*is finitely $R$-generated.*

*Proof.* By [14, 2.1] we have that $N = [M, G]$ is $R$-Artinian. Set $S_0 = \{0\} \leq N$ and define $S_k$ inductively for $k > 0$ by $S_{k+1}/S_k = \operatorname{soc}(N/S_k)$. ($\{S_k\}$ is the upper socle series of $N$.) Then $N$ is the union of the $S_k$ (use Hopkin's Theorem) and each $S_k$ is $R$-Noetherian as well as $R$-Artinian. Clearly $S_k G \leq S_k$ for each $k$.

Suppose $X = \{x_1, x_2, \ldots, x_{\mathfrak{m}}\}$ and $G = \langle g_1, g_2, \ldots, g_n \rangle$. Then each $x_i(g_j - 1)$ lies in $N = \bigcup_{k \geq 0} S_k$, so there exists $k$ with $x_i(g_j - 1) \in S_k$ for all $i$ and $j$. Then $RXG \leq RX + S_k$, so $RXG = RX + (RXG \cap S_k)$. Also $S_k$ is $R$-Noetherian, so $RXG \cap S_k$ is finitely $R$-generated. Consequently so too is $RXG$. □

**6.2**. *Let $G$ be a finitely generated subgroup of $\operatorname{Aut}_R M$ for $M$ a module over the commutative ring $R$. Under each of the following three conditions the group $G$ is residually finite.*

(a) *$M$ is finitely $R$-generated.*
(b) *$G \leq F \operatorname{Aut}_R M$.*
(c) *$G \leq F_1 \operatorname{Aut}_R M$.*

Theorem 4 follows at once from 6.2.

*Proof.* (a) By [7, 13.4] we may assume that $R$ too is finitely generated (as a ring) and hence that $R$ is Noetherian. Then $G$ is quasi-linear (by [8] or [9, §6]) and hence $G$ is residually finite by the linear case [7, 4.2].

(b) By [13, 2.3(c)] we may assume that $M$ is finitely $R$-generated. The claim then follows from Part (a).

(c) By 6.1 the group $G$ acts residually on the finitely $R$-generated $R$-$G$ sub-bimodules of $M$. Thus again we may assume that $M$ is finitely generated and apply Part (a). □

**6.3**. EXAMPLE. Let $J = \mathbb{Z}[1/2] \leq \mathbb{Q}$, let $N$ be the $\mathbb{Z}$-submodule of the matrix ring $J^{3\times 3}$ of all matrices with zeros above the diagonal and with equal integers in the $(1,1)$ and $(3,3)$ positions, let

$$H = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \leq \mathrm{GL}(3, J),$$

set $M = N/\mathbb{Z}e_{31}$ as $\mathbb{Z}$-module, where $\{e_{ij}\}$ denotes the set of standard matrix units, and put $G = H/\langle(1 + e_{31})\rangle$. Then $G$ is a 3-generator, soluble (even nilpotent-of-class-2 by cyclic) group, whose center is a Prüfer $2^\infty$-group. In particular $G$ is not residually finite.

The group $G$ acts faithfully on $M$ via right multiplication of $H$ on $N$, so we can regard $G$ as a subgroup of $\mathrm{Aut}_{\mathbb{Z}} M$. It is easy to check that $M$ has Krull dimension 1 and Krull codimension 1, for $M$ has a series of length 8 with four Prüfer $2^\infty$-factors and four infinite cyclic factors. Thus we have

$$G \leq \mathrm{Aut}_{\mathbb{Z}} M = F_2 \mathrm{Aut}_{\mathbb{Z}} M = F^2 \mathrm{Aut}_{\mathbb{Z}} M = F_\infty \mathrm{Aut}_{\mathbb{Z}} M,$$

and trivially $\mathbb{Z}$ is Noetherian. Also $G$ does not embed into either $F \mathrm{Aut}_S L$ or $F_1 \mathrm{Aut}_S L$ for any module $L$ over any commutative ring $S$, by 6.2.

Of course the integer 2 in the above construction can be replaced by any integer prime.

The first conclusion of the following remark is slightly stronger than local residual finiteness.

**6.4**. *Let $M$ be a module over the commutative ring $R$.*

(a) *If $M$ is Artinian, then* $\mathrm{Aut}_R M$ *is residually linear-of-finite-degree.*

(b) *If $M$ is locally Artinian, then* $F \mathrm{Aut}_R M$ *and* $F_1 \mathrm{Aut}_R M$ *are both residually nilpotent-by-finitarily linear.*

*Proof.* In either case, define $S_k \leq M$ for each $k \geq 0$ by $S_0 = \{0\}$ and $S_{k+1}/S_k = \mathrm{soc}(M/S_k)$. Then $\bigcup_k S_k = M$ and $\bigcap_k C_G(S_k) = \langle 1 \rangle$. Thus we may assume that $M = S_k$ for some $k$.

(a) Since $M$ is Artinian, each $S_{i+1}/S_i$ has finite composition length and so $M = S_k$ is Noetherian. Consequently $\mathrm{Aut}_R M$ is quasi-linear and hence clearly residually linear-of-finite-degree.

(b) Since $M = S_k$, each Noetherian submodule of $M$ is Artinian and conversely. Thus $F \mathrm{Aut}_R M = F_1 \mathrm{Aut}_R M, = G$ say. Also $M$ is a direct sum of its primary components (see Section 3), so we may suppose that $M$ is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of $R$. Assume the notation of 2.7 and its proof.

Then $\mathfrak{m}_0 = \mathfrak{m}$ and $A_{0,k} = M$. Hence $N = C_G(A_{0,1})$ stabilizes the series $\{A_{0,j}\}$ and consequently $N$ is nilpotent of class less than $k$ (assuming $M \neq \{0\}$, so $k \geq 1$). Finally $G/N$ embeds into $\mathrm{FGL}(V_0)$. The proof is complete. □

## References


[1] P. M. Cohn, *Algebra. Vol. 2*, second ed., John Wiley & Sons Ltd., Chichester, 1989. MR **91b:**00001

[2] P. Hall and B. Hartley, *The stability group of a series of subgroups*, Proc. London Math. Soc. (3) **16** (1966), 1–39. MR 33#173

[3] H. Matsumura, *Commutative algebra*, W. A. Benjamin, Inc., New York, 1970. MR 42#1813

[4] U. Meierfrankenfeld, R. E. Phillips, and O. Puglisi, *Locally solvable finitary linear groups*, J. London Math. Soc. (2) **47** (1993), 31–40. MR **94c:**20064

[5] D. W. Sharpe and P. Vámos, *Injective modules*, Cambridge University Press, London, 1972. MR 50#13153

[6] M. Shirvani and B. A. F. Wehrfritz, *Skew linear groups*, London Mathematical Society Lecture Note Series, vol. 118, Cambridge University Press, Cambridge, 1986. MR **89h:**20001

[7] B. A. F. Wehrfritz, *Infinite linear groups. An account of the group-theoretic properties of infinite groups of matrices*, Springer-Verlag, New York, 1973. MR 49#436

[8] ———, *Automorphism groups of Noetherian modules over commutative rings*, Arch. Math. (Basel) **27** (1976), 276–281. MR 53#13367

[9] ———, *Lectures around complete local rings*, Queen Mary College Department of Pure Mathematics, London, 1979. MR **80k:**13014

[10] ———, *Locally soluble finitary skew linear groups*, J. Algebra **160** (1993), 226–241. MR **94m:**20080

[11] ———, *Locally finite finitary skew linear groups*, Proc. London Math. Soc. (3) **83** (2001), 71–92. MR **2002a:**20054

[12] ———, *Nilpotence in finitary groups over commutative rings*, Comm. Algebra **30** (2002), 1417–1435. MR **2003b:**20075

[13] ———, *Finitary automorphism groups over commutative rings*, J. Pure Appl. Algebra **172** (2002), 337–346. MR **2003b:**20076

[14] ———, *On generalized finitary groups*, J. Algebra **247** (2002), 707–727. MR **2002m:**20085



School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, England

*E-mail address*: b.a.f.wehrfritz@qmul.ac.uk