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L-FREE GROUPS

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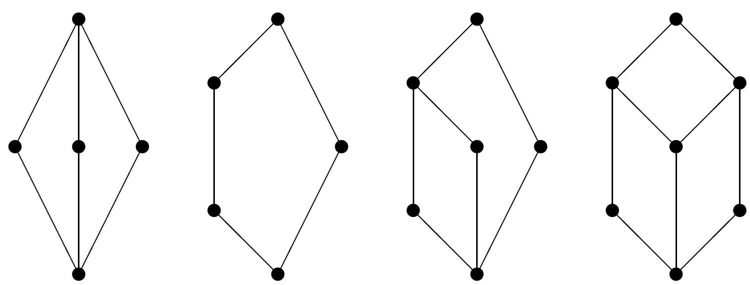
Abstract. Let L be a lattice. A lattice is called L-free if it has no sublattice isomorphic to L. In this paper we study finite groups whose subgroup lattices are L-free for certain lattices L.

Introduction

A lattice L is called primitive if the class $\mathfrak{C}(L)$ of lattices containing no sublattice isomorphic to L is a variety [2, p. 129]. A well-known example of such a lattice is the nonmodular lattice N_5 with 5 elements for which $\mathfrak{C}(N_5)$ is the variety of modular lattices; see Figure 1.



M_5 N_5 L_6 L_7

Figure 1

Not every variety has a primitive lattice and there are also other interesting classes of lattices characterized by the fact that their members do not contain certain lattices as sublattices. Therefore we introduce the following concept.

Definition. Let L be a lattice.

(a) A lattice is called *L-free* if it has no sublattice isomorphic to L.
(b) A group G is called *L-free* if its subgroup lattice $L(G)$ is L-free.

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For example, a lattice is modular if and only if it is N_5-free, and it is distributive if and only if it is N_5-free and M_5-free. We shall show in Section 1 that a subgroup lattice is distributive if and only if it is M_5-free.

It is a theorem of Krempa and Terlikowska-Osłowska [4] that a finite lattice is globally permutable if and only if it is L_6-free and L_7-free (see Figure 1); Baginski and Sakowicz [1] determined the finite groups with such a subgroup lattice. They showed that a finite group is L_6-free and L_7-free if and only if it is a direct product of $\tilde{P}$-groups and modular p-groups with pairwise relatively prime orders. Here a group X is called a $\tilde{P}$-group if $X = A\langle b\rangle$ where A is an elementary abelian normal p-subgroup of X, $o(b) = q^n$ for primes p and q such that q divides $p-1$ and $n \in \mathbb{N}$, and b induces a nontrivial power automorphism β in A. Note that a $\tilde{P}$-group X is a P-group if $o(b) = q$ and a P^*-group if $o(\beta) = q$; so the finite L_6- and L_7-free groups are not far from being modular (see [6, Theorem 2.4.4]).

In this paper we study the question what happens if a finite group has just one of the two properties occurring in the Baginski-Sakowicz Theorem. It turns out that these groups are also near to being modular. We show that a finite group is L_6-free if and only if its subgroup lattice is upper semimodular; the structure of these groups was given independently by Jones and Sato (see [7, p. 24]). And a finite group is L_7-free if and only if it is a direct product of groups with pairwise relatively prime orders which are either modular p-groups or of the form $X = AB$, where A is an elementary abelian normal p-subgroup of X and B a q-group, p and q are primes, and one of the following holds:

- (i) B is cyclic and every subgroup of B either is irreducible on A or induces a power automorphism in A.
- (ii) B is a quaternion group (of order 8) operating faithfully on A, $|A| = p^2$ and $p \equiv 3 \pmod 4$.

In particular, L_7-free groups are in general not supersoluble.

Our notation is standard (see [3] or [6]) except that we write $H \cup K$ for the group generated by the subgroups H and K of the group G. For basic results on subgroup lattices we refer the reader to [6].

1. M_5-free groups

Let us denote the chain with n elements by K_n. Then a lattice is K_n-free if and only if it has length at most $n-2$, and it is $(K_2 \times K_2)$-free if and only if it is a chain. So the smallest lattice which is interesting in the study of L-free groups is M_5.

THEOREM 1.1. *The group G is M_5-free if and only if $L(G)$ is distributive (and hence G is locally cyclic).*

Proof. If $L(G)$ is distributive, it is clearly M_5-free. Conversely, suppose that G is M_5-free. We show that G is abelian; then $L(G)$ is modular and M_5-free and hence distributive.

So suppose, for a contradiction, that G is not abelian. Then there exist $x, y \in G$ such that $xy \neq yx$. If $\langle x\rangle \cap \langle y\rangle = 1$ for all such pairs (x, y) of noncommuting elements, it would follow that $\langle x\rangle \cap \langle xy\rangle = 1 = \langle y\rangle \cap \langle xy\rangle$ and so $\{1, \langle x\rangle, \langle y\rangle, \langle xy\rangle, \langle x, y\rangle\}$ would be a sublattice of $L(G)$ isomorphic to M_5. Hence there exist $x, y \in G$ such that $xy \neq yx$ and $\langle x\rangle \cap \langle y\rangle \neq 1$. Then $|\langle x\rangle : \langle x\rangle \cap \langle y\rangle|$ and $|\langle y\rangle : \langle x\rangle \cap \langle y\rangle|$ are finite and we choose x, y with $xy \neq yx$ such that $|\langle x\rangle : \langle x\rangle \cap \langle y\rangle|$ is minimal.

Let $H = \langle x, y\rangle$ and let p be a prime dividing $|\langle x\rangle : \langle x\rangle \cap \langle y\rangle|$. Then $x^p y = y x^p$ and hence $N := \langle x^p\rangle \leq Z(H)$. If $\langle x\rangle \neq \langle x^y\rangle$, the group $\langle x, x^y\rangle/N$ would have two different subgroups of order p and hence would not be cyclic. It would follow that $\langle x, x^y\rangle \neq \langle xx^y\rangle N$ and so $\{N, \langle x\rangle, \langle x^y\rangle, \langle xx^y\rangle N, \langle x, x^y\rangle\}$ would be a sublattice of $L(G)$ isomorphic to M_5.

Thus $\langle x\rangle = \langle x^y\rangle \trianglelefteq H = \langle x, y\rangle$. If H/N were not abelian, it would contain a nonabelian section of order pq for some prime q; again $L(G)$ would have a sublattice isomorphic to M_5. So, finally, $H/N = \langle x\rangle/N \times \langle y\rangle N/N$ is a finite abelian group and $|\langle y\rangle N/N|$ is prime to p since H/N is M_5-free. Therefore H/N is cyclic and since $N \leq Z(H)$, it follows that H is abelian, the desired contradiction. $\square$

2. L_6-free groups

Both lattices L_6 and L_7 contain sublattices isomorphic to N_5. Therefore every modular (that is, N_5-free) lattice clearly is L_6-free and L_7-free. For finite p-groups the converse also holds.

Lemma 2.1. *The following properties of a finite p-group G are equivalent.*

(a) *G is L_6-free.*
(b) *G is L_7-free.*
(c) *$L(G)$ is modular.*

Proof. As mentioned above, (c) implies (a) and (b). Conversely assume that (a) or (b) holds and suppose, for a contradiction, that $L(G)$ is not modular. Then by [6, Lemma 2.3.3], G has a section X isomorphic to the dihedral group of order 8 or to the nonabelian group of order p^3 and exponent p for $p > 2$. If A_1 and A_2 are different noncyclic maximal subgroups of X and B_i, C_i are different minimal subgroups of A_i different from $Z(X)$, then $\{1, B_1, C_1, A_1, B_2, X\}$ and $\{1, B_1, A_1, Z(X), B_2, A_2, X\}$ are sublattices of $L(X)$ isomorphic to L_6 and L_7, respectively. This contradiction proves the lemma. $\square$

We shall need another general property of L_6-free or L_7-free lattices.

Lemma 2.2. *Let M and N be lattices and let $k \in \{6, 7\}$. If M and N are L_k-free, then so is $M \times N$.*

Proof. It is well-known that L_k is subdirectly irreducible (see [5, p. 34]), that is, there exist $a, b \in L_k$ such that $a \neq b$ and $a^\varphi = b^\varphi$ for every homomorphism φ of L_k which is not injective. (It is easy to see that one can take for a the least element of L_k and for b an atom which is not an antiatom if $k = 6$ and the intersection of the two antiatoms if $k = 7$.) If L_k were a sublattice of $M \times N$, then since M and N are L_k-free, the projections φ_1 and φ_2 of L_k into M and N, respectively, could not be injective; hence $a^{\varphi_i} = b^{\varphi_i}$ for $i = 1, 2$ and so $a = b$, a contradiction. □

We show next that finite L_6-free groups are supersoluble. For this we need a lemma which will also be used later.

Lemma 2.3. *Let G be a finite group, $N \trianglelefteq G$ and $H \leq G$ such that $G = NH$ and $N \cap H = 1$. If G is L_6-free, then H normalizes every subgroup of N.*

Proof. Suppose that this is false. Then there exist $B \leq N$ and $x \in H$ such that $B^x \neq B$; let B be minimal with this property. Then

$$E := B \cap B^x < B < B \cup B^x \leq N$$

and hence $E^x = E$. It follows that $N \cap E\langle x \rangle = E(N \cap \langle x \rangle) = E$ and therefore $\{E, B, B^x, B \cup B^x, E\langle x \rangle, \langle B, x \rangle\}$ is a sublattice of $L(G)$ isomorphic to L_6. □

Lemma 2.4. *Let G be a finite group. If G is L_6-free, then G is supersoluble.*

Proof. We use induction on $|G|$. Then every proper subgroup and factor group of G is supersoluble. By Huppert's theorem (see [3, p. 718]), G is soluble. Let N be a minimal normal subgroup of G. Then G/N is supersoluble, $|N|$ is a power of some prime p and $N \cap Z(P) \neq 1$ for every Sylow p-subgroup P of G. By Lemma 2.3, a subgroup X of order p of $N \cap Z(P)$ is normalized by every p'-subgroup of G and hence $X \trianglelefteq G$. So $N = X$ has order p and G is supersoluble. □

We can now prove our main result on L_6-free groups. For the sake of simplicity, we introduce the following notation.

Definition 2.5. The finite group G is called a *Q-group* if there exist pairwise different primes $p_1, \dots, p_k$ ($k \in \mathbb{N}$) and q such that $G = (P_1 \times \cdots \times P_k)Q$ and for all $i, j \in \{1, \dots, k\}$,

(1) P_i is an elementary abelian normal p_i-subgroup of G,
(2) Q is a cyclic q-group inducing nontrivial power automorphisms in P_i,
(3) $C_Q(P_i) \neq C_Q(P_j)$ for $i \neq j$.

THEOREM 2.6. *The following properties of the finite group G are equivalent.*

(a) *G is L_6-free.*
(b) *$L(G)$ is upper semimodular.*
(c) *G is a direct product of Q-groups and modular p-groups with pairwise relatively prime orders.*

Proof. It is well-known (see [7, p. 24]) that (b) and (c) are equivalent. We prove that (a) implies (b) and that (c) implies (a).

So assume, for a contradiction, that G is L_6-free but $L(G)$ is not upper semimodular and let G be minimal with this property. Since $L(G)$ is not upper semimodular, there exist subgroups H and K of G such that $H \cap K$ is a maximal subgroup of H but K is not maximal in $H \cup K$. We choose such a pair with $|H|$ minimal and $|K|$ maximal for this H. The minimality of G implies that $H \cup K = G$. Since K is not maximal in G, there exist maximal subgroups M of G and L of M such that $K \leq L$. Since $H \not\leq L$, we have $H \cap L = H \cap K$ and the maximality of K implies that $L = K$. By Lemma 2.4, G is supersoluble and hence maximal subgroups have prime index; so there exist primes p, q, r such that

$$|G : K| = pr \tag{4}$$

and $|H : H \cap K| = q$. If x is a q-element of minimal order in H such that $H = \langle x \rangle \cup (H \cap K)$, then $x^q \in H \cap K$ and so $\langle x \rangle \cap K = \langle x^q \rangle$ is maximal in $\langle x \rangle$ and $\langle x \rangle \cup K = H \cup K = G$. The minimality of H implies that

$$H = \langle x \rangle \text{ is cyclic of order } q^n \text{ for some } q \in \mathbb{P}, n \in \mathbb{N}. \tag{5}$$

Suppose that $H^G \neq G$. If $E := H^G \cap K$, then $H \cap E = H \cap K$ is maximal in H. The minimality of G implies that $L(H^G)$ is upper semimodular and so E is maximal in $E \cup H =: B$. Since $|H^G : E| = |G : K| = pr$, it follows that $E \lessdot B \lessdot H^G$; furthermore, $H \leq B < H^G$ implies that B is not normal in $G = H \cup K$ and hence there exists $y \in K$ such that $B \neq B^y$. Since $E \trianglelefteq K$, it follows that $\{E, B, B^y, H^G, K, G\}$ is a sublattice of $L(G)$ isomorphic to L_6. This contradiction shows that

$$H^G = G. \tag{6}$$

The supersoluble group G has a normal t-complement N for the smallest prime t dividing $|G|$ (see [3, p. 716]). Since $H^G = G$, it follows that $NH = G$ and so $t = q$ and H is a Sylow q-subgroup of G. By Lemma 2.3, every subgroup of N is normalized by H and any of its conjugates and hence is normal in G. In particular, $N \cap K \trianglelefteq G$ and since $(N \cap K)H \cap K = (N \cap K)(H \cap K)$ is maximal in $(N \cap K)H$, the lattice $L(G/N \cap K)$ is not upper semimodular. The minimality of G implies that $N \cap K = 1$. Therefore $|N|$ divides $|G : K| = pr$ and K is a q-group properly containing $H \cap K$. Since Sylow q subgroups are cyclic, it follows that $H \cap K \trianglelefteq H \cup K = G$ and the minimality of G now implies that $H \cap K = 1$. Since $L(G)$ is not modular, we finally get that

$$G = NH \text{ where } |N| = pr \text{ and } |H| = q = |K|,\ q < r \leq p. \tag{7}$$

Furthermore G is not a P-group, but every subgroup of N is normal in G. So if $p = r$, then N is cyclic. Since $H \cup K = G$, we have $PH \neq PK$ if P is the subgroup of order p of N. So if we choose $y \in P$ such that $H \neq H^y$, then $\{1, H, H^y, PH, K, G\}$ is a sublattice of $L(G)$ isomorphic to L_6. Therefore, finally, $p \neq r$ and then $N = P \times R$ where $|P| = p$ and $|R| = r$. Since $p > r > q$, there exist three subgroups Q_i of order q different from H in HP. Since $Q_i \cup Q_j = HP$ for $i \neq j$, the groups PK and RK each contain at most one of the Q_i and hence there exists Q_j such that $K \cup Q_j = G$. Now $\{1, H, Q_j, HP, K, G\}$ is a sublattice of $L(G)$ isomorphic to L_6. This contradiction proves that (a) implies (b).

We next show that (c) implies (a) and consider a minimal counterexample G to this assertion. Then [6, Lemma 1.6.4] and Lemma 2.2 together with 2.1 imply that G is a Q-group; we use the notation of Definition 2.5. Since G is a minimal counterexample, there exists a sublattice of $L(G)$ isomorphic to L_6, and its maximal element is G; note that property (c) is inherited by subgroups and factor groups since it is equivalent to (b).

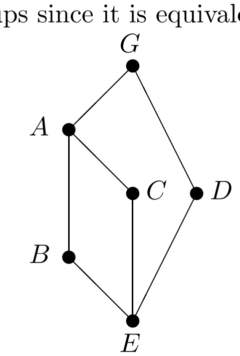

FIGURE 2

In the notation of Figure 2, clearly, $E_G = 1$. We will show that also

$$B_G = C_G = D_G = 1. \tag{8}$$

So suppose that $N \trianglelefteq G$ and $N \leq B$. If $NE = B$, it would follow that $BD = NED = ND = DN = DEN = DB$; so $G = BD$ and $A = B(A \cap D) = B$, a contradiction. Therefore $NE < B$ and since $A \cap ND = N(A \cap D) = NE$ and $B \cap NC = N(B \cap C) = NE$, it follows that $\{NE, B, NC, A, ND, G\}$ is a sublattice of $L(G)$ isomorphic to L_6. The minimality of G implies that $N = 1$. So $B_G = 1$ and, similarly, $C_G = 1$.

Now let $N \trianglelefteq G$ such that $N \leq D$. As above, if $NE = D$, then $BD = DB$ and we get the same contradiction $A = B$. Therefore $NE < D$ and since

$NA \cap D = N(A \cap D) = NE$ and $NA/N \simeq A/A \cap N$ with $A \cap N \leq E$, it follows that $\{NE, NB, NC, NA, D, G\} \simeq L_6$. So $N = 1$ and (8) holds.

It follows from (2) of Definition 2.5 that B, C, and D are q-groups. Hence they are cyclic and E is centralized by $B \cup D = G$. Thus

$$E = 1. \tag{9}$$

Let $M := P_1 \times \cdots \times P_k$ so that $G = MQ$. Since G/M is a cyclic q-group and $G = B \cup D$, one of MB or MD must be G and hence one of the groups B and D is a Sylow q-subgroup S of G and the other is contained in S^x for some $x \in M$. Thus

$$G = B \cup D \leq S \cup S^x \leq S \cup \langle x \rangle = \langle x \rangle S$$

and hence $M = \langle x \rangle$ is cyclic of order $p_1 \dots p_k$. By (8), $S_G = 1$ and by (3) of Definition 2.5, the minimal subgroup $\Omega(S)$ of S centralizes all but one of the P_i. Hence if $T := \Omega(S)^G$ is the group generated by all the subgroups of order q of G, then $|T| = p_j q$ for some $j \in \{1, \dots, k\}$. Now $B \cap C = 1$ implies that $B \cap T \neq C \cap T$ and hence $T \leq B \cup C = A$. But then $A \cap D \geq T \cap D \neq 1$, a final contradiction. $\square$

Remark 2.7. The author originally proved the equivalence of properties (a) and (c) of Theorem 2.6. He is grateful to A. Leone (Napoli) who pointed out to him that (c) is the characterization of finite groups with upper semimodular subgroup lattice given in [7]. Unfortunately, he was not able to find a simple direct proof for the equivalence of (a) and (b). The reason for this might be that the corresponding statement for lattices is wrong. The lattices N_5 or L_7, for example, are L_6-free but not upper semimodular. And the lattice in Figure 3 is upper semimodular but not L_6-free.

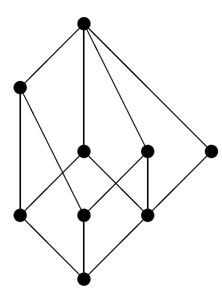

Figure 3

3. L_7-free groups

We first determine the $\{p,q\}$-groups with this property.

LEMMA 3.1. *Let p and q be different primes, $n \in \mathbb{N}$, and assume that $G = PQ$ where P is an elementary abelian normal subgroup of order p^n of G and $Q = \langle x \rangle$ is a cyclic q-group. Then the following properties are equivalent.*

(a) *G is L_7-free.*
(b) *Every subgroup Q_0 of Q either is irreducible on P or normalizes every subgroup of P.*
(c) *One of the following holds.*
  (i) *$G = P \times Q$ or G is a $\tilde{P}$-group, that is, x induces a (possibly trivial) power automorphism in P.*
  (ii) *$q \mid p-1$, $|P| = p^q$, and x induces an automorphism of order q^{k+1} (resp. at least $q^{k+1} = 4$ in case $q = 2$ and $k = 1$) in P where k is the largest integer such that $q^k \mid p-1$.*
  (iii) *$n \geq 2$, $q \nmid p^r - 1$ for $1 \leq r < n$, $q^m \mid p^n - 1$, and x induces an automorphism of order q^m in P ($m \in \mathbb{N}$).*

Proof. We use induction on $|G|$ to show that (a) implies (b). For this we may assume that $Q_0 = Q$ and that $Q = \langle x \rangle$ is not irreducible on P. By Maschke's theorem (see [3, p. 122]), P is completely reducible under Q, that is, $P = N_1 \times \cdots \times N_r$ with minimal normal subgroups N_i of G. If $r > 2$, then x induces power automorphisms in $N_1 \times N_i$ for all $i \in \{2,\ldots,r\}$ and therefore also in P. So, finally, suppose that $P = N_1 \times N_2$.

Let $1 \neq a \in N_1$ and M be a maximal subgroup of $\langle a \rangle \times N_2$ different from N_2 such that $a \notin M$. Then $P = N_1 M$ and $N_1 \cap M = 1$. Suppose, for a contradiction, that $M^x \neq M$ and let $L := (M \cup Q) \cap P$. Then $L^x = L$, hence $M < L$ and so $L = (N_1 \cap L)M$ with $N_1 \cap L \neq 1$. Since $(N_1 \cap L)^x = N_1 \cap L$ and N_1 is a minimal normal subgroup of G, it follows that $N_1 \leq L$ and so $M \cup Q = G$. Then $\{1, M, Q, N_1, N_1Q, P, G\}$ is a sublattice of $L(G)$ isomorphic to L_7. This contradiction shows that $M^x = M$. Since N_2 is a minimal normal subgroup of G, it follows that $N_2 \cap M = 1$ and hence $|N_2| = p$. Similarly, $|N_1| = p$. Now our argument shows that every diagonal M in the direct product $N_1 \times N_2$ is Q-invariant and so x induces a power automorphism in P.

We show next that (b) implies (a). If G satisfies (b), then every subgroup and factor group of G also satisfies (b) or is abelian. Thus if $\mathcal{L} \simeq L_7$ is a sublattice of $L(G)$, then by induction, its greatest element has to be G and hence the antiatoms A and B of $\mathcal{L}$ (see Figure 4 below) have to be contained in two different maximal subgroups of G. But if Q is irreducible on P, then the maximal subgroups of G are $P\phi(Q)$ and the conjugates of Q; since $L(Q)$ is a chain, it cannot contain a sublattice isomorphic to $K_2 \times K_2$. And if Q normalizes every subgroup of P, then G is abelian or a $\tilde{P}$-group

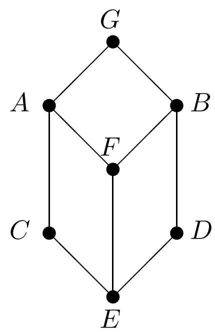

FIGURE 4

and by [1, Proposition 2.12], it is L_7-free. Since the proof there is rather long, note that a simple proof can be given similar to that of Theorem 2.6. Indeed, since $CD \neq DC$, the same argument as in the proof of (8) shows that $C_G = 1 = D_G$. Hence C and D are q-groups generating G, so one of them is a Sylow q-subgroup S of G and the other is contained in S^g for some $g \in P$. Then $G = C \cup D \leq \langle g \rangle S$ and hence S is a maximal subgroup of G, a contradiction.

We finally show that (b) and (c) are equivalent. If G satisfies (b) and x normalizes every subgroup of P, then (i) holds; conversely, the groups in (i) satisfy (b). So suppose that P is irreducible under Q, $|P| = p^n$ with $n \geq 2$, and x induces an automorphism of order q^m in P. Then it is well-known (see [3, p. 166]) that n is the smallest integer such that $q^m \mid p^n - 1$. If $x^{q^{m-1}}$ is irreducible on P, then $q \nmid p^r - 1$ for $1 \leq r < n$ and (iii) holds. So suppose that $x^{q^{m-1}}$ is not irreducible on P. Then by (b) there exists $k \in \{1, \ldots, m-1\}$ such that $x^{q^{m-k}}$ induces a power automorphism of order q^k in P and $x^{q^{m-k-1}}$ is irreducible on P. Then $q^k \mid p - 1$, but $q^{k+1} \nmid p^r - 1$ for $1 \leq r < n$; in particular, k is the largest integer such that $q^k \mid p - 1$. So $p = sq^k + 1$ where $q \nmid s$, hence $p^i \equiv 1 + isq^k \pmod{q^{k+1}}$ for $i \geq 1$, and

$$\frac{p^r - 1}{p - 1} = \sum_{i=0}^{r-1} p^i \equiv r + \frac{r(r-1)sq^k}{2} \pmod{q^{k+1}} \tag{10}$$

for $r \geq 2$. Since $q^{k+1} \mid p^n - 1$, it follows that $q \mid n$. On the other hand, since $q^{k+1} \mid p^q - 1$, the irreducible $GF(p)$-modules for a cyclic group of order q^{k+1} have order p^q and hence $n = q$. If $q > 2$, then (10) shows that $\frac{p^q-1}{p-1} \equiv q \pmod{q^{k+1}}$ and hence $q^{k+2} \nmid p^q - 1$; it follows that $m = k + 1$. The same is true if $q = 2$ and $k > 1$. Thus (ii) holds.

Conversely, if G satisfies (ii) or (iii) of (c), then P is completely reducible under Q. The arithmetical conditions in (iii) imply that the irreducible $GF(p)$-modules for a group of order q have order p^n. Thus P is irreducible under every nontrivial automorphism induced by elements of Q and (b) holds. In case (ii), as shown above, the irreducible $GF(p)$-modules for a cyclic group of order q^{k+1} have order p^q and hence P is irreducible under Q. Again it is well-known (see [3, p. 166]) that x operates on $P = GF(p^q)$ by multiplication with an element of order q^{k+1} (resp. q^m, $m \geq k+1$, if $q = 2$ and $k = 1$) of the multiplicative group of $GF(p^q)$. The q-th (resp. q^{m-1}-th) power of this element lies in $GF(p)$ and therefore fixes every subgroup of P. Again (b) holds. □

Note that groups with the properties (ii) or (iii) in (c) do exist. Since x operates irreducibly on P in these cases, these groups are L_7-free and not supersoluble. We shall show that there is just one further class of L_7 -free $\{p, q\}$-groups, namely the following.

Lemma 3.2. *Let p be a prime such that $p \equiv 3 \pmod 4$ and let $G = PQ$ be the semidirect product of an elementary abelian group P of order p^2 by a quaternion group Q of order 8 operating faithfully on P. Then G is L_7-free.*

Proof. By Lemma 3.1, PH is L_7-free for every subgroup H of order 4 of Q. So if $L(G)$ has a sublattice isomorphic to L_7, the greatest element of this lattice is G and we choose the notation as in Figure 4. If $C \leq P\phi(Q)$, then D must contain a Sylow 2-subgroup of G since $C \cup D = G$. But this is impossible since Q is a maximal subgroup of G. Hence $C \not\leq P\phi(Q)$ and therefore C contains a subgroup H^x of order 4 where $H \leq Q$ and $x \in P$. Since $[G/H] = \{H, Q, PH, G\}$, it follows that $C = H^x$ has order 4. Similarly, $|D| = 4$ and since C and D are contained in different Sylow 2-subgroups, $E = C \cap D = 1$. Since Q has only one minimal subgroup, it follows that $A \neq Q^x$ and so $A = (PH)^x = PC$. Similarly, $B = PD$ and $F = PC \cap PD = P\phi(Q)$. But then $F \cap C \neq E$, a contradiction. □

Lemma 3.3. *If G is L_7-free and $|G|$ is divisible by at most two different primes, then G is nilpotent or one of the groups occurring in Lemmas 3.1 or 3.2.*

Proof. We use induction on $|G|$ and may assume that G is not nilpotent; hence $|G| = p^\alpha q^\beta$, where p and q are different primes and $\alpha, \beta \in \mathbb{N}$. We have to show that G has a normal elementary abelian Sylow p-subgroup, say, and the Sylow q-subgroups of G are cyclic if G is not one of the groups in Lemma 3.2. Then by Lemma 3.1, G is one of the groups in (c) of that lemma.

First suppose, for a contradiction, that G has only nonnormal (nontrivial) Sylow subgroups. By Burnside's $\{p, q\}$-Theorem, G is soluble; let $M \trianglelefteq G$ be such that $|G : M| = q$, say. Then the Sylow p-subgroups of M are not normal in M and hence, by induction, $M = NP$ where N is a normal elementary abelian q-subgroup of M and P is a quaternion group or a cyclic p-group which either is irreducible on N or induces a nontrivial power automorphism in N. Let Q be a Sylow q-subgroup of G. Then Q/N is not normal in G/N

and hence, again by induction, M/N is elementary abelian and so has order p. The Frattini argument shows that $G = M \cdot N_G(P) = N \cdot N_G(P)$ and hence $Q = N(N_G(P) \cap Q)$. If $N_G(P) \cap N \neq 1$, then $N_G(P) \cap N$ and hence N would be centralized by P, a contradiction. Therefore $N_G(P) \cap N = 1$, so $|N_G(P) \cap Q| = q$ and Q is generated by elements of order q. By Lemma 2.1, $L(Q)$ is modular and hence by [6, Lemma 2.3.5], Q is elementary abelian. It follows that N is centralized by $Q^G = G$ and so $M = P \times N$. This contradiction shows that

(11) G has a normal Sylow p-subgroup P,

say. Again let Q be a Sylow q-subgroup of G and suppose, for a contradiction, that P is not elementary abelian. Then if H is a maximal subgroup of Q, the induction assumption implies that PH is nilpotent. Hence $H \trianglelefteq G$ and if $H \neq 1$, again by induction, G/H would be nilpotent, a contradiction since, by (11), Q is not normal in G. So $H = 1$, that is, $|Q| = q$. Let $N := \phi(P)$. Then $1 \neq N \trianglelefteq G$ and since G is not nilpotent, Q does not centralize P/N (see [3, p. 275]). Thus G/N is one of the groups in Lemma 3.1 and therefore $[P, Q] = P$. Let Q^x be a conjugate of Q such that $Q^x \neq Q$. If $Q \cup Q^x = G$, then $\{1, Q, Q^x, N, NQ, NQ^x, G\}$ would be a sublattice of $L(G)$ isomorphic to L_7. Hence $Q \cup Q^x < G$ and by induction, $Q \cup Q^x = MQ$ with elementary abelian normal p-subgroup M of $Q \cup Q^x$. So $Q \cup Q^x \leq \Omega(P)Q$ and it follows that $Q^G \leq \Omega(P)Q$. But then $[P, Q] \leq P \cap Q^G \leq \Omega(P)$ and $\Omega(P)$ is a proper subgroup of P since $L(P)$ is modular. This contradiction yields that

(12) P is elementary abelian.

It remains to be shown that Q is cyclic or G is one of the groups in Lemma 3.2. So suppose that Q is not cyclic and let H be a maximal subgroup of Q. Then by induction, PH is nilpotent or H operates nontrivially on P and therefore is cyclic or a quaternion group. In the first case, $H \leq C_G(P)$ and hence $H \cap K \trianglelefteq G$ if K is a maximal subgroup of Q different from H. Again by induction, $G/H \cap K$ would be nilpotent if $H \cap K \neq 1$. This is not the case, hence $H \cap K = 1$, that is,

(13) Q is elementary abelian of order q^2

in this case. If H is a quaternion group, Lemma 2.1 and [6, Theorem 2.3.8] show that $Q = H \times Z$ where $|Z| = 2$. Then Q contains a noncyclic abelian maximal subgroup of order 8, which cannot happen as we have just shown. Thus either (13) holds or every maximal subgroup of Q is cyclic and operates nontrivially on P. It is well-known (see [3, p. 311]) that then $|Q| = q^2$ or Q is a quaternion group. In the latter case, $Q/C_Q(P)$ is noncyclic and therefore at least one of the maximal subgroups of Q does not operate as a group of power automorphisms on P. Then Lemma 3.1 implies that this subgroup is irreducible on P and $|P| = p^2$. Hence $p \equiv 3 \pmod 4$ and G is one of the groups in Lemma 3.2.

So we finally have to show that (13) is impossible. But if (13) holds, then since $|C_Q(P)| \leq q$, there exist subgroups C and D of order q in G such that $PC \neq PD$ and $C \cup D$ is not a q-group. Then $P \cup C \cup D = G$ and hence $F := P \cap (C \cup D)$ is a Sylow p-subgroup of $C \cup D$ and $FC \neq C \cup D \neq FD$. It follows that $\{1, C, D, F, FC, FD, C \cup D\}$ is a sublattice of $L(G)$ isomorphic to L_7. $\square$

We show next that every finite L_7-free group is soluble. For this (and also later) we need the following simple property of (soluble) L_7-free groups.

Lemma 3.4. *Sylow subgroups for different primes of a finite soluble L_7-free group permute.*

Proof. Let G be such a group, p and q different primes and suppose that $P \in \operatorname{Syl} p(G)$ and $Q \in \operatorname{Syl} q(G)$. We use induction on $|G|$ to show that $PQ = QP$ and may assume that $P \cup Q = G$. Let N be a minimal normal subgroup of G. Since G is soluble, N is an r-group for some prime r. If $p \neq r \neq q$, then $\{1, P, Q, N, NP, NQ, G\}$ is a sublattice of $L(G)$ isomorphic to L_7. Thus $r = p$, say, so $N \leq P$ and, by induction, $(P/N)(QN/N) = (QN/N)(P/N)$. It follows that

$$PQ = PNQ = PQN = QNP = QP,$$

as desired. $\square$

Lemma 3.5. *Every finite L_7-free group is soluble.*

Proof. Let G be a minimal counterexample. Then, clearly, G is a non-abelian simple group. Suppose, for a contradiction, that G has a Sylow p-subgroup P which is not elementary abelian. Then $N_G(P) = PH$ with a complement H to P in $N_G(P)$ and for any prime q dividing $|H|$, every q-element $x \in H$ generates together with P a $\{p, q\}$-group which is nilpotent, by Lemma 3.3. Thus $x \in C_G(P)$ and hence $N_G(P) = P \times H$. By Lemma 2.1, $L(P)$ is modular and hence by [6, Theorem 2.3.1], $\exp P' < \exp P =: p^n$. So $P \cap N_G(P)' = P' \leq \Omega_{n-1}(P)$ and $P \cap (P')^g \leq \Omega_{n-1}(P)$ for all $g \in G$; by Grün's First Theorem (see [3, p. 423]), it follows that $P \cap G' \leq \Omega_{n-1}(P) < P$. But this implies that $G' < G$, a contradiction. Thus every Sylow subgroup of G is elementary abelian.

By Burnside's criterion (see [3, p. 419]), $N_G(S) \neq C_G(S)$ for every Sylow subgroup S of G. Let q be the smallest prime dividing $|G|$ for which there exists a Sylow subgroup S of G such that q divides $|N_G(S)/C_G(S)|$; let Q be a Sylow q-subgroup of $N_G(S)$ and let S be a p-group, $p \in \mathbb{P}$. By Lemma 3.3, Q is cyclic (or quaternion) and hence $|Q| = q$. Therefore if Q were a Sylow q-subgroup of G, the choice of q would imply that $N_G(Q) = C_G(Q)$, a contradiction. So if T is a Sylow q-subgroup of G containing Q and K is a complement to Q in T, then $K \neq 1$. If $S \cup K = G$, then $\{1, S, K, Q, SQ, T, G\}$

would be a sublattice of $L(G)$ isomorphic to L_7. Thus $S \cup K < G$ and hence $S \cup K$ is soluble. Since S and K are normalized by Q, $(S \cup K)Q = S \cup T$ is also soluble. By Lemma 3.4, it follows that $ST = TS$ is an L_7-free $\{p, q\}$-group, but neither S nor T is normal in ST. This contradicts Lemma 3.3. □

We can now prove our main result.

THEOREM 3.6. *A finite group is L_7-free if and only if it is a direct product of modular p-groups and groups occurring in Lemmas 3.1 or 3.2 with pairwise relatively prime orders.*

Proof. If G is such a direct product, then by Lemmas 2.1, 3.1, and 3.2, the subgroup lattices of the direct factors are L_7-free. Again Lemma 2.2 and [6, Lemma 1.6.4] yield that the same holds for $L(G)$.

Conversely, suppose that G is L_7-free and consider a minimal counterexample to the assertion of the theorem. Then by [6, Theorem 1.6.5], $L(G)$ is directly indecomposable and Lemma 3.3 together with Lemma 2.1 implies that $|G|$ is divisible by at least three different primes. By Lemma 3.5, G is soluble. So if N is a normal subgroup of prime index t, say, in G, the minimality of G implies that N has at least two different normal Sylow subgroups, or at least one if N is a Hall t'-subgroup of G. In any case, one of these Sylow subgroups is a normal Sylow p-subgroup P of G. Since P is not a direct factor of G, there exist a prime $q \neq p$ and a Sylow q-subgroup Q of G that does not centralize P. By Lemma 3.3, P is elementary abelian and Q is cyclic or a quaternion group. Again PQ is not a direct factor of G and hence there exist a prime r such that $p \neq r \neq q$ and a Sylow r-subgroup R of G which does not centralize PQ. By Lemma 3.4, PQR is a subgroup of G and, of course, a counterexample to the theorem; hence $PQR = G$.

Since Q does not centralize P, there exists $x \in P$ such that $Q^x \neq Q$. Again by Lemma 3.4, $H := QR$ and Q^xR both are Hall $\{q, r\}$-subgroups of G and hence there exists $y \in P$ such that $H^y = Q^xR$ (see [3, p. 662]). Then $R \leq H \cap H^y = C_H(y)$ (see [6, Lemma 4.1.1]) and $y \neq 1$ since $|Q \cup Q^x|$ is divisible by p. It follows from 3.3 and 3.1 that PR is nilpotent. In particular, $R \trianglelefteq PR \leq C_G(P) \trianglelefteq G$ and $C_G(P) < G$. If $PR = C_G(P)$, then $R \trianglelefteq G$. If $PR < C_G(P)$, then $|Q| \geq q^2$ and since Q is cyclic or quaternion, Lemma 3.3 implies that $R \trianglelefteq QR$.

So in both cases, $R \trianglelefteq G$, hence $PR = P \times R$ and Q operates nontrivially on P and on R. If Q is a quaternion group, then Q is faithful on P and R; if Q is cyclic, then $C_Q(P) \leq C_Q(R)$, say, and the minimality of G implies that $C_Q(P) = 1$. Thus in any case, $C_Q(P) = 1$ and there exists $x \in P$ such that $Q \cap Q^x = 1$. Let $y \in R$ such that $[y, Q] \neq 1$ and consider $G_0 := Q \cup Q^{xy}$. By [6, Lemma 4.1.1], $G_0 = [xy, Q]Q$ and so p and r divide $|G_0|$. Hence $R_0 := R \cap G_0 \neq 1$ and since

$$Q \cap Q^{xy} \leq R_0Q \cap R_0Q^{xy} \leq RQ \cap RQ^x = R,$$

it follows that $\{1, Q, Q^{xy}, R_0, R_0Q, R_0Q^{xy}, G_0\}$ is a sublattice of $L(G)$ isomorphic to L_7, a final contradiction. $\square$

References

[1] C. Baginski and A. Sakowicz, *Finite groups with globally permutable lattice of subgroups*, Colloq. Math. **82** (1999), 65–77.

[2] R. Freese, J. Ježek, and J.B. Nation, *Free lattices*, Math. Surveys and Monographs, vol. 42, Amer. Math. Soc., Providence, RI, 1995.

[3] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.

[4] J. Krempa and B. Terlikowska-Osłowska, *On uniform dimension of lattices*, Contributions to General Algebra, vol. 9 (Linz, 1994), Hölder-Pichler-Tempsky, Wien, 1995, pp. 219–230.

[5] R. McKenzie, *Equational bases and nonmodular lattice varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.

[6] R. Schmidt, *Subgroup lattices of groups*, De Gruyter, Berlin, 1994.

[7] M. Suzuki, *Structure of a group and the structure of its lattice of subgroups*, Springer, Berlin-Heidelberg-New York, 1956.

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