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ON DEVIATION IN GROUPS

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Abstract. The main result of the paper states that a metabelian group G has deviation for normal subgroups if and only if it has a finite series of normal subgroups each of whose factor meets the maximal or the minimal condition for G-invariant subgroups.

1. Introduction

Let Ω be a set of subgroups of a group G. The group G is said to satisfy the *minimal (maximal) condition* Min-Ω (Max-Ω) *for subgroups from* Ω if G has no infinite descending (ascending) chain of subgroups from Ω, and the *weak minimal (maximal) condition* Min-∞-Ω (Max-∞-Ω) *for subgroups from* Ω if $|H_i : H_{i+1}| < \infty$ ($|H_{i+1} : H_i| < \infty$) for almost all i for any infinite descending (ascending) chain $\{H_i\}$ of subgroups from Ω. If Ω is the set of all subgroups of G, then Ω is usually omitted from these notations. The weak minimal and maximal conditions were introduced by R. Baer and D. I. Zaitsev.

By a result of S. N. Černikov (see [10, 5.4.23]) a soluble group G has Min if and only if G is a finite extension of a direct product of finitely many of quasicyclic groups; nowadays such groups are called Černikov groups. A soluble group G has Max if and only if it is polycyclic, that is, if G has a finite subnormal series with cyclic factors (see [10, 5.4.12]).

A group G is said to be *minimax* if it has a finite subnormal series each of whose factors has either Min or Max. It follows from the above mentioned description of soluble groups with Min and Max that a soluble group G is minimax if and only if it has a finite subnormal series each of whose factors is either cyclic or quasicyclic. Minimax groups were introduced by R. Baer [1] and investigated by R. Baer [2], D. J. S. Robinson [8], [9] and D. I. Zaitsev [14], [15].

In [14] D. I. Zaitsev proved that a locally soluble group G has Min-∞ or Max-∞ if and only if G is a soluble minimax group. The fact that such different chain conditions as Min-∞ and Max-∞ lead to the same algebraic structure suggests that there is another chain condition which generalizes both

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Min-∞ and Max-∞. In the present paper we consider what probably is the most general (but still fruitful) chain condition, which involves the notion of deviation.

Let Ω be a partially ordered set (poset for short). For any elements $a, b \in \Omega$ we put $a/b = \{x \in \Omega \mid a \leq x \leq b\}$. By [6, 6.1.2], the *deviation* of Ω, $\operatorname{dev} \Omega$, is defined as follows:

(1) If Ω is trivial, then $\operatorname{dev} \Omega = -\infty$.
(2) If Ω has the minimal condition, then $\operatorname{dev} \Omega = 0$.
(3) For a general ordinal β, $\operatorname{dev} \Omega = \beta$ if
  (i) $\operatorname{dev} \Omega \neq \alpha < \beta$, and
  (ii) in any descending chain $\{a_i\}$ of elements of Ω all but finitely many factors a_i/a_{i+1} have deviation less than β.

We should note that there exist posets without deviation.

Let R be a ring. If Ω is the set of all submodules of an R-module M, then $\operatorname{dev} \Omega$ is called the *Krull dimension* of M and denoted by $K_R(M)$. The Krull dimension of the right module R_R is called the Krull dimension of the ring R and denoted by $K(R)$. The notion of Krull dimension was introduced in [7] and plays a very important role in the theory of rings and modules. Thus it is desirable to extend this notion to the theory of groups. If G is a group and Ω is a set of subgroups of G, then $\operatorname{dev}_\Omega G$ denotes the deviation of Ω, which is a poset with respect to inclusion. If Ω is the set of all subgroups of G, then we put $\operatorname{dev}_\Omega G = \operatorname{dev} G$. If Ω is the set of all normal subgroups of G, then we call $\operatorname{dev}_\Omega G$ the Krull dimension of G and denote it by $K(G)$.

Evidently, if a group G has Min-∞-Ω, then $\operatorname{dev}_\Omega G \leq 1$. The arguments of the proof of [6, 6.1.8] show that if a group G has Max-∞-Ω, then $\operatorname{dev}_\Omega G$ exists. Thus, the condition of existence of $\operatorname{dev}_\Omega G$ generalizes both Min-∞-Ω and Max-∞-Ω. However, Lemma 4.4 below shows that a soluble group has $\operatorname{dev} G$ if and only if G is minimax. A group G is said to be *G-minimax* if it has a finite series of normal subgroups each of whose factors satisfies either the minimal or the maximal condition for G-invariant subgroups. It follows from [17, Theorem 4.2] that any metabelian group with the weak minimal condition for normal subgroups (Min-∞-G) is G-minimax. But there exist G-minimax metabelian groups which do not have Min-∞-G. Our main result, Theorem 4.5, states that *a metabelian group G has Krull dimension if and only if it is G-minimax.*

A module is said to be *minimax* if it has a finite series of submodules each of whose factors is either artinian or Noetherian. The main theorem follows from Theorem 4.3 which states that if a group G is abelian minimax, then a $\mathbb{Z}G$-module has Krull dimension if and only if it is minimax. The proof of Theorem 4.3 in turn depends strongly on Theorem 3.4, which deals with commutative domains generated by minimax groups of units. Methods and ideas for studying such domains were introduced in [12] and further developed in [13].

2. Module theoretic lemmas

Let R be a ring and let M be a nonzero R-module with Krull dimension. The module M is said to be *critical* if $K_R(M/N) < K_R(M)$ for any nonzero submodule N of M.

LEMMA 2.1. *Let R be a commutative ring and let M be a critical R-module. Then:*

(i) $\mathrm{Ann}_R(x)$ *is a prime ideal of R for any element $0 \neq x \in M$.*
(ii) $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(y) = \mathrm{Ann}_R(M)$ *for any nonzero elements $x, y \in M$.*

Proof. (i) Suppose that there is an element $0 \neq x \in M$ such that $I = \mathrm{Ann}_R(x)$ is not prime. Then $L = R/I \cong xR$ is a critical L-module. Let a be a zero divisor of L. Then $N = \mathrm{Ann}_L(a) \neq 0$ and $aL \cong L/N$. Since L is critical, $K_L(aL) = K_L(L)$ and $K_L(L/N) < K_L(L)$, but this contradicts $aL \cong L/N$.

(ii) If $\mathrm{Ann}_R(x) \neq \mathrm{Ann}_R(y)$ for some nonzero elements $x, y \in M$, then there is an element $\alpha \in R$ such that $0 \neq \mathrm{Ann}_M(\alpha) \neq M$. The map $\varphi : M \to M$ given by $\varphi : x \to x\alpha$ is a nonzero endomorphism and $\mathrm{Ker}\,\varphi = \mathrm{Ann}_M(\alpha) \neq 0$, but this contradicts [6, 6.2.13]. □

LEMMA 2.2. *Let R be a commutative ring and let $\{B_i \mid i = \overline{1,m}\}$ be a finite set of prime ideals of R. Let A be a prime ideal of R such that $A \not\subseteq B_i$ for any i. Then:*

(i) *There is an element $a \in A \setminus \bigcup_{i=1}^m B_i$.*
(ii) *If M is an R-module with an ascending chain*

$$0 = M_0 \leq M_1 \leq \cdots \leq M_\alpha \leq M_{\alpha+1} \leq \cdots \leq M_\gamma \leq M_{\gamma+1},$$

whose factors are n-critical, $\mathrm{Ann}_R(M_{\alpha+1}/M_\alpha) \in \{B_i \mid i = \overline{1,m}\}$ if $\alpha + 1 \leq \gamma$, and $\mathrm{Ann}_R(M_{\gamma+1}/M_\gamma) = A$, then M has an n-critical submodule whose annihilator coincides with A.

Proof. (i) If $B_i \subseteq B_j$ for some i and j, then the ideal B_i can be omitted in the union $\bigcup_{i=1}^m B_i$. Thus we can assume that $B_i \not\subseteq B_j$ for any i and j. Let $X = \bigcap_{i=1}^m B_i$ and let D be the image of A in the quotient ring $L = R/X$ and C_i the image of B_i in L. Then it is sufficient to show that there is an element $b \in D \setminus \bigcup_{i=1}^m C_i$. Since $0 = \bigcap_{i=1}^m C_i$, we have $L \leq \bigoplus_{i=1}^m L_i$, where $L_i = L/C_i \cong R/B_i$ and $C_i = (\bigoplus_{j\neq i} L_J) \cap L$. As D is not contained in any C_i, we have $\mathrm{Pr}_{L_i}(D) \neq 0$ for each i. Since the C_i are prime ideals of L, the relation $(\prod_{j\neq i} C_j) \leq C_i$ implies that $C_j \leq C_i$ for some i and j, but this is impossible because $B_i \not\subseteq B_j$ for any i and j. Thus, there is an element $x_i \in (\prod_{j\neq i} C_j) \setminus C_i$, and hence $Dx_i = F_i \leq D \cap L_i \neq 0$, and we can put $b = \sum_{i=1}^m b_i$, where $0 \neq b_i \in F_i$.

(ii) By (i), there is an element $a \in A \setminus \bigcup_{i=1}^m B_i$. The map $\varphi : M_{\gamma+1} \to M_{\gamma+1}$ given by $\varphi : x \mapsto xa$ is an endomorphism. Since $a \in A \setminus \bigcup_{i=1}^m B_i$, it is easy to see that $\operatorname{Ker}\varphi \cap M_\gamma = 0$. If $\operatorname{Ker}\varphi \neq 0$, then $\operatorname{Ann}_R(\operatorname{Ker}\varphi) = A$ and $\operatorname{Ker}\varphi$ is an n-critical submodule. If $\operatorname{Ker}\varphi = 0$, then $N = \varphi(M_{\gamma+1}) \cong M_{\gamma+1}$ and hence N contains a submodule T such that N/T is an n-critical R-module and $\operatorname{Ann}_R(N/T) = A$. On the other hand, as $N \leq M_\gamma$, N has an ascending chain $\{N_i\}$ with n-critical factors whose annihilators coincide with some B_i. Let β be the maximal ordinal such that $N_\beta \leq T$. Then $(N_{\beta+1} + T)/T \neq 0$ is isomorphic to a quotient module of $N_{\beta+1}/N_\beta$. Since N/T and $N_{\beta+1}/N_\beta$ are n-critical modules, this implies that $(N_{\beta+1} + T)/T \cong N_{\beta+1}/N_\beta$. But this is impossible because $A \neq B_i$ for all i. Thus a contradiction is obtained. □

Let R be a ring and let M be an R-module. The module M is said to have *finite uniform dimension* if it contains no direct sum of infinitely many of submodules. By [6, 2.2.9], if the module M has finite uniform dimension, then there exists a positive integer n such that M contains no direct sum of more than n submodules; the minimal n with this property is called the *uniform dimension of* M and denoted by $\operatorname{u-dim} M$.

Let R be a ring and let M be an R-module with Krull dimension. For any ordinal α the submodule $\tau_\alpha(M) = \sum\{A \mid A \leq M, K_R(A) \leq \alpha\}$ is called the *α-torsion submodule* of M and, by [6, 6.2.18], $K_R(\tau_\alpha(M)) \leq \alpha$. Suppose that $K_R(M) = n$ is an integer and $\tau_{n-1}(M) = 0$. Let L be an n-critical submodule of M and let $N = \tau_{n-1}(M/L)$. Then it is easy to see that N is a maximal n-critical submodule of M. By [6, 6.2.6], the module M has finite uniform dimension and hence, by [6, 2.2.8], M has an essential submodule $U = \bigoplus_{i=1}^m U_i$, where the U_i are uniform submodules. Evidently, the submodules U_i can be chosen to be n-critical.

Lemma 2.3. *Let R be a commutative ring and let M be an R-module with finite Krull dimension n such that $\tau_{n-1}(M) = 0$. Let $U = \bigoplus_{i=1}^m U_i$ be an essential submodule of M, where the U_i are n-critical submodules, and let $B_i = \operatorname{Ann}_R(U_i)$. Let M_0/U be the $(n-1)$-torsion submodule of M/U. Let*

$$M_0 \leq M_1 \leq \cdots \leq M_\alpha \leq M_{\alpha+1} \leq \cdots \leq M_\gamma = M$$

be an ascending chain of submodules of M such that $M_{\alpha+1}/M_\alpha$ is a maximal n-critical submodule of M/M_α and let $A_\alpha = \operatorname{Ann}_R(M_{\alpha+1}/M_\alpha)$. Then:

(i) *Each A_α coincides with some B_i.*

(ii) *If γ is an infinite ordinal, then there is a submodule $L \leq M$ with an ascending chain $\{L_i \mid i \in \mathbb{N}\}$ such that $L = \bigcup_{i\in\mathbb{N}} L_i$, L_{i+1}/L_i is a maximal n-critical submodule of L/L_i and $\operatorname{Ann}_R(L_{i+1}/L_i) = B$ for each i, where B is a prime ideal of R.*

Proof. (i) The proof is by transfinite induction on γ. Evidently, we can assume that γ is a non-limit ordinal and that the assertion holds for $M_{\gamma-1}$.

Suppose that $A_{\gamma-1} \neq B_i$ for any i. By Lemma 2.2(ii), there is an n-critical submodule $V \leq M$ such that $\mathrm{Ann}_R(V) = A_\gamma$ and hence $U \cap V = 0$, but this is impossible because the submodule U is essential.

(ii) It follows from (i) that M has an ascending chain $\{M_\alpha\}$ of submodules such that $M_{\alpha+1}/M_\alpha$ is a maximal n-critical submodule of M/M_α and each A_α coincides with some B_i, where $A_\alpha = \mathrm{Ann}_R(M_{\alpha+1}/M_\alpha)$ and $\{B_i \mid i = 1, m\}$ is a finite set of prime ideals of R. Evidently, there is an ideal $B \in \{B_i \mid i = 1, m\}$ such that infinitely many of the A_i coincide with B. Put $L_1 = 0$, and let t be the first index such that $A_t = B$. Then it follows from Lemma 2.2(ii) that M_{t+1} has an n-critical submodule whose annihilator coincides with B. Therefore M_{t+1} has a maximal n-critical submodule L_2 whose annihilator coincides with B. The arguments of the proof of [6, 6.2.21] show that the quotient module M_{t+1}/L_2 has a finite series of submodules with n-critical factors whose annihilators coincide with ideals from $\{B_i \mid i = 1, m\}$. Passing to the quotient module M/L_2 and repeating the above arguments, the assertion follows. □

Let R be a ring and let G be a group. An RG-module is said to be *faithful* if $C_G(M) = 1$.

LEMMA 2.4. *Let k be a finite field of characteristic p, let G be an abelian Černikov group and let M be a faithful kG-module with Krull dimension. Then $|M| < \infty$ and $|G| < \infty$ if one of the following conditions holds:*

(i) *G has a p-subgroup P of finite index.*

(ii) *$M = kG$.*

Proof. (i) Suppose that $|M| = \infty$. It follows from [6, 6.2.10] that the module M has an ascending chain $\{M_i\}$ of submodules whose factors are critical and hence, by Lemma 2.1, annihilated by prime ideals of kG. Since kP has a unique augmentation prime ideal, all factors of the chain $\{M_i\}$ are centralized by P. Then it follows from Maschke's theorem that each factor of the chain $\{M_i\}$ is a direct sum of finite simple submodules, and by [6, 6.2.6] all factors of the chain $\{M_i\}$ are finite. Thus the module M contains an infinite submodule N which has ascending chain $\{N_i \mid i = \mathbb{N}\}$ with finite factors. Since the group G is Černikov, it contains a divisible subgroup $H \leq P$ of finite index. As H has no proper subgroups of finite index, the submodule N is centralized by H. Therefore $|P : C_P(N)| < \infty$, and hence N has a finite chain $\{L_i\}$ of submodules whose factors are centralized by P. As N is infinite, the chain $\{L_i\}$ has an infinite factor which, by Maschke's theorem, is a direct sum of finite simple submodules, but this contradicts [6, 6.2.6]. Thus, $|M| < \infty$ and hence $|G| < \infty$.

(ii) Suppose that the group G is infinite. Since kG has Krull dimension, kF also has Krull dimension for any subgroup $F \leq G$. Therefore, there is no harm in assuming that G is a quasicyclic p-group. If $p = \mathrm{char}\, k$, then the assertion follows from (i). If $p \neq \operatorname{char} k$, then it follows from Maschke's theorem that kG does not have finite uniform dimension, but this contradicts [6, 6.2.6]. □

3. Krull dimension in commutative domains generated by minimax groups of units

A finite series of ideals of a commutative ring R is said to be a composition series if all its factors are simple R-modules. Let G be an abelian minimax group; the *spectrum* $\mathrm{Sp}(G)$ consists of primes p such that G has an infinite p-quotient group. If the group G is torsion, then $\pi(G)$ denotes the set of prime divisors of orders of elements of G.

Lemma 3.1. *Let f be a field of characteristic zero and let T be a Černikov subgroup of the multiplicative group of f. Let K be a subring of f generated by T and let n be a positive integer whose prime divisors do not belong to* $\mathrm{Sp}(T)$. *Then the quotient ring K/Kn has a composition series.*

Proof. Since the additive group of the ring K is torsion-free, the quotient ring K/nK has a finite series of ideals each of whose factors is isomorphic to the quotient ring K/pK for some prime divisor p of n. Therefore, it is sufficient to consider the case where $n = p$ is a prime.

Let D be a finite subgroup of T containing primitive roots of 1 of degree q^2 for all $q \in \mathrm{Sp}(T)$ and such that $\pi(T/D) = \mathrm{Sp}(T)$. By [13, Lemma 7], $K = U \bigotimes_{\mathbb{Z}D} \mathbb{Z}T$, where U is a subring of f generated by D. Since the quotient ring U/Up is finite, it has a finite series of ideals each of whose factors is isomorphic to the quotient ring $L = U/P$ for some maximal ideal P of U containing pU. Then K/pK has a finite series of ideals each of whose factors is isomorphic to $F = (U/P) \bigotimes_{\mathbb{Z}D} \mathbb{Z}T$. It follows from the choice of n and D that the Sylow p-subgroup T_p of T is contained in D. Since L is a field of characteristic p, T_p centralizes L and hence also F. Therefore we can assume that $p \notin \pi(T)$.

Since the group T is locally cyclic, $D = \langle x \rangle$, where x is a primitive root of 1 of degree $m = |D|$. We now show that the image y of x in the quotient ring $L = U/P$ is a primitive root of 1 of degree m. It is sufficient to show that $y^t \neq 1$ for any proper divisor t of m. Suppose that $y^t = 1$. Then y is a root of a polynomial $X^t - 1 \in \mathbb{F}_p[X]$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $A(X) \in \mathbb{Z}[X]$ be the cyclotomic polynomial of degree m and let $P(X)$ be the image of $A(X)$ in $\mathbb{F}_p[X]$. Then y is a root of $P(X)$. The polynomial $X^m - 1 \in \mathbb{F}_p[X]$ may be written in the form $X^m - 1 = (X^t - 1)P(X)Q(X)$, where $Q(X)$ is a polynomial in $\mathbb{F}_p[X]$. Then, as y is a root of the polynomials $X^t - 1$ and $P(X)$, the polynomial $X^m - 1$ has repeated roots. On the other hand, as p is not a divisor of m, the derivative mX^{m-1} of $X^m - 1$ does not have common roots with $X^m - 1$ and hence the polynomial $X^m - 1$ does not have repeated roots.

This contradiction shows that y is a primitive root of 1 of degree m. Let J/Kp be a maximal ideal of K/Kp containing P/Up. Since y is a primitive root of 1 of degree m, it follows from [13, Lemma 7] that $L/J = (U/P) \bigotimes_{\mathbb{Z}D} \mathbb{Z}T = F$. Hence F is a simple $\mathbb{Z}T$-module. □

A ring is said to be *Hilbert* if each of its prime ideals is an intersection of maximal ideals. A ring is said to be *absolutely Hilbert* if it is commutative, Noetherian and Hilbert, and all its field images are locally finite. The *dimension* of a ring is the maximal length of a chain of prime ideals.

LEMMA 3.2. *Let f be a field of characteristic zero, let T be a Černikov subgroup of the multiplicative group of f and let Q be a subring of f consisting of fractions whose denominators are products of powers of primes from* $\mathrm{Sp}(T)$. *Let L be a subring of f generated by T and Q. Then the ring L is*

(i) *Noetherian,*
(ii) *critical and satisfies $K(L) = 1$,*
(iii) *absolutely Hilbert of dimension* 1.

Proof. Let R be a subring of L generated by T. Then $L = R \bigotimes_{\mathbb{Z}} Q$ and the additive group of L is p-divisible for any $p \in \mathrm{Sp}(T)$. Let I be a proper ideal of L. Then there is a positive integer $n \in I$. As the additive group of L is p-divisible for any $p \in \mathrm{Sp}(T)$, prime divisors of n do not belong to $\mathrm{Sp}(T)$. Since $\pi(L/R) \subseteq \mathrm{Sp}(T)$, it is not difficult to see that $L/nL \cong R/nR$. Then it follows from Lemma 3.1 that the quotient ring L/nL has a finite composition series. This implies that for any proper ideal I of L the quotient ring L/I is Noetherian and $K(L/I) = 0$. Therefore, the ring L is Noetherian, critical and satisfies $K(L) = 1$.

If J is a maximal ideal of K, then K/J is an elementary abelian group and hence the field K/J is locally finite. As for any nonzero ideal I of K the quotient ring K/I has a finite composition series, it follows from [3, Chap IV, §2, Proposition 9] that any prime ideal of K is maximal. Therefore, K is an absolutely Hilbert ring of dimension 1. □

Let I be an ideal of a group ring RG. We denote by $I^\dagger$ the normal subgroup $(I + 1) \cap G$ of G.

LEMMA 3.3. *Let f be a field of characteristic zero and let G be a subgroup of the multiplicative group of f such that $G = T \times H$, where H is a finitely generated free abelian group and T is a Černikov group. Let K be a subring of f generated by G. Then the ring K has Krull dimension if and only if one of the following conditions holds:*

(i) *The additive group of K is p-divisible for any $p \in \mathrm{Sp}(T)$.*
(ii) *K is a Noetherian ring, and $K(K) \leq r(H) + 1$.*

Proof. Suppose that the additive group of K is p-divisible for any $p \in \mathrm{Sp}(T)$. Then K contains a subring Q consisting of fractions whose denominators are products of powers of primes from $\mathrm{Sp}(T)$. Let L be a subring of K generated by Q and T. By Lemma 3.2, the ring L is Noetherian and $K(L) = 1$. It is not difficult to see that K is a quotient ring of the group ring LH. By [4, Theorem 1], LH is a Noetherian ring and hence so is K. Then, by [6, 6.2.3], K has Krull dimension and, by [6, 6.6.1], $K(K) \leq r(H) + 1$. If the ring K is Noetherian, then, by [6, 6.2.3], it has Krull dimension.

Suppose now that the ring K has Krull dimension and that for some $p \in \mathrm{Sp}(T)$ the additive group of K is not p-divisible. By [13, Lemma 5], there is a finitely generated subgroup $D \leq G$ such that $H \leq D$, $\pi(G/D) = \mathrm{Sp}(T)$ and $K = U \bigotimes_{\mathbb{Z}D} \mathbb{Z}G$, where U is a subring of K generated by D. Evidently, the additive group of U is not p-divisible and hence there is a maximal ideal J of U such that U/J is a p-group. It follows from [5, Theorem 3.1] that $|U/J| < \infty$ and hence $|D/J^{\dagger}| < \infty$. As $p \in \mathrm{Sp}(T)$, there is a subgroup F of G such that $D \leq F$ and F/D is an infinite p-group. Hence $F/J^{\dagger}$ is an infinite Černikov group which contains a p-subgroup of finite index. Then $K = V \bigotimes_{\mathbb{Z}F} \mathbb{Z}G$, where V is a subring of K generated by F. Thus the ring V has Krull dimension. As $V = U \bigotimes_{\mathbb{Z}D} \mathbb{Z}F$, we have $V/JV = (U/J) \bigotimes_{\mathbb{Z}D} \mathbb{Z}F$. Therefore V/JV is an infinite $\mathbb{F}_p(F/J^{\dagger})$-module with Krull dimension, but this contradicts Lemma 2.4(i). □

THEOREM 3.4. *Let f be a field and let G be a minimax subgroup of the multiplicative group of f. Let K be a subring of f generated by G. The ring K has Krull dimension if and only if one of the following conditions holds:*

(i) *$G = T \times F$, where T is a torsion group and F is a free abelian group, and if* $\mathrm{char}\, f = 0$, *then the additive group of K is p-divisible for any $p \in \mathrm{Sp}(T)$.*

(ii) *The ring K is Noetherian and satisfies $K(K) \leq r(F) + 1$.*

Proof. Suppose that the ring K has Krull dimension. Put $\pi = \mathrm{Sp}(G) \setminus \mathrm{char}\, f$ and let K_1 be the ring obtained by adding to K the primitive roots of 1 of degree p^2 for all $p \in \pi$. Then K_1 is a finitely generated K-module and, as the ring K has Krull dimension, it follows from [6, 6.2.5] that the ring K_1 also has Krull dimension. Thus there is no harm in assuming that $K = K_1$. The group G can be presented in the form $G = T \times F$, where T is a torsion group and F is a torsion-free group. Then it follows from [13, Lemma 7] that there is a dense subgroup $H \leq G$ such that $H = A \times B$, where A is a finite subgroup of T, $B \leq F$, $\mathrm{Sp}(B) \subseteq \{\mathrm{char}\, f\}$, $\pi(F/B) \subseteq \mathrm{Sp}(F) \setminus \mathrm{char}\, f$ and $K = R \bigotimes_{\mathbb{Z}H} \mathbb{Z}G$, where R is a subring of K generated by H.

If $\mathrm{char}\, f = 0$, then H is finitely generated. Suppose that $\mathrm{char}\, f = p$ and that the subgroup H is not finitely generated. As $K = R \bigotimes_{\mathbb{Z}H} \mathbb{Z}G$, the ring R has Krull dimension. Let S be a subring of K generated by B. By [13, Lemma 5], there is a finitely generated dense subgroup $C \leq B$ such that $|S/S(1-C)| = \infty$. The ring R is obtained by adding the primitive roots of 1 to S and, as 1 is the leading coefficient of any cyclotomic polynomial, it is not difficult to show that the ring R can be presented in the form $R = S \bigoplus (\bigoplus_{i=1}^{t} Sa_i)$, where $a_i \in R$. Therefore, $R(1-C) \cap S = S(1-C)$ and hence $|R/R(1-C)| = \infty$. As H/C is a Černikov group having a p-subgroup of finite index and $R/R(1-C)$ is an infinite $\mathbb{F}_p(H/C)$-module with Krull dimension, we obtain a contradiction to Lemma 2.4. Hence the subgroup H is finitely generated.

Suppose that $B \neq F$. Let $X = A \times F$. Then $K = Y \bigotimes_{\mathbb{Z}X} \mathbb{Z}G$, where Y is a subring of K generated by X and hence Y has Krull dimension. By [13, Proposition 1], there is a maximal ideal I of S such that $\pi(B/I^\dagger) \cap \pi(F/B) = \emptyset$. As $Y = R \bigotimes_{\mathbb{Z}H} \mathbb{Z}X$, we have $Y = R \bigotimes_{\mathbb{Z}B} \mathbb{Z}F$ and hence $Y/IY = (R/IR) \bigotimes_{\mathbb{Z}B_1} \mathbb{Z}F_1$, where $B_1 = B/I^\dagger$ and $F_1 = F/I^\dagger$. Since $\pi(B/I^\dagger) \cap \pi(F/B) = \emptyset$, there is a subgroup $F_2 \leq F_1$ such that $F_1 = F_2 \times B_1$. The relation $Y/IY = (R/IR) \bigotimes_{\mathbb{Z}B_1} \mathbb{Z}F_1$ shows that $Y/IY = (R/IR)F_2$ is a group ring with Krull dimension, but this contradicts Lemma 2.4(ii) because F_2 is an infinite Černikov group. Thus the subgroup $F = B$ is finitely generated.

If $\operatorname{char} f = 0$, then, by Lemma 3.3, the ring K is Noetherian, satisfies $K(K) \leq r(F)+1$, and the additive group of K is p divisible for any $p \in \operatorname{Sp}(T)$.

If $\operatorname{char} f = p > 0$, then the subring L of K generated by T is a field and the ring K can be presented as a quotient ring of the group ring LF. Then it follows from [4, Theorem 1] that the ring K is Noetherian and, by [6, 6.6.1], $K(K) \leq r(F)$.

If the ring K is Noetherian, then, by [6, 6.2.3], it has Krull dimension. □

4. Modules over abelian minimax groups and metabelian groups with Krull dimension

Let G be a soluble group of finite rank. Evidently, there exists a unique minimal characteristic subgroup $P(G)$ of G such that the quotient group $G/P(G)$ is free abelian.

Lemma 4.1. *Let G be a minimax abelian group and let M be a $\mathbb{Z}G$-module with Krull dimension. Then:*

(i) *Any cyclic critical submodule L of M is Noetherian and $K(L) \leq r(G/P(G)) + 1$.*
(ii) *$K(M) \leq r(G/P(G)) + 1$.*

Proof. (i) Since $L \cong K = \mathbb{Z}G/\operatorname{Ann}_{\mathbb{Z}G}(K)$, where K is a domain generated by a minimax group $G/(\operatorname{Ann}_{\mathbb{Z}G}(K))^\dagger$, it follows from Theorem 3.4 that the ring K is Noetherian and $K(L) \leq r(G/P(G)) + 1$.

(ii) It follows from [6, 6.2.10] that the module M has an ascending chain $\{M_i\}$ whose factors are cyclic critical and $\cup M_i = M$. Then (ii) follows from (i) and [6, 6.2.17]. $\square$

LEMMA 4.2. *Let G be a minimax abelian group and let M be a non-artinian minimax $\mathbb{Z}G$-module. Then the module M has a maximal submodule.*

Proof. Since the module M is minimax, it is sufficient to consider the case where M has a non-artinian critical cyclic submodule N such that M/N is an artinian module and $\tau_0(M) = 0$. It follows from Lemma 2.3 that the module M has an ascending chain of submodules $N = M_0 \leq M_1 \leq \cdots \leq M_\alpha \leq M_{\alpha+1} \leq \cdots \leq M_\gamma = M$ such that $M_{\alpha+1}/M_\alpha$ is a simple submodule of M/M_α and $A_\alpha = \mathrm{Ann}_R(M_{\alpha+1}/M_\alpha)$ coincides with some B_i, where $(B_i \mid i = 1, m)$ is a finite set of maximal ideals of $\mathbb{Z}G$.

It follows from Theorem 3.4 and Lemma 3.2 that N can be considered as an LF-module, where L is an absolutely Hilbert ring of dimension at most 1 and F is a finitely generated abelian group. Then it follows from [11, Main Theorem (2nd version)] that the submodule N is residually simple. Therefore, there is a maximal submodule T of N such that $B = \mathrm{Ann}_{\mathbb{Z}G}(N/T) \neq B_i$ for any i. By Lemma 2.2, there is an element $a \in B \setminus (\bigcup_{i=1}^m B_i)$. The map $\varphi : M/T \to M/T$ given by $\varphi : x \mapsto xa$ is an endomorphism. Since $a \in B \setminus (\bigcup_{i=1}^m B_i)$, it is easy to see that $\mathrm{Ker}\,\varphi = (N/T)$ and hence $\varphi(M/T) \cong M/N$. As $\varphi(M/N) = M/N$, this implies that $M/T = N/T \oplus \varphi(M/T)$ and the assertion follows. $\square$

THEOREM 4.3. *Let G be a minimax abelian group and let M be a $\mathbb{Z}G$-module. The module M has Krull dimension if and only if it is minimax. If the module M has Krull dimension, then $K_{\mathbb{Z}G}(M) \leq r(G/P(G)) + 1$.*

Proof. If the module M is minimax, then it follows from [6, 6.2.3] that it has Krull dimension. By Lemma 4.1 we have $K_{\mathbb{Z}G}(M) \leq r(G/P(G)) + 1$ and we can use induction on $K_{\mathbb{Z}G}(M)$. If $K_{\mathbb{Z}G}(M) = 0$, then the module M is artinian.

Suppose that $K_{\mathbb{Z}G}(M) = n > 0$. Then by the induction hypothesis the $(n-1)$-torsion submodule $\tau_{n-1}(M)$ of M is minimax. So, passing to the quotient module $M/\tau_{n-1}(M)$ we can assume that $\tau_{n-1}(M) = 0$. Then it follows from Lemma 2.2 that M has an ascending chain $\{M_\alpha\}$ of submodules such that $M_{\alpha+1}/M_\alpha$ is a maximal n-critical submodule of M/M_α and each A_α coincides with some B_i, where $A_\alpha = \mathrm{Ann}_{\mathbb{Z}G}(M_{\alpha+1}/M_\alpha)$ and $\{B_i \mid i = 1, m\}$ is a finite set of prime ideals of $\mathbb{Z}G$. Each factor $N_\alpha = M_{\alpha+1}/M_\alpha$ has a cyclic n-critical submodule F_α such that $K_{\mathbb{Z}G}(N_\alpha/F_\alpha) \leq n - 1$. By Lemma 4.1(i), F_α is a Noetherian module and, by the induction hypothesis, the quotient module N_α/F_α is minimax. Thus, it is sufficient to show that the chain $\{M_\alpha\}$ is finite.

Suppose that the chain $\{M_\alpha\}$ is infinite. Then, by Lemma 2.3(ii), there is a submodule $L \leq M$ with an ascending chain $\{L_i \mid i \in \mathbb{N}\}$ such that $L = \bigcup_{i\in\mathbb{N}} L_i$, L_{i+1}/L_i is a maximal n-critical submodule of L/L_i and $\mathrm{Ann}_{\mathbb{Z}G}(L_{i+1}/L_i) = B$ for each i, where B is a prime ideal of $\mathbb{Z}G$. By Lemma 4.2, the submodule L_2 has a maximal submodule T_2 and, by [16, Theorem 2.4], the quotient group $G/C_G(L_2/T_2)$ is torsion and L_2/T_2 is an elementary abelian p_2-group.

Suppose that $r((G/B^\dagger)/P(G/B^\dagger)) \geq 1$. Then there is an element $z_2 \in G \setminus B^\dagger$ such that $L_2(1-z_2) \leq T_2$ and $A_2 \cap L_2 = L_2(1-z_2)$, where $A_2 = M(1-z_2)$. Putting $A_i = M$, we see that $L_2 \cap A_1 \not\subset L_1 + A_2$, where $L_1 = 0$. As $\{L_i \cap A_2 \mid i \in \mathbb{N}\}$ is an ascending chain with n-critical factors, the above arguments show that there are a maximal ideal T_3 of $(L_3 \cap A_2)$ and an element $z_3 \in G \setminus B^\dagger$ such that $(L_2 \cap A_2) \leq T_3$, $L_3(1-z_3) \leq T_3$ and $A_3 \cap L_3 = L_3(1-z_3)$, where $A_3 = A_2(1-z_3)$. It easily follows that $L_3 \cap A_2 \not\subset L_2 + A_3$. Hence, by moving up the chain $\{L_i \mid i \in \mathbb{N}\}$ we can construct a descending chain $\{A_i \mid i \in \mathbb{N}\}$ such that $L_{i+1} \cap A_i \not\subset L_i + A_{i+1}$ for each i. But, by [6, 6.2.16], in this case the module M has no Krull dimension and a contradiction is obtained.

If $r((G/B^\dagger)/P(G/B^\dagger)) = 0$, then it follows from Theorem 3.4 that $n = 1$ and the quotient group L_{i+1}/L_i is torsion-free for each i. Then, putting $A_1 = M$ and $A_2 = Mp_2$ we see that $L_2 \cap A_1 \not\subset L_1 + A_2$. By Lemma 2.3, there are a maximal ideal T_3 of $(L_3 \cap A_2)$ and a prime p_3 such that $(L_2 \cap A_2) \leq T_3$, $L_3p_3 \leq T_3$ and $A_3 \cap L_3 = L_3p_3$, where $A_3 = A_2p_3$, and hence $L_3 \cap A_2 \not\subset L_2 + A_3$. Hence, as in the above case, we can construct a descending chain $\{A_i \mid i \in \mathbb{N}\}$ and obtain a contradiction. □

Lemma 4.4. *Let G be a soluble group. Then* $\operatorname{dev} G$ *exists if and only if the group G is minimax. Furthermore, if* $\operatorname{dev} G$ *exists, then* $\operatorname{dev} G \leq 1$, *and* $\operatorname{dev} G = 0$ *if and only if the group G is Černikov.*

Proof. Evidently, by using induction on the solvability length of G, we can assume that the group G is abelian. Then the group G can by considered as a $\mathbb{Z}$-module and, as G has Krull dimension, it follows from [6, 6.2.6] that $\text{u-dim}\, G < \infty$. Therefore $r(G) < \infty$ and hence there is a finitely generated dense subgroup H of G. The above arguments show that $\text{u-dim}(G/H) < \infty$, which implies that the quotient group G/H is Černikov and hence the group G is minimax.

If G is a minimax group, then it satisfies Min-∞ and hence $\operatorname{dev} G \leq 1$. Moreover, $\operatorname{dev} G = 0$ if and only if G satisfies Min and hence the group G is Černikov. □

Theorem 4.5. *A metabelian group G has Krull dimension if and only if it is G-minimax. If the group G has Krull dimension, then $K(G) \leq r(G/P(G)) + 1$.*

Proof. The derived subgroup M of G can be considered as a $\mathbb{Z}(G/M)$-module, where G/M acts on M by conjugation. Then the assertion follows from Lemma 4.4 and Theorem 4.3. $\square$

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