Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 485–513
S 0019-2082

BAER SUBPLANES

HELMUT SALZMANN

ABSTRACT. The real plane is a Baer subplane of the complex projective plane in the following sense: each complex line contains a real point and, dually, each complex point is on a real line. In the first part of this survey main aspects of the general theory will be reviewed. The second part is concerned with finite planes. Part 3 deals rather completely with Baer subplanes of compact, connected topological projective planes.

Introduction

In the year 1946 Reinhold Baer [3] published a paper *Projectivities with fixed points on every line of the plane* in the Bulletin of the AMS. (Today one prefers to speak of *collineations* or *automorphisms* of a projective plane $\mathcal{P}$ rather than of projectivities.) A collineation $\beta \neq \mathbb{1}$ with the property studied by Baer (and, in particular, each involution) either fixes all points of some line (which is then called the axis of β), or there is a quadrangle (four points in general position) of fixed points and the fixed elements of β form a proper subplane $\mathcal{F}_\beta$ of $\mathcal{P}$. In the latter case, β is said to be a *Baer collineation*. According to Theorem 1 in Baer's paper, β is then also a Baer collineation of the dual plane. This motivates the following definition:

Definition. A proper subplane $\mathcal{B}$ of a (not necessarily Desarguesian) projective plane $\mathcal{P}$ is called a *Baer subplane* if each line of $\mathcal{P}$ contains a point in $\mathcal{B}$ and, dually, each point of $\mathcal{P}$ is incident with a line in $\mathcal{B}$.

A familiar example is the real plane considered as a subplane of the complex projective plane $\mathcal{P}_\mathbb{C}$. It is the fixed plane of the involution ι of $\mathcal{P}_\mathbb{C}$ induced by complex conjugation of the coordinates. The complex plane is a compact, connected topological projective plane, and continuity of ι implies that the real plane is closed in $\mathcal{P}_\mathbb{C}$. Note, however, that a Baer subplane does not necessarily consist of the fixed elements of a collineation; cf. 3.10.

Received June 1, 2002.
2000 *Mathematics Subject Classification.* Primary 51-02. Secondary 51A35, 51E15, 51H10.

©2003 University of Illinois

A closed subplane $\mathcal{B}$ of an arbitrary compact projective plane $\mathcal{P}$ of positive topological dimension 2ℓ is a Baer subplane of $\mathcal{P}$ if, and only if, the point space of $\mathcal{B}$ is ℓ-dimensional. Such subplanes play an important rôle in the study of compact, connected projective planes.

In the first part of this survey I will review main aspects of the general theory. The second part is concerned with finite planes. Finally, I will deal more completely with the topological situation.

Personal Remark. Reinhold Baer came to Frankfurt University on July 4, 1956, and he stayed there to the end of Summer 1967. From the very first day until he left, I had the great privilege to belong to the ever-growing group of young mathematicians whose mentor he became. In the fifties, Baer had become more absorbed by group-theoretical questions than by geometry; nevertheless, he remained strongly interested in my problems concerning compact projective planes, and constantly encouraged me to continue my work. During those 11 years he guided me in a fatherly way, mathematically and otherwise. Whatever I may have accomplished later at the University of Tübingen, I could hardly have succeeded without what I have learned from him. As homage to Reinhold Baer, I will describe here what has grown out of one of his ideas.

1. Baer subplanes of general projective planes

In this section, $\mathcal{P} = (P, \mathfrak{L})$ will always denote a projective plane with point set P and line set $\mathfrak{L}$. Recall that in a projective plane any two distinct points are joined by a unique line and any two distinct lines intersect in a unique point. Though it somewhat obscures the notion of duality, lines are usually considered as sets of points. The duals of lines are then *pencils* $\mathfrak{L}_p = \{L \in \mathfrak{L} \mid p \in L\}$. See [68, Chap. 2] for a short summary of projective planes or [36] for a fuller account; [22, Chap. 3] gives special emphasis to finite planes. The incidence relation, i.e., the set of *flags* will be denoted by $F = \{(p, L) \in P{\times}\mathfrak{L} \mid p \in L\}$. The projective plane $\mathcal{P}_F$ over a field F has as points the 1-dimensional subspaces of the vector space F^3, and as lines the 2-dimensional subspaces or the kernels of linear forms.

1.1. Subplanes. A *projective plane* $\mathcal{B} = (B, \mathfrak{B})$ is a *subplane* of $\mathcal{P}$, in symbols $\mathcal{B} \leq \mathcal{P}$, if $B \subseteq P$ and $\mathfrak{B} = \{\, L \cap B \;\big|\; L \in \mathfrak{L},\, 1 < |L \cap B| \}$. With respect to $\mathcal{B}$, a line $L \in \mathfrak{L}$ is said to be an *interior line* if $L \cap B \in \mathfrak{B}$; otherwise L is called an *exterior line*. Analogously, the points in B are called *interior points*, and the others *exterior points*.

A *proper* subplane $\mathcal{B}$ of $\mathcal{P}$ is a *Baer subplane*, $\mathcal{B} \lessdot \mathcal{P}$, if $L \cap B \neq \emptyset$ for each line $L \in \mathfrak{L}$ and, dually, each pencil $\mathfrak{L}_p$ contains an interior line; cf. also [51].

The definition implies immediately:

1.2. Lemma. *If L is an interior line with respect to $\mathcal{B} \mathrel{<\!\!\!\bullet} \mathcal{P}$ and if $p \in L \setminus B$, then $\lambda_p = (x \mapsto px) : B \setminus L \to \mathfrak{L}_p \setminus \{L\}$ is a bijection.*

1.3. Examples. If $K = F(\delta)$ is a quadratic extension of the field F, then $\mathcal{P}_F \mathrel{<\!\!\!\bullet} \mathcal{P}_K$. In fact, an equation $\sum_{\nu=1}^{3} (a_\nu + \delta b_\nu)\, x_\nu = 0$ with $a_\nu, b_\nu \in F$ has always a non-trivial solution $(x_\nu)_\nu \in F^3$. Hence any line of the plane $\mathcal{P}_K$ contains a point of $\mathcal{P}_F$ and dually.

1.4. Maximality. *Any Baer subplane $\mathcal{B}$ of $\mathcal{P}$ is a maximal subplane of $\mathcal{P}$.*

Proof. If $\mathcal{B} < \mathcal{S} \leq \mathcal{P}$, then $\mathcal{S}$ contains a full pencil of $\mathcal{P}$ by Lemma 1.2, and hence $\mathcal{S} = \mathcal{P}$. □

Remark. The converse is not true: if K is a cubic extension of the (finite) field F, then F is maximal in K and $\mathcal{P}_F$ is maximal in $\mathcal{P}_K$, but $\mathcal{P}_F$ is not a Baer subplane; see 2.1.

The point set of a Baer subplane is a special example of a more general notion; cf. 2.2–2.4:

1.5. Definition: Blocking sets. Let $\mathcal{P} = (P, \mathfrak{L})$ be a projective plane. A subset $B \subseteq P$ is called a *blocking set* if, and only if, $\emptyset \neq B \cap L \neq L$ for each line $L \in \mathfrak{L}$. Obviously, the complement of a blocking set is also a blocking set.

Assume now that $\mathcal{B} = (B, \mathfrak{B})$ is a sub*plane* of $\mathcal{P}$. Then $\mathcal{B} \mathrel{<\!\!\!\bullet} \mathcal{P}$ if, and only if,

(i) B is a blocking set in $\mathcal{P}$, and
(ii) $\mathfrak{B}$ is a blocking set in the dual of $\mathcal{P}$.

1.6. Proposition. *In projective planes in general, even in Desarguesian planes, conditions* (i) *and* (ii) *of* 1.5 *are independent.*

(They are equivalent, however, in finite planes and in the topological case; see 2.1, 2.2 and 3.9.)

Proof. According to [21, Theorems 5.9.1 and 5.9.2], there exist skew field extensions $K|F$ of any given characteristic such that the dimension of K as a *right* vector space over F is $[K{:}F]_r = 2$, and as a *left* F-vector space K has dimension $[K{:}F]_l = s$, where s is a prescribed number in the range $3 \leq s \leq \infty$. Consider the subplane $\mathcal{P}_F = (B, \mathfrak{B})$ of $\mathcal{P}_K$. As in Ex. 1.3 it follows that B is a blocking set in $\mathcal{P}_K$. If a point p has coordinates $x_\nu \in K$ which are left linearly independent over F, then p is on no interior line of $\mathcal{P}_F$, and $\mathfrak{B}$ is not a blocking set in the dual of $\mathcal{P}_K$. □

Baer subplanes play a crucial rôle in the so-called *derivation* process, one of the most fruitful methods for constructing new projective planes, in particular in the finite case. For finite planes this method is due to Ostrom [53], and in the infinite case it is due to Johnson [38].

1.7. Derivation. Let $\mathcal{A} = \mathcal{P}^W = (A{=}P\setminus W,\ \mathfrak{A}{=}\mathfrak{L}\setminus\{W\})$ be an affine subplane of the projective plane $\mathcal{P}$. A non-empty set $D \subset W$ is called a *derivation set* if for each pair of distinct points $a, b \in A$ such that $ab \cap W \in D$ there is a Baer subplane $\langle a, b, D\rangle \mathrel{<\!\!\bullet} \mathcal{P}$ intersecting W exactly in D. In this case, all these Baer subplanes and all lines in $\mathfrak{A}$ which do not meet the set D form the line set of a new ('derived') affine plane $D\mathcal{A} = (A,\ D\mathfrak{A})$.

Proof. $D\mathcal{A}$ is indeed an affine plane: by construction and the maximality of Baer subplanes, any two distinct points of A are on a unique line in $D\mathfrak{A}$. Any ordinary line intersects each Baer subplane $\langle a, b, D\rangle$ in an affine point. It remains to show that the parallel axiom holds for the new lines. Suppose that $\mathcal{B} = \langle a, b, D\rangle$ and $\mathcal{Q} = \langle p, q, D\rangle$ have no affine point in common. Then the interior line of $\mathcal{B}$ through p contains a point $z \in D$ and hence is also an interior line of $\mathcal{Q}$. Moreover, all common interior lines pass through the same point z. If there is an affine point x, however, which belongs both to $\mathcal{B}$ and to $\mathcal{Q}$, then all lines dx with $d \in D$ are common interior lines of $\mathcal{B}$ and $\mathcal{Q}$. It follows that there exists a unique parallel $\langle p, q, D\rangle$ to $\mathcal{B}$. □

Remarks. (1) Obviously, each automorphism of $\mathcal{A}$ which maps D onto itself yields an automorphism of $D\mathcal{A}$. In particular, $D\mathcal{A}$ is a translation plane if $\mathcal{A}$ has a transitive group of translations with axis W.

(2) Coordinatize the affine plane $\mathcal{A}$ by K and identify W with $K \cup \{\infty\}$ in such a way that the common point at infinity of all lines of slope s corresponds to the element $s \in K \cup \{\infty\}$. If K is a quadratic extension of the field F, then $D = F \cup \{\infty\}$ is a derivation set.

(3) In the situation (2), the derived plane $D\mathcal{A}$ is not Desarguesian: each homology α of $\mathcal{A}$ with center a induces an automorphism of $D\mathcal{A}$, but $\langle a, b, D\rangle$ need not be α-invariant, and the homology group of $D\mathcal{A}$ is not transitive.

(4) In the topological case, a sequence of points $z_\nu \in W\setminus D$ may converge to a point $d \in D$, but the lines az_ν do not converge to a Baer subplane. Therefore, derivation of a topological plane cannot be expected to yield again a topological plane.

(5) Algebraic aspects of the derivation process are described in [29, VII.5].

1.8. Theorem. *If D is a derivation set as in* 1.7*, then all the Baer subplanes $\langle a, b, D\rangle$ are Desarguesian planes.*

This has been proved in the finite case in [59]; the proof in the general case (by representing the affine part of a Baer plane $\langle a, b, D\rangle$ as a plane in a 3-dimensional affine space) is due to Cofman [20]; see also [49, Th. 51.1]. □

1.9. Repeated derivation. The line at infinity of the derived plane $D\mathcal{A}$ consists of the points in $W\setminus D$ and the set $\widehat{D}$ of the parallel classes of new lines. By the proof of 1.7, each original line of $\mathcal{A}$ whose point at infinity belongs to D is the point set of an affine Baer subplane of $D\mathcal{A}$. Hence $\widehat{D}$ is a derivation set in $D\mathcal{A}$, and $\widehat{D}D\mathcal{A} = \mathcal{A}$. (For the last result it is essential that $W \setminus \widehat{D} = W \setminus D$.)

More results related to derivation will be discussed in the next section. The derivation process has been thoroughly investigated by Johnson in his book *Subplane covered nets* [40].

The existence of many Baer subplanes forces the full plane to be nice. In fact, the following has been proved in [19, Th. B]:

1.10. Theorem. *The plane $\mathcal{P}$ is a Moufang plane* (see 1.12)*, provided there exists a family* $\mathbf{B}$ *of Baer subplanes such that:*

(a) *Any quadrangle of $\mathcal{P}$ is contained in a unique subplane $\mathcal{B} \in \mathbf{B}$.*
(b) *Whenever two subplanes $\mathcal{B}$ and $\mathcal{B}'$ in* $\mathbf{B}$ *have* 3 *distinct points of a line L of $\mathcal{P}$ in common, then $B \cap L = B' \cap L$.*

Coordinates from a so-called *ternary field* can be introduced in any affine plane. This has first been done by M. Hall in 1943. As algebraic structures, general ternary fields are hardly manageable, however. More useful are some special types of ternary fields, most notably those defined next.

1.11. Quasi-fields. A structure $(K, +, \cdot, 0, 1)$ is called a quasi-field if, and only if:

(a) $(K, +)$ is a commutative group with neutral element 0.
(b) $(a, b) \mapsto a \cdot b$ is a *multiplication* on K and $1 \cdot x = x \cdot 1 = x$ for each $x \in K \setminus \{0\}$.
(c) Each map $x \mapsto x \cdot c$ with $c \neq 0$ is a bijection of K and $0 \cdot c = 0$.
(d) The *left* distributive law $c \cdot (a + b) = c \cdot a + c \cdot b$ holds.
(e) Each map $x \mapsto a \cdot x - b \cdot x$ with $a \neq b$ is a bijection of K.

A quasi-field K which also satisfies the right distributive law is said to be a *semi-field* or a division algebra; if multiplication in K is associative, then K is a (planar) *near-field*. Both additional properties together obviously characterize (skew) fields.

The relevance of quasi-fields is explained by the following definition:

1.12. Translation planes. An affine plane $\mathcal{A}$ is a translation plane, if it admits a sharply point-transitive group of collineations which map each line onto a parallel (*translations*). Equivalently, $\mathcal{A}$ can be coordinatized by a quasi-field in such a way that the point set of $\mathcal{A}$ is $K \times K$ and the lines are given by equations $y = s \cdot x + t$ or $x = c$. The translations are then of the form $(x, y) \mapsto (x+a, y+b)$, and the properties of a quasi-field just reflect the fact that this construction actually yields an affine translation plane. If, and only if, K is a semi-field, the plane $\mathcal{A}$ is also a dual translation plane (with translation center the infinite point on the vertical lines). A *Moufang plane* is a projective plane such that each of its affine subplanes is a translation plane.

Note. Some authors write the equation of a line of *slope* s in the form $y = x \cdot s + t$ and consequently interchange all factors in the definition of a quasi-field.

There exists a vast variety of quasi-fields and translation planes, even of finite or locally compact, connected ones; cf. [49] or [44]. With regard to Baer subplanes, near-fields play a special rôle. In contrast to quasi-fields in general, the class of near-fields is much more restricted; see [30, XI.6] for a short review, examples, and more references.

1.13. Kernel. The *kernel* K of a quasi-field Q consists of all elements $c \in Q$ such that

$$(x + y) \cdot c = x \cdot c + y \cdot c \text{ and } (x \cdot y) \cdot c = x \cdot (y \cdot c) \text{ for all } x, y \in Q\ .$$

Obviously, K is a field or a skew field, and Q may be considered as a right vector space over K.

Each *near-field* N with kernel K having (left and right) dimension 2 over K yields a projective plane $\mathcal{H}$ with a Desarguesian Baer subplane $\mathcal{D} \cong \mathcal{P}_K$ such that any translation of $\mathcal{D}$ extends to a translation of $\mathcal{H}$. The finite planes of this kind have been found in 1957 by Hughes [35]; the general construction is due to Dembowski [23] and (in corrected form) Biliotti [8]. A plane $\mathcal{H}$ having a Baer subplane with the property just mentioned will be called a *Hughes plane*. Write $\dot{N} = (N \setminus \{0\}, \cdot\,)$ for the multiplicative group of the near-field $(N, +, \cdot\,)$. The additive group N^3 has nearly the properties of a vector space. This motivates the following construction:

1.14. Hughes planes. Let N be a near-field of dimension 2 over its kernel K. The point set of $\mathcal{H} = \mathcal{H}_N$ consists of the classes $\langle \mathfrak{x} \rangle = \dot{N} \cdot \mathfrak{x}$, where $0 \neq \mathfrak{x} = (x, y, z) \in N^3$, and the lines in the pencil through $\langle (0, 1, -1) \rangle$ are given by an equation $x \cdot s + y + z = 0$ or $x = 0$. All other lines are obtained by applying the elements of the group $\Gamma = \mathrm{GL}_3 K$ to the vectors $\mathfrak{x} \in N^3$ from the right. The point set of the subplane $\mathcal{D}$ is $D = \{\langle \mathfrak{x} \rangle \mid 0 \neq \mathfrak{x} \in K^3\}$. (Planes constructed in this way are also known as 'Generalized Hughes planes'.)

A full proof that this construction in fact leads to a Hughes plane can be found in [23] and [8] (interchange left and right!). Here, only some typical arguments will be given:

(1) Because multiplication in N is associative, Γ maps points to points, and $D^\Gamma = D$.

(2) A line L_s with an equation $x \cdot s + y + z = 0$ intersects D in more than one point (i.e., L_s is an interior line) if, and only if, $s \in K$. Therefore, $L_s \cap D$ is a point or an ordinary line of $\mathcal{P}_K$, and $\mathcal{D} \cong \mathcal{P}_K$. Moreover, Γ induces on $\mathcal{D}$ the full projective linear group.

(3) Any interior point is joined to each other point by a unique line, since $\langle 0,1,-1\rangle$ and $\langle 1,u,v\rangle$ are on a unique line L_s.

(4) If $h \in N \setminus K$, then $K + h \cdot K = N$ because N has dimension 2. Consequently, the orbit $\langle 0,1,h\rangle^\Gamma$ consists of all exterior points, i.e., the group Γ is transitive on the set of all exterior points. Hence each point is incident with an interior line.

(5) From (2) it follows that each line intersects the set D of interior points, and $\mathcal{D}$ is a Baer subplane of $\mathcal{H}$.

(6) The matrices $\begin{pmatrix} c & a & b \\ & 1 & \\ & & 1 \end{pmatrix}$ with $c \neq 0$ induce collineations of $\mathcal{H}$ with axis L_∞ given by $x = 0$. Therefore, each axial collineation of $\mathcal{D}$ is induced by an axial collineation of $\mathcal{H}$. Moreover, Γ acts even transitively on the set of all exterior flags.

(7) It remains to be shown that $\mathcal{H}$ is indeed a projective plane, i.e., that any two distinct lines intersect in a unique point and that any two exterior points are joined by some line. This follows by elementary but more involved calculations; see [23, 3.11] and [8]. □

1.15. Remark. With respect to any interior line L, a Hughes plane $\mathcal{H}_N$ is a semi-translation plane in the sense of [22, p. 136]. The properties of the translation group T with axis L show immediately that each orbit $a^\mathsf{T} \neq a$ is the point set of an affine Baer subplane.

1.16. Transitivity of Hughes groups. *Assume that each translation (each axial collineation) of the Desarguesian Baer subplane $\mathcal{B} <\!\!\bullet\, \mathcal{P}$ is induced by a translation (an axial collineation) of $\mathcal{P}$. Then the global stabilizer* $\mathsf{B} = \{\beta \in \operatorname{Aut}\mathcal{P} \mid \mathcal{B}^\beta = \mathcal{B}\}$ *of $\mathcal{B}$ is transitive on the set of all exterior flags (of pairs of distinct exterior points on an exterior line).*

Both assertions are immediate consequences of the assumptions. Sharper results can be obtained in the finite and the topological case; cf. 2.16 and the proof of 3.17 and 3.19.

The example of Remark (2) in 1.7 can be vastly generalized [28]:

1.17. Derivation sets in translation planes. *Suppose that the quasifield Q is a 2-dimensional left vector space over its subfield K. If for each $q \in Q \setminus K$ and $c \in K \setminus \{0\}$ the injective map $(a,b) \mapsto q(ac) + bc : K \times K \to Q$ is bijective, then the set of all points at infinity on lines of slope $k \in K \cup \{\infty\}$ is a derivation set for the translation plane over Q.*

1.18. Corollary ([28, Satz 3]). *A proper Moufang plane is not derivable.*

1.19. Baer groups. For a given Baer subplane $\mathcal{B} <\!\!\bullet\, \mathcal{P}$, the group $\Gamma_\mathcal{B}$ consisting of all collineations of $\mathcal{P}$ which fix each element of $\mathcal{B}$ is called the *Baer group* of $\mathcal{B}$. The group $\Gamma_\mathcal{B}$ is said to be transitive, if it acts transitively on the set of exterior points of one (and then of every) *interior* line. Planes with a transitive Baer group have also been called *tangentially transitive*; cf. [60]. Finite examples of transitive Baer groups are given in 2.8 ff. Biliotti and Johnson [9] studied Baer groups of finite planes which leave a second Baer subplane invariant. In the topological situation, Baer groups of closed Baer subplanes are rather small; see 3.12 and 3.14. They are never transitive.

1.20. Definition. A translation plane with a transitive Baer group $\Gamma_{\mathcal{B}}$ is a *generalized Hall plane* if the translation axis is an interior line of $\mathcal{B}$.

The name is motivated by results in 2.8–2.12. A detailed treatment can be found in [43].

2. Baer subplanes of finite projective planes

All projective planes in this section will be assumed to be finite. If one and hence every line of $\mathcal{P}$ carries $n{+}1$ points, then $\mathcal{P}$ is said to be a plane of *order* n. Such a plane has exactly $n^2{+}n{+}1$ points and equally many lines.

2.1. Baer subplanes. *Assume that $\mathcal{B} \le \mathcal{P}$ and that $\mathcal{B}$ is a plane of order m. Then $\mathcal{B} \lessdot \mathcal{P}$ if, and only if, $\mathcal{P}$ has order m^2.*

Proof. If $\mathcal{B}$ is a Baer subplane, then Lemma 1.2 shows immediately that $n = m^2$. Conversely, λ_p (as defined in 1.2) is always injective, and $n = m^2$ implies that λ_p is also surjective. Thus, $L \cap B \neq \emptyset$ whenever L contains a point p on an interior line. Suppose now that q is not on an interior line. Then the map $(x \mapsto qx) : B \to \mathfrak{L}_q$ is again injective, but $\operatorname{card} B > \operatorname{card} \mathfrak{L}_q$. □

2.2. Theorem. *Let $\mathcal{P}$ be a projective plane of order m^2. Each blocking set B in $\mathcal{P}$ which consists of exactly $m^2 + m + 1$ elements is the point set of a Baer subplane of $\mathcal{P}$.*

Proof. By the definition of a blocking set, the map $\lambda_p = (x \mapsto px) : B \to \mathfrak{L}_p$ is surjective for each point $p \notin B$. Because $|\mathfrak{L}_p| = m^2 + 1$, it follows that each line L intersects B in at most $m + 1$ points. Choose a line L which intersects B in a maximal number of points, and assume that $|L \cap B| = k \le m$. This will lead to a contradiction by counting suitable interior flags in two different ways as in [15]. Consider the set of flags $\gimel = (B \times \mathfrak{X}) \cap \digamma$, where $\mathfrak{X} = \{\, X \in \mathfrak{L} \smallsetminus \{L\} \mid X \cap L \notin B \wedge |X \cap B| > 1 \,\}$ contains all interior lines intersecting L in a point outside B, and $\digamma$ is the incidence relation in $\mathcal{P}$. For $p \in L \smallsetminus B$, the restriction of $\gimel$ to the line pencil $\mathfrak{L}_p$ shall be denoted by $\gimel_p$. Then $|\gimel_p| \le 2(m{+}1{-}k)$ because $|\mathfrak{X} \cap \mathfrak{L}_p| \le m{+}1{-}k$ and the number $|\gimel_p|$ is largest if $|X \cap B| = 2$ for each $X \in \mathfrak{X} \cap \mathfrak{L}_p$. Thus one has the upper bound

$$|\gimel| = \textstyle\sum_{p \in L \smallsetminus B} |\gimel_p| \le 2\,(m + 1 - k)\,(m^2 + 1 - k).$$

To obtain a lower bound, let $a \in B \smallsetminus L$. Since at most $k(k{-}1)$ of the $m(m{+}1)$ points of $B \smallsetminus \{a\}$ are on lines in $\mathfrak{L}_a \smallsetminus \mathfrak{X}$, it follows that $|\mathfrak{X} \cap \mathfrak{L}_a|\,(k - 1) \ge$

$m(m+1)-k(k-1)=(m+1-k)(m+k)$. Summing up over all $a \in B \setminus L$, one gets

$$|\mathfrak{I}|\,(k-1) \geq (m+1-k)\,(m+k)\,(m^2+m+1-k).$$

Hence $(m+k)\,(m^2+m+1-k) < 2k\,(m^2+1-k)$. This contradiction shows that $k=m+1$. Consequently, λ_p is bijective on $B \setminus L$ and $\mathfrak{X}=\emptyset$, so that each line meets B only in one point or in exactly $m+1$ points, and B is indeed the point set of a Baer subplane. □

2.3. *A blocking set in a projective plane of order m^2 contains a least m^2+m+1 points.*

Proof. Suppose that $m^2+m+1-|B|=t>0$. As before, λ_p is surjective and $|L \cap B| \leq m+1-t$ for each line L. Hence, a set of t collinear points may be added to B to obtain a blocking set which is not the point set of a Baer subplane. This contradicts Theorem 2.2. □

More generally, the following is proved in [16], [17], and [18]:

2.4. Theorem: Blocking sets. *Each blocking set B in a projective plane $\mathcal{P}$ of order n satisfies $n+\sqrt{n}+1 \leq b := |B| \leq n^2-\sqrt{n}$.*

Proof. The upper bound follows from the first inequality by taking complements. In the case $n=2$, no triangle is a blocking set and hence no blocking set exists. Suppose now that $n>2$ and that $b<n+\sqrt{n}+1$. Then $|L \cap B| < \sqrt{n}+1$ for each line $L \in \mathfrak{L}$ (because $|\mathfrak{L}_p|=n+1$). Put $k_\nu = \big|\{\, L \in \mathfrak{L} \,\big|\, |L \cap B| = \nu \,\}\big|$. Since B is a blocking set,

$$\sum_\nu k_\nu = |\mathfrak{L}| = n^2+n+1, \quad \sum_\nu \nu k_\nu = |B \times \mathfrak{L} \cap F| = b\,(n+1),$$

and

$$\sum_\nu \nu(\nu-1)k_\nu = \big|\{(a,b,L) \mid a,b \in B \cap L,\ a \neq b\,\}\big| = b\,(b-1).$$

The assumption implies $k_\nu = 0$ for $\nu \geq \sqrt{n}+1$, and hence

$$h := \sum_\nu (\nu-1)(\nu-\sqrt{n}-1)k_\nu \leq 0.$$

An easy calculation shows that

$$\begin{aligned} h &= b\,(b-1) - b\,(n+1)(\sqrt{n}+1) + (n^2+n+1)(\sqrt{n}+1) \\ &= (n+\sqrt{n}+1-b)\,(n\sqrt{n}+1-b) > 0, \end{aligned}$$

a contradiction. □

More on blocking sets can be found in the survey [11] and in [75]. Homogeneous blocking sets are studied in [10].

We return to Baer subplanes. The next result should be compared with 3.24:

2.5. Disjoint Baer subplanes. *If q is any prime power, the Pappian plane $\mathcal{P}$ over the field $F = \mathbb{F}_{q^2}$ is a* disjoint *union of q^2-q+1 Baer subplanes.*

Proof. Identify the vector space F^3 with $\mathbb{F}_{q^6}$, recall that the multiplicative group $\mathbb{F}_{q^6}^{\times} = \langle c \rangle$ is cyclic, and write $P = \{xF \mid x \in \langle c \rangle\}$. Then $x \mapsto xc$ induces on P a collineation γ, and $\langle \gamma \rangle$ is a sharply transitive collineation group of $\mathcal{P}$ of order q^4+q^2+1, called a Singer cycle. The set $B = \{xF \mid x \in \mathbb{F}_{q^3}^{\times}\}$ is the point set of a Baer subplane coordinatized by $\mathbb{F}_q$. The cyclic group $\mathbb{F}_{q^3}^{\times}$ is generated by $b = c^{q^3+1}$, and B is an orbit of $\langle \gamma^{q^2-q+1} \rangle$. Because γ is a collineation, the sets $B\gamma^{\nu}$ with $0 \leq \nu < q^2-q+1$ are pairwise disjoint point sets of Baer subplanes; their union is P. See also [76]. □

2.6. *The order of a finite derivable plane is a square prime power.*

Proof. This is an immediate consequence of 1.8 and 2.1. □

In the finite case, the surjectivity condition in 1.17 is a consequence of $[Q{:}K]_l = 2$, and the criterion holds in a simpler form:

2.7. Derivable translation planes. *If the quasi-field Q is a 2-dimensional left vector space over its subfield $K \cong \mathbb{F}_q$, then $K \cup \{\infty\}$ corresponds to a derivation set D of the translation plane $\mathcal{A}$ over Q.*

This leads to many classes of finite derivable translation planes. There exist also derivable dual translation planes and even derivable semi-translation planes. A few examples shall be described in the following.

2.8. Derived Pappian planes. *Assume in* 2.7 *that $Q \cong \mathbb{F}_{q^2}$ is a field. Then the derived plane $D\mathcal{A}$ has a transitive Baer group (cf.* 1.19*).*

Proof. The line L of $\mathcal{A}$ with the equation $x = 0$ (and slope ∞) is the common axis of all collineations of $\mathcal{A}$ of the form $(x, y) \mapsto (ax,\ y+bx)$ with $a, b \in \mathbb{F}_q$ and $a \neq 0$. These collineations map the derivation set D consisting of all slopes in $\mathbb{F}_q \cup \{\infty\}$ onto itself. Hence they yield a group Γ of automorphisms of the derived plane $D\mathcal{A}$ (recall that $\mathcal{A}$ and $D\mathcal{A}$ have the same point set). By 1.9, the points of L are the affine points of a Baer subplane $\mathcal{B}$ of $D\mathcal{A}$, and Γ acts trivially on $\mathcal{B}$. Therefore, Γ acts freely on the $q(q-1)$ exterior points of any interior line. Obviously, Γ has order $q(q-1)$, and the claim follows. □

2.9. Remark. The same conclusions can be drawn if Q is a semi-field of order q^2 with a subfield $K \cong \mathbb{F}_q$ such that $(xk)y = x(ky)$ holds for all $k \in K$ and $x, y \in Q$; see also [39].

Planes with a transitive Baer group can also be obtained by the following construction [33]:

2.10. Hall quasi-fields. *Let f be an irreducible quadratic polynomial over a finite field $K = \mathbb{F}_q$ of order $q > 2$. Then there is a unique quasi-field Q of order q^2 with kernel K such that K is in the center of Q and $f(x) = 0$ for each $x \in Q \setminus K$.*

Proof. Choose a basis $\{1, j\}$ of Q over K and write $f(z) = z{\cdot}z - rz + s$. The condition that $f(a+bj) = 0$ for $b \neq 0$ implies $(a+bj){\cdot}j = -b^{-1}f(a) + (r-a)j$. It can easily be verified that this determines the multiplication of a quasi-field having the desired properties. □

Remark. Mäurer [50] has given the following elegant description of the Hall quasi-fields: the set of left multiplications consists of the conjugacy class in $\mathrm{GL}_2\, K$ with characteristic polynomial f together with the scalar multiples of the identity. Clearly, this set is sharply transitive.

2.11. Hall planes. The *Hall planes* are the translation planes over the quasi-fields described in 2.10.

The center K of Q coordinatizes an affine Baer subplane; its Baer group is transitive on the set of all points at infinity outside the Baer subplane.

Proof. Left multiplication in Q by an element $c = a+bj \in Q \setminus K$ is a linear map given by the matrix $\widehat{c}$ with left column $(a, b)'$ and right column $(-b^{-1}f(a),\, r-a)'$, determined by $\operatorname{tr}\widehat{c} = r$ and $\det\widehat{c} = s$. Each regular matrix $H \in K^{2\times 2}$ which maps the unit element $e = (1, 0)' \in K$ onto itself induces an automorphism of $(Q, +, \cdot)$. In fact,

$$H(c \cdot z) = H\widehat{c}H^{-1}Hz = \widehat{d}\,Hz = Hc \cdot Hz$$

since conjugation preserves trace and determinant and $d = \widehat{d}\,e = H\widehat{c}H^{-1}e = H\widehat{c}\,e = Hc$. These automorphisms fix K pointwise and are transitive on $Q \setminus K$. They induce Baer collineations on the affine plane $\mathcal{A}_Q$, and they act transitively on the points at infinity which do not belong to $\mathcal{A}_K$. □

The relations between the quasi-fields coordinatizing $\mathcal{A}$ and $D\mathcal{A}$ lead to the following result due to Albert [1]:

2.12. Theorem. *The Hall planes are exactly the derived Pappian planes.*

A proof of the following sharper result can be found in [49, §52]:

2.13. Theorem (Prohaska). *Let D be a derivation set for the finite affine plane $\mathcal{A}$ and put $\Delta = \{\delta \in \operatorname{Aut}\mathcal{A} \mid D^\delta = D\}$. If the stabilizer Δ_a has exactly two affine point orbits besides $\{a\}$, then one of the planes $\mathcal{A}$ and $D\mathcal{A}$ is Pappian and the other one is a Hall plane.*

Let $\mathsf{B} = \{\beta \in \operatorname{Aut}\mathcal{P} \mid \mathcal{B}^\beta = \mathcal{B}\}$ denote the global stabilizer of a Baer subplane $\mathcal{B} \lessdot \mathcal{P}$. Whereas the Hall planes have the property that the kernel $\Gamma_\mathcal{B}$ of the action of B on $\mathcal{B}$ is as large as possible, the *Hughes planes* have a very large induced group $\mathsf{B}/\Gamma_\mathcal{B}$. Indeed, B is doubly transitive on B, and then $\mathcal{B}$ is Desarguesian and $\mathsf{B}/\Gamma_\mathcal{B}$ contains all translations of $\mathcal{B}$; cf. [56]. Moreover, it is required that each translation of $\mathcal{B}$ is induced by a *translation* of $\mathcal{P}$.

As has been shown in 1.14, near-fields can be used to construct Hughes planes. All finite near-fields have been determined in [78]; cf. also [34, §20.7]; for an explicit description of the finite near-fields see, e.g., [22, §5.2].

2.14. Dickson near-fields. Let $(Q, +, \ )$ be a field, A its automorphism group, and $\alpha : Q^\times \to \mathsf{A}$ a map. Define a new multiplication on Q by $x \circ y = xy^{\alpha(x)}$. If α satisfies suitable conditions, then $(Q, +, \circ)$ is a near-field. All near-fields which can be obtained in this way are called *Dickson near-fields.*

2.15. Theorem (Zassenhaus). *Each finite near-field is either a Dickson near-field or one of seven exceptional near-fields of order p^2, where p is one of the primes* $5, 7, 11, 23, 29$*, or* 59.

Combined with the next result, this provides a complete classification of all finite Hughes planes.

2.16. Theorem (Lüneburg). *Assume that $\mathcal{P}$ is a non-Desarguesian finite Hughes plane (i.e., there is a Desarguesian Baer subplane $\mathcal{B}$ of order q such that each translation of $\mathcal{B}$ is induced by a translation of $\mathcal{P}$). Put $\Gamma = \operatorname{Aut}\mathcal{P}$ and let Δ denote the group generated by all translations. Then:*

(a) *$\mathcal{B}^\Gamma = \mathcal{B}$ and Δ is transitive on the set of all exterior flags.*
(b) *Either $\Delta \cong \mathrm{PSL}_3\,\mathbb{F}_q$ or $q = 7$ and $\Delta \cong \mathrm{SL}_3\,\mathbb{F}_7$.*
(c) *There is a near-field N of order q^2 such that $\mathcal{P} \cong \mathcal{H}_N$ is a plane of the kind constructed in* 1.14.
(d) *Each homology of $\mathcal{B}$ extends to a (unique) homology of $\mathcal{P}$.*

Notes on the proof. (1) If $B^\Gamma \neq B$, then some interior line is mapped to an exterior one and it is an easy consequence of 1.16 that Γ is flag-transitive on $\mathcal{P}$ and then even doubly transitive on P. By [22, 4.4.10 or 4.4.20], the plane $\mathcal{P}$ would be Desarguesian. Hence $\mathcal{B}^\Gamma = \mathcal{B}$.

(2) As in 1.16, the group Δ is transitive on the exterior flags. By an idea of Freudenthal [27, §6], the exterior geometry $\mathcal{E}$ can be described as follows: choose a line L in $\mathcal{E}$ and an exterior point $p \in L$, and let $\Pi = \Delta_p$ and $\Lambda = \Delta_L$ denote the corresponding stabilizers in Δ. Identify p^α and L^β with the cosets

$\Pi\alpha$ and $\Lambda\beta$, respectively. Incidence $p^\alpha \in L^\beta$ is then equivalent with the condition $\Pi\alpha \cap \Lambda\beta \neq \emptyset$. The interior elements can easily be adjoined. Thus, it suffices to determine the triple (Δ, Π, Λ) up to isomorphism.

(3) The kernel $\Gamma_{\mathcal{B}}$ coincides with the centralizer $\mathrm{Cs}_\Gamma \Delta$: in fact, a given $\gamma \in \Gamma_{\mathcal{B}}$ fixes center and axis of each translation τ, and γ maps some point a and its image a^τ onto itself. Therefore, $\tau^\gamma = \tau$. The converse is obvious. In particular, Δ is a central extension of the simple group $\Delta^* = \Delta|_{\mathcal{B}} \cong \mathrm{PSL}_3\,\mathbb{F}_q$ induced by Δ on $\mathcal{B}$.

(4) The group Δ is *perfect*, i.e., the commutator group Δ' is equal to Δ. This follows from the fact that all translations are conjugate (because Δ is transitive on quadrangles of $\mathcal{B}$) and that some non-trivial translation can be written as a commutator of two translations σ and τ such that the center of σ is on the axis of τ.

(5) Perfect central extensions or *coverings* of a perfect group are discussed in [2, §33]. A covering of a simple group L is said to be quasi-simple. If $L \cong \mathrm{PSL}_n\,\mathbb{F}_q$ and if Z is the center of a covering group of L, then the order d of Z divides n and $q{-}1$; see [2, p. 251]. Thus, Δ is simple or $\Delta \cong \mathrm{SL}_3\,\mathbb{F}_q$.

(6) The number of pairs of exterior lines intersecting in an exterior point is the same as the order of the group $\Delta \cong \mathrm{SL}_3\,\mathbb{F}_q$. Based on all these facts, it can be shown that each possible triple (Δ, Π, Λ) is realized in one of the planes $\mathcal{H}_N$, where N is a finite near-field. For the lengthy details and more information see [48]. □

In the last proof, the groups Π and Λ play a symmetric rôle. This implies [61]:

2.17. Corollary. *Each finite Hughes plane is self-dual.*

2.18. Hughes planes are derivable. *If $\mathcal{P}$ and $\mathcal{B}$ are as in* 2.16 *and if W is an interior line, then $W \cap B$ is a derivation set.*

The proof uses coordinate methods. It relies mainly on the fact that $\mathcal{P}$ is of the form $\mathcal{H}_N$, where the near-field N has dimension 2 over its kernel; cf. [8] or [22, 5.4.4].

2.19. Multiple derivations. If D and D' are disjoint derivation sets for an affine plane $\mathcal{A}$, then D' is also a derivation set for the derived plane $D\mathcal{A}$, and one may form $D'D\mathcal{A} = DD'\mathcal{A}$. The process can be extended to any family of pairwise disjoint derivation sets. Multiply derived translation planes are again translation planes.

2.20. Example: Pappian planes of odd order. *The line W at infinity of the affine plane $\mathcal{A}$ over the field $\mathbb{F}_{q^2}$ contains a family of $q{-}1$ pairwise disjoint derivation sets.*

Proof. The projective line W over $\mathbb{F}_{q^2}$ may be considered as point set of the Miquelian inversive plane $\mathcal{M}(q)$ whose circles are the projective line $D = \mathbb{F}_q \cup \{\infty\}$ over $\mathbb{F}_q$ and all its images D^λ with $\lambda \in \Lambda = \mathrm{PGL}_2\,\mathbb{F}_{q^2}$. Each circle is a derivation set for $\mathcal{A}$ since Λ is induced by a group of automorphisms of $\mathcal{A}$. In $\mathcal{M}(q)$ there exist 'flocks' of mutually disjoint circles covering all but two points. This follows from an alternative description of $\mathcal{M}(q)$ as the geometry of plane sections of a non-ruled quadric O in the projective 3-space $\mathrm{PG}_3\,(q)$; see [22, p. 257]. Two distinct tangent planes of O meet in a line L. All other planes containing L intersect O in a circle, and these circles obviously form a flock. $\square$

Thus a finite Pappian plane yields several multiply derived translation planes. The question which of these planes admit a group $\mathrm{SL}_2\,\mathbb{F}_5$ is dealt with in [13].

Finally, Baer sub*spaces* should be mentioned. Recall that each finite projective space of dimension $n \geq 3$ is a geometry $\mathrm{PG}_n\,(q)$ over a field $\mathbb{F}_q$ (see [22, 1.4.1–1.4.4]).

2.21. Baer subspaces. The injection $\mathbb{F}_q \to \mathbb{F}_{q^2}$ yields a canonical embedding $\mathrm{PG}_n\,(q) \to \mathrm{PG}_n\,(q^2)$. The Baer subspaces of the larger space are all images $\mathrm{PG}_n\,(q)^\gamma$, where γ is an automorphism of $\mathrm{PG}_n\,(q^2)$. Such Baer subspaces have been studied in [6], [7], and [74].

3. Baer subplanes of compact, connected planes

Assume from now on that P and $\mathfrak{L}$ are compact, connected Hausdorff spaces and that join and intersection of distinct elements are continuous operations. The projective plane $\mathcal{P}$ will then be called a *topological plane* for short; see [68, Chap. 4] for basic properties of such planes. The point space of a topological plane has a countable basis for the topology ([68, Theorem 41.8]). There are several notions of *dimension* of a topological space; for a space X with a countable basis most of the familiar dimensions agree, in particular the inductive dimension $\mathrm{ind}\,X$ and the covering dimension $\dim X$; see [57] or [68, §92]. One can speak, therefore, without ambiguity of *the dimension* of a topological plane.

It has been conjectured that the dimension of any topological plane is finite. In all known examples the lines are even topological *manifolds*, and the topological properties of arbitrary finite-dimensional planes are quite close to those of topological planes whose lines are manifolds; cf. [68, Chap. 5].

The *classical planes* are the Pappian planes $\mathcal{P}_\mathbb{R}$ and $\mathcal{P}_\mathbb{C}$, the Desarguesian plane $\mathcal{P}_\mathbb{H}$ over the locally compact, connected quaternion skew field $\mathbb{H}$, and the projective closure $\mathcal{P}_\mathbb{O}$ of the affine plane over the octonion algebra $\mathbb{O}$. (In fact, $\mathcal{P}_\mathbb{O}$ is the only compact proper Moufang plane; see [30, XI.7.9].)

The coordinate spaces $\mathbb{K}$ of these planes are homeomorphic to $\mathbb{R}^\ell$ with ℓ dividing 8, and the lines are one-point compactifications of $\mathbb{K}$, hence they are homeomorphic to $\mathbb{S}_\ell$. A detailed treatment of the classical planes can be found in [68, Chap. 1]. Examples of other topological planes will be given in 3.1.

The *automorphism group* $\Sigma = \operatorname{Aut}\mathcal{P}$ of a topological plane $\mathcal{P}$ consists of all *continuous* collineations. There is a canonical topology to be used on Σ: the *compact-open* topology. It coincides with the topology of uniform convergence on P because P is compact. Equipped with this topology, Σ becomes a topological transformation group of P; cf. [68, 96.3–96.7]. If $\mathcal{P}$ is a classical plane, then the connected component of Σ is a simple Lie group of dimension 8, 16, 35, or 78, respectively.

3.0. Finite-dimensional planes: Review. *Let $\mathcal{P} = (P, \mathfrak{L})$ be a topological projective plane with a compact, connected point space P of finite topological dimension $\dim P = d$. Then:*

(a) *Each line L is homotopy equivalent to a sphere $\mathbb{S}_\ell$, where $\dim L = \ell \mid 8$ and $d = 2\ell$.*
(b) *P and L are homology manifolds, and the (singular) homology of P coincides with that of the point space of the classical plane with the same dimension.*
(c) *Each automorphism fixes a point and a line, and each involution is a reflection or a Baer involution.*
(d) *Taken with the compact-open topology, Σ is a locally compact group, and $\dim \Sigma < \infty$.*
(e) *Any closed subset A of $X \in \{L, P\}$ such that $\dim A = \dim X$ has a non-void interior.*
(f) *If the lines are manifolds, then they are homeomorphic to a sphere $\mathbb{S}_\ell$.*
(g) *If $d \le 4$, then the lines of $\mathcal{P}$ are spheres.*
(h) *If the lines are spheres, then the spaces P and $\mathfrak{L}$ are homeomorphic.*

Proof. (a) [68, (54.11)]; (b) [68, (54.10) and (52.13)]; (c) [68, (55.19) and (55.29)]; (d) [68, (44.3)]; (e) [68, (51.21a)]; (f) [68, (52.3)]; (g) [68, (53.7)]; (h) [45]. □

Throughout this chapter, it will be assumed that the dimension of $\mathcal{P}$ is finite. In order to avoid topological subtleties, one may always imagine the lines to be manifolds. Many examples can be obtained by some slight modification of the multiplication or the addition of a classical coordinate system. In the 2-dimensional case, there are also easy geometric constructions; see, e.g., [52] or [68, Chap. 3]. It suffices to describe an affine plane or its coordinate structure K. If the latter is locally compact and connected and if all relevant operations are continuous, then K yields a compact, connected topological projective plane $\mathcal{P}_K$; see [62, Th. 7.15] or [68, (43.5)]. More precisely, the *affine* plane $\mathcal{A}_K$ has the point space $K \times K$, the lines of finite *slope*

a are written in the form $y = \tau(a, x, b)$, and the properties of the *ternary field* (K, τ) reflect the axioms of an affine plane. $\mathcal{P}_K$ is then the projective closure of $\mathcal{A}_K$. If $\mathcal{A}_K$ has a transitive group of 'vertical' translations, then $\tau(a, x, b)$ can be expressed as $a \circ x + b$; see [30, XI Part A] or [68, Chap. 2] for more details; cf. also 1.11 and 1.12. In the case $K \approx \mathbb{R}^\ell$, continuity of $\tau : K^3 \to K$ suffices for $\mathcal{P}_K$ to be a *topological* plane ([68, Th. 43.6]).

3.1. Topological planes: Examples.
(a) If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is a homeomorphism of $\mathbb{R}$, then the verticals $\{c\}\times\mathbb{R}$ and all curves in $\mathbb{R}^2$ with an equation $y - b = f(x - a)$ are the lines of a 2-dimensional topological affine plane, the so-called *shift* plane $\mathcal{E}_f$.

(b) Let ρ be a homeomorphism of $\mathbb{R}$ which is the identity on $[0, \infty)$. Instead of the usual multiplication of $\mathbb{C} = \mathbb{R}^2$ define a new multiplication by

$$(a, b) \circ (x, y) = (ax - b^\rho y,\ bx + ay).$$

An easy calculation shows that left and right multiplications in $(\mathbb{C}\smallsetminus\{0\}, \circ)$ are bijective. Hence $\mathbb{C}_\rho = (\mathbb{C}, +, \circ)$ is a topological quasi-field. It coordinatizes a 4-dimensional topological translation plane; cf. [68, (64.13)]. The elements $(x, 0)$ form a closed subfield of $\mathbb{C}_\rho$, which obviously is isomorphic to $\mathbb{R}$ and coordinatizes a Baer subplane.

(c) Similarly, define a new multiplication on $\mathbb{K} \in \{\mathbb{H}, \mathbb{O}\}$ as follows: choose a fixed real number $t > 1/2$ and put $c \circ z = t\cdot cz + (1-t)\cdot zc$. For $t \neq 1$, the so-called *mutation* $\mathbb{K}_t = (\mathbb{K}, +, \circ)$ of $\mathbb{K}$ is a non-associative topological division algebra, and distinct parameters yield non-isomorphic planes; see [68, (87.7) and (82.21)].

In the topological setting, the adequate notion of a substructure is that of a *closed* subplane. However, arbitrary subplanes will also be considered. By definition, $\mathcal{S} = (S, \mathfrak{S})$ is a closed subplane of $\mathcal{P}$ if, and only if, $\mathcal{S} \le \mathcal{P}$ and S is a closed subspace of P.

3.2. Closure of a subplane. If $\mathcal{S} = (S, \mathfrak{S})$ is any subplane of $\mathcal{P}$, then $\overline{S}$ is the point space of a closed subplane $\langle S\rangle$ (proof by an easy convergence argument). More generally, if S is an arbitrary subset of P and if S contains a quadrangle, then $\langle S\rangle$ is the intersection of all closed subplanes of $\mathcal{P}$ whose point set contains S.

3.3. The real plane. *The classical plane $\mathcal{P}_\mathbb{R}$ has no Baer subplane whatsoever.*

Proof. Assume that $\mathcal{B} \lessdot \mathcal{P}_\mathbb{R}$. Then $\mathcal{B} = \mathcal{P}_S$, and $\mathbb{R}$ is a quadratic Galois extension of the field S, but $\operatorname{Aut}\mathbb{R} = \mathbb{1}$. □

Though Baer subplanes are maximal, they need not be closed:

3.4. Baer collineations of the complex plane $\mathcal{P}_\mathbb{C}$. A transcendency basis T of $\mathbb{C}$ over $\mathbb{Q}$ has cardinality $\operatorname{card} T = \operatorname{card}\mathbb{C} = \aleph$. Each bijection of T extends to an automorphism $\alpha \in \Gamma = \operatorname{Aut}\mathbb{C}$, and $\operatorname{card}\Gamma = 2^\aleph$. Denote the fixed field of α by F_α, and write again $\iota = z \mapsto \overline{z}$ for the usual conjugation. If $\rho \in \Gamma$ is an involution, then F_ρ is a real-closed field and carries a unique ordering (by the Artin–Schreier theory). In particular, $F_\iota = \mathbb{R}$, and from $\operatorname{Aut}\mathbb{R} = \mathbb{1}$ it follows that $\langle\iota\rangle$ is its own centralizer. Consequently, ι has $2^\aleph$ distinct conjugates in Γ. If $\rho = \iota^\alpha \neq \iota$, then $F_\rho \cong \mathbb{R}$ and F_ρ is dense in $\mathbb{C}$. Applied to the coordinates, ρ induces a Baer collineation of $\mathcal{P}_\mathbb{C}$. The corresponding Baer subplane is everywhere dense in the topological plane $\mathcal{P}_\mathbb{C}$.

For detailed information on $\operatorname{Aut}\mathbb{C}$ see [41] or [71]; cf. also [4].

3.5. More on Baer collineations of $\mathcal{P}_\mathbb{C}$. By a theorem of Steinitz [73, §23], an algebraically closed field A of characteristic 0 is isomorphic to $\mathbb{C}$ if, and only if, $\operatorname{card} A = \aleph$. Let $F^\natural$ denote the algebraic closure of the field F. Choose a transcendency basis $T \subseteq \mathbb{R}$ of $\mathbb{C}$ over $\mathbb{Q}$, and map T bijectively onto a *proper* subset $S \subset T$. Then $\mathbb{C}_1 = \mathbb{Q}(S)^\natural < \mathbb{Q}(T)^\natural = \mathbb{C}_2$ and $\mathbb{C}_1 \cong \mathbb{C} \cong \mathbb{C}_2$.

(a) If $\mathbb{C}_2$ is identified with $\mathbb{C}$, then the ordinary conjugation of $\mathbb{C}$ induces on $\mathbb{C}_1$ an involution whose fixed field does not contain the elements of $T \setminus S$, and which is, therefore, not complete with respect to its unique ordering. Hence $\mathcal{P}_\mathbb{C}$ has (many) Baer subplanes which are isomorphic to proper subplanes of $\mathcal{P}_\mathbb{R}$.

(b) If, on the other hand, $\mathbb{C}_1$ is identified with $\mathbb{C}$, then, under the isomorphism $\mathbb{C} \cong \mathbb{C}_2$, the usual conjugation of $\mathbb{C}$ yields an involution of $\mathbb{C}_2$ with a fixed field which properly contains $\mathbb{R}$ and thus is a non-Archimedian ordered field. Consequently, $\mathcal{P}_\mathbb{C}$ has also many non-Archimedian Baer subplanes.

Remark. Phenomena like those described in 3.4 and 3.5 are rare. In fact, of all the topological planes that have been studied so far, only the complex plane admits discontinuous collineations; see also [68, (55.22)] and [69].

3.6. Closed Baer subplanes of the classical planes. *For $\mathbb{K} = \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, each closed Baer subplane $\mathcal{B} \lessdot \mathcal{P}_\mathbb{K}$ is the fixed plane of a Baer involution induced by an automorphism β of $\mathbb{K}$, and $\mathcal{B} \cong \mathcal{P}_\mathbb{F}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, respectively. In the first case, β is complex conjugation. In the other cases, β is conjugate to the map $(z \mapsto z^i = \overline{i}zi)$ of $\mathbb{H}$ or the map $\eta = (x,y) \mapsto (x,-y)$ of the octonion algebra $\mathbb{O} = \mathbb{H}\times\mathbb{H}$.*

Proof. Choose coordinates with respect to a quadrangle $\mathfrak{e}$ in $\mathcal{B}$. The subplane $\langle\mathfrak{e}\rangle$ (as defined in 3.2) is isomorphic to the real plane.

(a) If $\mathbb{K} = \mathbb{C}$, then $\langle\mathfrak{e}\rangle = \mathcal{B}$ is indeed the fixed plane of complex conjugation.

(b) There is a closed subfield $S < \mathbb{H}$ such that $\mathcal{B} = \mathcal{P}_S$, and $S = \langle\mathbb{R}, h\rangle$ for some quaternion h with $h^2 = -1$. The group $\operatorname{Aut}\mathbb{H}$ consists of all inner automorphisms of $\mathbb{H}$ and is transitive on the 2-sphere $\{c \in \mathbb{H} \mid c^2 = -1\}$; see [68, Prop. (11.29)] or [24, Chap. 7].

(c) The octonion algebra $\mathbb{O}$ can be described as $\mathbb{H}\times\mathbb{H}$ with the usual addition and the multiplication $(a,b)\,(x,y) = (ax - \overline{y}b,\, ya + b\overline{x})$. Its automorphism group $\mathrm{G}_2 = \operatorname{Aut}\mathbb{O}$ fixes the center $\mathbb{R}$ of $\mathbb{O}$ and acts transitively on $\{\,(u,v) \in \mathbb{O}^2 \mid u^2 = v^2 = -1 \wedge uv + vu = 0\,\}$ and, hence, also on the set of all quaternion subfields of $\mathbb{O}$; see [68, (11.30)]. Each collineation of $\mathcal{P}_{\mathbb{O}}$ is continuous, and the group $\mathsf{E} = \operatorname{Aut}\mathcal{P}_{\mathbb{O}}$ is known to be transitive on quadrangles; cf. [68, Th. 17.6]. From these results it follows that E is transitive on the set of all closed Baer subplanes. Obviously, η is an involution of $\mathbb{O}$ with fixed point field $\mathbb{H} = \mathbb{H}\times\{0\}$. □

The topological version of 1.2 is noteworthy:

3.7. Lemma. *Let $\mathcal{B} = (B, \mathfrak{B})$ be a closed Baer subplane of the topological plane $\mathcal{P}$. If p is an exterior point of the interior line L, then the map $\lambda_p = (x \mapsto px) : B\setminus L \approx \mathfrak{L}_p\setminus\{L\}$ is a homeomorphism.*

Proof. Being a restriction of the continuous join map, λ_p is continuous. Suppose, conversely, that the lines $L_\nu = px_\nu$ $(\nu \in \mathbb{N})$ converge to a line $H \neq L$. Because B is compact, the x_ν accumulate at a unique point $c = B \cap H$. Therefore, λ_p^{-1} is also continuous. □

3.8. Theorem. *Assume that $\mathcal{B} = (B, \mathfrak{B})$ is a closed subplane of the topological plane $\mathcal{P} = (P, \mathfrak{L})$, and that* $\dim P = 2\ell < \infty$. *Then $\mathcal{B} \lessdot \mathcal{P}$ if, and only if,* $\dim B = \ell$.

Proof. If $\mathcal{B}$ is a Baer subplane, then Lemma 3.7 immediately implies that $\dim B = \dim \mathfrak{L}_p = \ell$. The converse is proved in full generality in [68, (55.5)]. If the lines of $\mathcal{P}$ are manifolds, then the pencil $\mathfrak{L}_p$ is homeomorphic to $\mathbb{S}_\ell$, and the proof reduces to the following arguments: suppose that $\dim B = \ell$ and that no interior line passes through the point p. Then $\varphi = (x \mapsto px) : B \to \mathfrak{L}_p$ maps the compact space B homeomorphically onto its image. From $\dim B^\varphi = \ell$ it follows that B^φ contains interior points of $\mathfrak{L}_p$; see [68, (92.14)] or [37, Th. IV 3]; cf. also 3.0(e). Consequently, B is an ℓ-dimensional manifold. By domain invariance, the map φ is open and B^φ is open in $\mathfrak{L}_p$. Because $\mathfrak{L}_p$ is connected, $\varphi : B \approx \mathfrak{L}_p$ is a homeomorphism, but the homology H_*B is different from $\mathrm{H}_*\mathbb{S}_\ell$; see [68, (52.14c)]. Hence each point $p \in P$ is on an interior line and, dually, each line $L \in \mathfrak{L}$ contains an interior point. □

3.9. Corollary. *Suppose that $\mathcal{B} = (B, \mathfrak{B})$ is a proper* closed *subplane of a finite-dimensional topological plane $\mathcal{P}$. If each line $L \in \mathfrak{L}$ contains a point of B, then each point of P is incident with an interior line. In other words, one of the two dual conditions suffices for $\mathcal{B}$ to be a Baer subplane.*

Proof. If no interior line passes through the point p, then $\varphi = (x \mapsto px) : B \to \mathfrak{L}_p$ would be a homeomorphism, but this is impossible, as stated at the end of the last proof. $\square$

Most of the non-classical topological planes that have been described explicitly have fairly large automorphism groups, and these groups are used in the construction of the planes. For this reason, Baer subplanes of these planes are often the fixed planes of continuous Baer collineations. There are other examples, however.

3.10. Closed Baer subplanes which are not fixed planes of automorphisms.

(a) Consider Example 3.1(b) with $b^\rho = b^3$ for $b < 0$. The reals $x = (x, 0)$ coordinatize a closed Baer subplane $\mathcal{B}$. Assume that $\mathcal{B} = \mathcal{F}_\beta$ for some automorphism β of the plane. Then β corresponds to an automorphism α of $\mathbb{C}_\rho$ which fixes exactly the reals. Put $j = (0, 1)$ and note that $\sqrt{-1} = \pm j$ in $\mathbb{C}_\rho$. Hence $j^\alpha = -j$ and $(0, t)^\alpha = (0, -t)$, but $t > 1$ implies $(0, t) \circ j = -t$ and $(0, -t) \circ (-j) = -t^3$. This contradicts $\alpha \in \operatorname{Aut} \mathbb{C}_\rho$.

(b) Another example is the complex shift plane $\mathcal{E}_f$ with lines $y - b = f(x - a)$ and $\{c\} \times \mathbb{C}$, where

$$f(w) = \begin{cases} w^2 & (v \geq 0) \\ w^2 + \gamma v^2 & (v < 0) \end{cases}$$

for $w = u + iv$ and $\gamma = \alpha + i\beta$, $4\alpha < -\beta^2$, $\beta > 0$; see [68, (74.30)]. Restriction to real coordinates yields a closed Baer subplane. The group $\operatorname{Aut} \mathcal{E}_f$ is connected (loc. cit.). Now use 3.14(a) and note that $\mathcal{E}_f$ has no Baer involution by [68, (55.21c)].

(c) The *semi-nuclear* division ring $S = (\mathbb{C}^2, +, \circ)$ with the multiplication

$$c \circ z = (a, b) \circ (x, y) = (ax - \overline{b}y,\ bx + \overline{a}y + i\overline{b}y)$$

yields an 8-dimensional example: note that S has no zero-divisors. This is obvious for $by = 0$. If $by \neq 0$, then $c \circ z = 0$ implies $a\overline{a} + b\overline{b} + ia\overline{b} = 0$. With $|ab^{-1}| = t$ it follows that $t^2 + 1 = t$, but this equation has no positive solution. Since multiplication in S is $\mathbb{R}$-linear, $(S, +, \circ)$ is in fact a division ring. The map $x \mapsto (x, 0) : \mathbb{C} \to S$ is an embedding, and $\mathbb{C}$ coordinatizes a closed Baer subplane of $\mathcal{P}_S$. By 3.12 below, the group

$$\Gamma = \{\gamma \in \operatorname{Aut} S \mid \gamma|_{\mathbb{C}} = \mathbb{1}\}$$

is compact. Put $j = (0, 1)$ and $j^\gamma = (u, v)$ for $\gamma \in \Gamma$. Then $j^{\gamma^\nu} = (u_\nu, v^\nu)$, and compactness of Γ shows that $v\overline{v} = 1$. From

$$(iu - 1,\ iv) = (j \circ j)^\gamma = j^\gamma \circ j^\gamma = (u^2 - v\overline{v},\ (u + \overline{u})v + iv\overline{v})$$

it follows that $u(u - i) = 0$ and $u + \overline{u} = i\,(1 - \overline{v})$, and then $u + \overline{u} = 0$ and $v = 1$. Hence $j^\gamma = u + j$ and $j^{\gamma^\nu} = \nu{\cdot}u + j$. Now compactness of Γ gives $u = 0$ and $\gamma = \mathbb{1}$. $\square$

3.11. Continuity. *If a collineation γ of the topological plane $\mathcal{P}$ fixes each element of a closed Baer subplane $\mathcal{B} <\!\bullet\, \mathcal{P}$, then γ is continuous, i.e., $\gamma \in \operatorname{Aut}\mathcal{P}$.*

Proof. Let p be an exterior point. By Lemma 3.7, the pencil $\mathfrak{L}_p$ is mapped homeomorphically onto $\mathfrak{L}_{p^\gamma}$. Continuity of γ on P is an immediate consequence. □

3.12. Compactness. *Let $\mathcal{P}$ be a topological plane and $\Sigma = \operatorname{Aut}\mathcal{P}$ its locally compact automorphism group. If $\mathcal{B} = (B, \mathfrak{B})$ is a closed Baer subplane of $\mathcal{P}$, then the corresponding Baer group, the pointwise stabilizer $\Gamma_{\mathcal{B}} := \bigcap_{x\in B} \Sigma_x$, is a compact subgroup of Σ.*

The proof uses Lemma 3.7 and the Arzela-Ascoli theorem. For the somewhat technical details see [68, Th. 83.6].

3.13. Baer groups of the classical planes. *Consider a closed ℓ-dimensional Baer subplane $\mathcal{B}$ of a 2ℓ-dimensional classical plane, $\ell \in \{2, 4, 8\}$.*

(a) *If $\ell = 2$, then $\Gamma_{\mathcal{B}} \cong \mathbb{Z}_2$.*
(b) *If $\ell = 4$, then $\Gamma_{\mathcal{B}} \cong \mathrm{SO}_2\,\mathbb{R}$.*
(c) *If $\ell = 8$, then $\Gamma_{\mathcal{B}} \cong \mathrm{SU}_2\,\mathbb{C}$.*

Proof. The coordinate system of $\mathcal{P}$ is an extension of the coordinate (skew) field of $\mathcal{B}$.

(a) This follows from the fact that $\operatorname{Aut}_{\mathbb{R}}\mathbb{C}$ is generated by complex conjugation.

(b) As mentioned in 3.6 (see [24, Chap 7, §3]), each automorphism of $\mathbb{H}$ is of the form $\gamma = (z \mapsto c^{-1}zc)$ with $c\bar{c} = 1$, and γ fixes $\mathbb{C}$ elementwise if, and only if, $c \in \mathbb{C}$.

(c) Representing $\mathbb{O}$ as in the proof of 3.6(c), it is not difficult to see that all automorphisms of $\mathbb{O}$ which induce the identity on $\mathbb{H}\times\{0\}$ have the form $(a, b) \mapsto (a, bq)$ with $q\bar{q} = 1$, hence they form a group homeomorphic to $\mathbb{S}_3$. □

Baer groups of other topological planes tend to be smaller than those of the classical planes of the same dimension. Examples are provided by the 16-dimensional Hughes planes discussed below.

3.14. Baer groups of 4- and 8-dimensional planes.

(a) *Any Baer group Γ of a 4-dimensional topological plane has order at most 2.*
(b) *A Baer group of an 8-dimensional topological plane is at most 1-dimensional.*

Proof. (a) Because of 3.0(g), an interior line L is homeomorphic to $\mathbb{S}_2$. The interior points on L form a circle S. By the Schoenflies theorem, $L \setminus S$ is a union of 2 disks D and D'. The elements of Γ which map D onto itself form a subgroup Δ of index at most 2, and Δ acts freely on $D \approx \mathbb{R}^2$; moreover, the action of Δ on D is orientation preserving. According to Brouwer's translation theorem (see [31] or [26]) each orbit $x^\Delta \neq x$ in D is unbounded, but Δ is compact and hence $\Delta = \mathbb{1}$.

(b) If the 8-dimensional plane $\mathcal{P}$ has a Baer subplane, then it follows from 3.7, 3.8, and 3.0 that each line of $\mathcal{P}$ is homeomorphic to $\mathbb{S}_4$. Let Δ denote the connected component of the given Baer group Γ. All compact, connected effective groups on $\mathbb{S}_4$ having an orbit of dimension > 1 have been determined by Richardson; cf. [68, Th. 96.34]. None of these groups fixes a 2-sphere pointwise. Hence $\dim \Gamma = \dim p^\Delta \leq 1$. □

Satisfactory results on Baer groups of arbitrary 16-dimensional planes have been obtained only for Lie groups. These results depend on the following theorem:

3.15. Uniqueness of Baer involutions. *Let α and β be commuting involutions of a topological plane. If α and β have the same subplane of fixed elements, then $\alpha = \beta$.*

It is not known whether or not this is true without the assumption that $\alpha\beta = \beta\alpha$.

Proof. Suppose that $\Gamma = \langle \alpha, \beta \rangle$ is a Baer group of order 4 with fixed plane $\mathcal{B} = (B, \mathfrak{B})$. If L is an interior line, then Γ fixes $L \cap B$ pointwise and Γ acts freely on the complement $L \setminus B$. Because of 3.0(a), the line L has the (integer) cohomology of a sphere $\mathbb{S}_\ell$, and 3.14 implies that $\ell \in \{4, 8\}$. Moreover, $L \cap B$ has the cohomology of a sphere $\mathbb{S}_k$, and $\ell = 2k$ by Theorem 3.8. A theorem of Smith [70, p. 407] asserts that Γ has k-periodic cohomology: $\mathrm{H}^{q+k}(\Gamma, \mathbb{Z}) \cong \mathrm{H}^q(\Gamma, \mathbb{Z})$ for all $q \geq 0$, but then Γ must be cyclic. In fact, the Künneth formula gives $\mathrm{H}^{2q}(\mathbb{Z}_2^2, \mathbb{Z}) \cong \mathbb{Z}_2^{q+1}$; see, e.g., [25, p. 18]; cf. also [68, (55.27)]. □

3.16. Baer groups of 16-dimensional planes. *Assume that Γ is a connected Baer group of a 16-dimensional topological plane $\mathcal{P}$.*

(a) *If Γ is a Lie group, then $\Gamma \cong \mathrm{SU}_2\,\mathbb{C}$ or $\Gamma \cong \mathrm{SO}_2\,\mathbb{R}$ or $\Gamma = \mathbb{1}$.*
(b) *In any case, $\dim \Gamma < 8$. If the lines of $\mathcal{P}$ are manifolds, then $\dim \Gamma \leq 5$.*

Proof. (a) Being a compact connected Lie group, Γ is a product of a central torus group and a semi-simple Lie group, and Γ does not contain a 2-torus by 3.15. Hence Γ is either almost simple without two commuting involutions, and then $\Gamma \cong \mathrm{SU}_2\,\mathbb{C} \cong \mathrm{Spin}_3\,\mathbb{R}$, or Γ is a torus group of dimension at most 1.

(b) Denote the point set of the Baer subplane by B and let L be an interior line. Each $x \in L \setminus B$ has a compact orbit $x^\Gamma \approx \Gamma$. If $\dim \Gamma = \dim L$, then x^Γ contains a non-empty open subset of L by 3.0(e). Being homogeneous,

x^Γ would be open in L, but $L \setminus B$ is connected and not compact. Hence $\dim \Gamma < 8$. Assume now that $L \approx \mathbb{S}_8$ and $\dim \Gamma \geq 6$. Then Γ is a Lie group by [14, Th. 10 and 11], and (a) would imply $\dim \Gamma \leq 3$. $\square$

As in the finite case, interesting results hold in the so-called Hughes situation: *some Baer subplane* $\mathcal{B} <\!\!\bullet\, \mathcal{P}$ *is left invariant by a fairly large group* Δ *of automorphisms of* $\mathcal{P}$. For topological planes, only closed subgroups $\Delta \leq \Sigma$ will be considered. The dimension $\dim \Delta$ is a very good measure for the size of Δ. Because the connected component Δ^1 has the same dimension as Δ, one may always assume Δ to be connected.

3.17. Theorem. *Suppose that* Δ *leaves a Baer subplane* $\mathcal{B}$ *of a* 4*-dimensional topological plane* $\mathcal{P}$ *invariant. If* $\dim \Delta \geq 7$*, then* $\Delta \cong \mathrm{PSL}_3 \mathbb{R}$ *and* $\mathcal{P}$ *is the classical complex plane.*

Proof. Write $\Delta|_{\mathcal{B}} = \Delta/\Phi$ for the group induced by Δ on $\mathcal{B}$. Here, the kernel Φ has order at most 2 by 3.14(a). Since Δ is connected and since a Baer involution of a 4-dimensional plane reverses the orientation of each fixed line, the kernel is trivial; see [68, (55.21bc)]. From the results in [68, §38] it follows that $\mathcal{B}$ is classical and that $\Delta \cong \mathrm{PSL}_3 \mathbb{R}$. Hence $\dim \Delta = 8$. If K and L are exterior lines intersecting in an exterior point p, then $\Delta_{K,L}$ fixes also the interior elements incident with K, L, and p and thus 3 collinear interior points. Therefore, each $\delta \in \Delta_{K,L}$ induces on $\mathcal{B}$ an axial collineation and fixes all interior lines through its center. Because Δ contains no Baer collineations, $\Delta_{K,L} = \mathbb{1}$ and Δ is transitive on the set of all such pairs $\{K, L\}$ by [68, (96.11)]. In particular, Δ is flag-transitive on the subgeometry $\mathcal{E}$ consisting of the exterior elements. It can be shown that the pair of stabilizers Δ_p and Δ_L is unique up to conjugation. Hence $\mathcal{E}$ is isomorphic to the exterior geometry in the complex plane; cf. [27, §6]. Details are given in [64] or [68, (72.3)]. $\square$

3.18. Remark: Affine Hughes groups of 4-dimensional planes. Assume in 3.17 that $\dim \Delta = 6$ instead of $\dim \Delta \geq 7$. Then Δ is isomorphic to the group $\mathbb{R}^2 \cdot \mathrm{GL}_2^+ \mathbb{R}$ of orientation preserving affine maps of the real plane (cf. [68, (33.6 and 33.8)] or [62, 4.3]). All 4-dimensional planes admitting such a group have been determined explicitly in two long papers [46] and [42]. Incidentally, this work completes the classification of 4-dimensional compact planes $\mathcal{P}$ satisfying $\dim \mathrm{Aut}\, \mathcal{P} \geq 6$.

The Hughes situation in 8- and 16-dimensional planes can be treated in a way similar to 3.17. However, the pair of stabilizers of an exterior line L and an exterior point $p \in L$ is not unique up to conjugation, and one obtains one-parameter families of topological Hughes planes. A full account of these planes can be found in [68, §86].

3.19. Eight-dimensional Hughes planes. *Suppose that Δ leaves a Baer subplane $\mathcal{B}$ of an 8-dimensional topological plane $\mathcal{P}$ invariant. If $\dim\Delta \ge 14$, then $\mathcal{B}$ is classical, and the commutator group $\Delta' \cong \mathrm{SL}_3\,\mathbb{C}$ induces on $\mathcal{B}$ the full projective linear group. For each $r \ge 0$ there exists a unique plane $\mathcal{H}_r$ admitting such an action. The plane $\mathcal{H}_0$ is isomorphic to $\mathcal{P}_{\mathbb{H}}$. If $r > 0$, then $\mathcal{H}_r$ is a proper Hughes plane, and its full automorphism group is 17-dimensional.*

Sketch of proof. (1) Write $\Delta^* = \Delta|_{\mathcal{B}} = \Delta/\Phi$. The kernel Φ of this action is at most 1-dimensional by 3.14(b), and $\dim\Delta^* > 12$. Consequently, $\mathcal{B}$ is classical; see [68, (72.8)]. From [68, (71.4) and (71.8)] it follows that $\Delta^* \cong \mathrm{PSL}_3\,\mathbb{C}$ is simple. Since $\dim\Delta \le 17$, a maximal semi-simple subgroup of Δ is 16-dimensional, and one may assume that the simply connected covering group of Δ is isomorphic to $\mathsf{A} := \mathrm{SL}_3\,\mathbb{C}$.

(2) The group A acts on $\mathcal{P}$, possibly with a kernel of order 3. If S is an interior line, then A_S has a subgroup $\Upsilon \cong \mathrm{SU}_2\,\mathbb{C}$. The theorem of Richarson [68, (96.34)] shows that the central involution of Υ acts trivially on S and hence is a reflection. All involutions of A being conjugate, there are no Baer involutions in A.

(3) Let K and L be exterior lines intersecting in an exterior point p. The stabilizer $\Psi = \mathsf{A}_{K,L}$ fixes the interior points $a \in K$ and $b \in L$, the interior line M through p and the intersection $ab \cap M$. Because $\mathcal{B}$ is classical, Ψ induces on $\mathcal{B}$ a group of axial collineations with axis ab, and each $\psi \in \Psi \smallsetminus \{\mathbb{1}\}$ has in $\mathcal{B}$ some center $z \in M$. Since ψ fixes all interior lines through z, the fixed elements of ψ form a Baer subplane, and ψ is contained in a compact subgroup of Ψ by 3.12. Since the commutator of two homologies with distinct centers and the same axis in $\mathcal{P}_{\mathbb{C}}$ is an elation, it follows that all elements in $\Psi|_{\mathcal{B}}$ have the same center $c \notin ab$ and that Ψ is a Baer group.

(4) According to step (2), the compact Lie group Ψ does not contain an involution. Hence $\dim\Psi = 0$ and Ψ is finite. With [68, (96.11)] it follows that A is transitive on the set of all admissible pairs $\{K, L\}$ and, dually, on the set of all pairs of exterior points with exterior join. The space of interior lines is simply connected, and so is the space of exterior points on a given interior line. Repeated application of [68, (94.4)] shows that A_p and Ψ are connected. Hence $\Psi = \mathbb{1}$ and $\Delta \cong \mathsf{A}$; moreover, Δ_L is sharply transitive on the set of pairs of exterior points on L.

(5) By a well-known theorem of Tits [77], there is a near-field $(\mathbb{H}, +, \circ_r)$, where $r \ge 0$ and

$$a \circ_r x = a\,|a|^{-ir} x\,|a|^{ir},$$

such that the action of Δ_L on $L \smallsetminus \{a\} \approx \mathbb{R}^4$ is equivalent to the group of linear maps $x \mapsto a \circ_r x + b$ of $\mathbb{H}$; see also [68, (64.19–23)]. If $r > 0$, then $(\mathbb{H}, +, \circ_r)$ has kernel $\mathbb{C}$ (because $x \in \mathbb{C}$ implies $a \circ_r x = ax$). As in 1.14, the near-field yields a projective plane $\mathcal{H}_r$ with points $\langle\mathfrak{x}\rangle = \mathbb{H}^\times \circ_r \mathfrak{x}$, $0 \ne \mathfrak{x} = (x, y, z) \in \mathbb{H}^3$. The lines through $\langle 0, 1, -1\rangle$ are given by an equation $x \circ_r s + y + z = 0$ or

$x = 0$, and the other lines are obtained by applying the group $\mathrm{GL}_3\,\mathbb{C}$ from the right. If $r \neq r'$ then $\mathcal{H}_r \ncong \mathcal{H}_{r'}$.

(6) It can be verified that each *Hughes plane* $\mathcal{H}_r$ as described in step (5) is indeed an 8-dimensional topological plane. Because of the distinction between interior and exterior elements, the proof of the continuity properties is not so easy; see [68, §86] for details.

(7) Obviously, $\mathcal{H}_0$ is the classical Desarguesian quaternion plane. In all other cases, $\mathrm{GL}_3\,\mathbb{C}$ induces on $\mathcal{H}_r$ the full automorphism group Σ, and $\dim \Sigma = 17$. □

3.20. Remark. The Hughes planes are the only 8-dimensional topological planes with an automorphism group of dimension at least 17 which are not translation planes or dual translation planes ([67]).

There are 16-dimensional analoga of the planes $\mathcal{H}_r$ even though corresponding near-fields do not exist ([30, XI.8.8]):

3.21. Sixteen-dimensional Hughes planes. *If $\mathcal{B}$ is a Δ-invariant closed Baer subplane of $\mathcal{P}$ and if $\dim \Delta > 30$, then Δ has a subgroup $\Gamma \cong \mathrm{SL}_3\,\mathbb{H}$ and $\dim \Delta \leq 38$. For each $r \geq 0$ there is a unique 16-dimensional topological plane $\mathcal{O}_r$ admitting a non-trivial (and then even effective) action of Γ. The element $\beta \neq \mathbb{1}$ in the center of Γ is a Baer involution. $\mathcal{O}_0$ is the classical Moufang plane; in all other cases, $\dim \mathrm{Aut}\,\mathcal{O}_r = 36$.*

Proof. From $\dim \Delta > 26$ it follows that Δ is a Lie group; see [58, Th. L]. Write again $\Delta^* = \Delta|_{\mathcal{B}} = \Delta/\Phi$ for the group induced on $\mathcal{B}$. By 3.16(a), the kernel Φ has dimension $\dim \Phi \leq 3$, and $\dim \Delta^* > 27$. As shown in [68, (83.26)], the stabilizer of a point or a line in $\mathrm{Aut}\,\mathcal{B}$ is at most 27-dimensional, and [68, Th. 84.27] implies that $\mathcal{B}$ is classical and that $\Delta^* \cong \mathrm{PSL}_3\,\mathbb{H} \cong \Gamma/\langle\beta\rangle$. In particular, Γ acts non-trivially on $\mathcal{P}$. Now the claim is a consequence of [68, (86.31) and (86.34–37)]. Originally, the proof is due to Hähl [32]. □

Remark. *Each reflection of $\mathcal{B}$ is induced by a reflection of $\mathcal{O}_r$, and $\mathcal{O}_r$ is a semi-translation plane; cf.* 1.15.

Proof. By [68, (55.40)] there is no non-trivial action of $\mathrm{SO}_5\,\mathbb{R}$ on any topological plane. Hence the central involution σ of a subgroup $\mathrm{U}_2\,\mathbb{H} \cong \mathrm{Spin}_5\,\mathbb{R}$ of Γ is a reflection of $\mathcal{O}_r$, and each reflection of $\mathcal{B}$ extends to a conjugate of σ. Because the product of different reflections with the same axis is a translation, each translation of $\mathcal{B}$ is induced by a translation of $\mathcal{O}_r$. □

For related characterizations of the 16-dimensional Hughes planes see [67].

3.22. Corollary. *If again* $\mathcal{B}^\Delta = \mathcal{B} \lessdot \mathcal{P}$ *and if* $\dim \Delta = 38$*, then* $\mathcal{P} \cong \mathcal{P}_{\mathbb{O}}$ *is classical.*

A *direct proof* of this result is given in [65, Th. 1].

In contrast to 2.5, closed Baer subplanes of a topological plane are never disjoint. This is a consequence of the following result by Löwen:

3.23. Theorem. *If* $\mathcal{B}$ *is a closed Baer subplane of* $\mathcal{P}$ *and if* L_x *denotes the unique interior line through the exterior point* x*, then the map* $\pi = (x \mapsto L_x) : P \smallsetminus B \to \mathfrak{B}$ *is a locally trivial fibering which does not admit a cross section.*

A proof can be found in [47]. It uses the version of Lefschetz duality given in Bredon's book on sheaf theory and will not be reproduced here. If the point space B of $\mathcal{B}$ is a manifold, the ordinary Lefschetz duality theorem [72, 6.2.19] suffices.

3.24. Corollary. *Two distinct closed Baer subplanes of a topological plane have least one point in common. Dually, there is also a common interior line.*

Proof. (1) Assume that $\mathcal{B}$ and $\mathcal{B}'$ are closed Baer subplanes of $\mathcal{P}$ with disjoint point sets B and B', respectively. If the subplanes have no interior line in common, then each line S with $S \cap B \in \mathfrak{B}$ intersects B' in a unique point $x = \sigma(S)$ and $\pi(x) = S$. Thus $\pi \circ \sigma = \mathbb{1}$ and σ would be a cross section for $\pi : P \smallsetminus B \to \mathfrak{B}$. Therefore, the planes $\mathcal{B}$ and $\mathcal{B}'$ have a common interior line, if their point sets are disjoint.

(2) If two Baer subplanes $\mathcal{B}$ and $\mathcal{B}'$ have a common interior line L but no common point, then each point $x \in B \smallsetminus L$ would lie on a unique interior line with respect to $\mathcal{B}'$ and this would set up a homeomorphism $B \smallsetminus L \approx \mathfrak{B}' \smallsetminus \{L\}$, but the first space is contractible ([68, (51.4)]), while the second is not; see 3.0(a) together with [68, (51.26)]. □

The possibilities for the intersection $\mathcal{B} \cap \mathcal{B}'$ are known only in the following case:

3.25. Intersections. *Let* α *and* β *be distinct commuting continuous Baer involutions of a topological plane* $\mathcal{P}$*, and put* $\mathcal{F}_\beta = \mathcal{B} = (B, \mathfrak{B})$*. Then either* $\mathcal{F}_\alpha \cap \mathcal{F}_\beta \lessdot \mathcal{B}$ *and* $\alpha\beta$ *is a Baer involution, or* $\alpha\beta$ *is a reflection of* $\mathcal{P}$ *with axis* L *and center* c *and the point set of* $\mathcal{F}_\alpha \cap \mathcal{F}_\beta$ *is exactly* $(L \cap B) \cup \{c\}$*.*

Proof. By 3.15 and 3.0(c), the collineation α induces on $\mathcal{B}$ either a Baer involution or a reflection. In the first case $\alpha\beta$ is also a Baer involution. In the second case α fixes in B the points of a line L and one additional point c, moreover, α and β act freely on $L \smallsetminus B$. As in the proof of 3.15, it follows that $\alpha|_L = \beta|_L$. Hence $\alpha\beta$ has axis L. □

References

[1] A. A. Albert, *The finite planes of Ostrom*, Bol. Soc. Mat. Mexicana (2) **11** (1966), 1–13. MR 37#6826

[2] M. Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 1986. MR **89b:**20001

[3] R. Baer, *Projectivities with fixed points on every line of the plane*, Bull. Amer. Math. Soc. **52** (1946), 273–286. MR 7,387e

[4] ______, *Die Automorphismengruppe eines algebraisch abgeschlossenen Körpers der Charakterkistik* 0, Math. Z. **117** (1970), 7–17. MR 42#7638

[5] A. Barlotti, *On the definition of Baer subplanes of infinite planes*, J. Geometry **3** (1973), 87–92. MR 49#6022

[6] A. Beutelspacher, *On Baer subspaces of finite projective spaces*, Math. Z. **184** (1983), 301–319. MR **85e:**51019

[7] A. Beutelspacher and J. Ueberberg, *On the intersection of Baer subspaces*, Arch. Math. (Basel) **56** (1991), 203–208. MR **92c:**51014

[8] M. Biliotti, *A Dembowski generalisation of the Hughes planes*, Boll. Un. Mat. Ital. B (5) **16** (1979), 674–693. MR **84d:**51024

[9] M. Biliotti and N. L. Johnson, *Maximal Baer groups in translation planes and compatibility with homology groups*, Geom. Dedicata **59** (1996), 65–101. MR **97d:**51003

[10] M. Biliotti and G. Korchmáros, *Blocking sets which are preserved by transitive collineation groups*, Forum Math. **4** (1992), 567–591. MR **93m:**51008

[11] A. Blokhuis, *Blocking sets in Desarguesian planes*, Combinatorics, Vol. 2 (Keszthely, 1993), János Bolyai Math. Soc., Budapest, 1996, pp. 133–155. MR **97h:**51012

[12] R. Bödi, *Smooth Hughes planes are classical*, Arch. Math. (Basel) **73** (1999), 73–80. MR **2000i:**51038

[13] A. Bonisoli, G. Korchmáros, and T. Szönyi, *Some multiply derived translation planes with* $\mathrm{SL}(2,5)$ *as an inherited collineation group in the translation complement*, Des. Codes Cryptogr. **10** (1997), 109–114. MR **98b:**51009

[14] G. E. Bredon, *Some theorems on transformation groups*, Ann. of Math. (2) **67** (1958), 104–118. MR 20#83

[15] A. Bruen, *Baer subplanes and blocking sets*, Bull. Amer. Math. Soc. **76** (1970), 342–344. MR 40#4856

[16] ______, *Blocking sets in finite projective planes*, SIAM J. Appl. Math. **21** (1971), 380–392. MR 46#2543

[17] A. A. Bruen and R. Silverman, *Arcs and blocking sets. II*, European J. Combin. **8** (1987), 351–356. MR **89d:**05043

[18] A. A. Bruen and J. A. Thas, *Blocking sets*, Geom. Dedicata **6** (1977), 193–203. MR 57#1256

[19] J. Cofman, *Baer subplanes of affine and projective planes*, Math. Z. **126** (1972), 339–344. MR 46#4359

[20] ______, *Baer subplanes and Baer collineations of derivable projective planes*, Abh. Math. Sem. Univ. Hamburg **44** (1975), 187–192. MR 53#3865

[21] P. M. Cohn, *Skew fields. Theory of general division rings*, Encyclopedia of Mathematics and its Applications, vol. 57, Cambridge University Press, Cambridge, 1995. MR **97d:**12003

[22] P. Dembowski, *Finite geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 44, Springer-Verlag, Berlin-New York, 1968. MR 38#1597

[23] ______, *Generalized Hughes planes*, Canad. J. Math. **23** (1971), 481–494. MR 45#4272

[24] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R. Remmert, *Numbers*, Graduate Texts in Mathematics, vol. 123, Springer-Verlag, New York, 1990. MR **91h:**00005

[25] L. Evens, *The cohomology of groups*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991. MR **93i:**20059

[26] A. Fathi, *An orbit closing proof of Brouwer's lemma on translation arcs*, Enseign. Math. (2) **33** (1987), 315–322. MR **89d:**55004

[27] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Univ. Utrecht, 1951; reprinted in Geom. Dedicata **19** (1985), 7–63. MR 13,433; MR **86k:**17018b

[28] T. Grundhöfer, *Eine Charakterisierung ableitbarer Translationsebenen*, Geom. Dedicata **11** (1981), 177–185. MR **82j:**51007

[29] E. G. Goodaire and M. J. Kallaher, *Systems with two binary operations, and their planes*, Quasigroups and Loops: Theory and Applications, Heldermann, Berlin, 1990, Chapter VII, pp. 161–195. MR 1 125 813

[30] T. Grundhöfer and H. Salzmann, *Locally compact double loops and ternary fields*, Quasigroups and Loops: Theory and Applications, Heldermann, Berlin, 1990, Chapter XI, pp. 313–355. MR 1 125 817

[31] L. Guillou, *Théorème de translation plane de Brouwer et généralisations du théorème de Poincaré-Birkhoff* [*Brouwer's plane translation theorem and generalizations of the Poincaré-Birkhoff theorem*], Topology **33** (1994), 331–351. MR **95h:**55003

[32] H. Hähl, *Charakterisierung der kompakten, zusammenhängenden Moufang-Hughes-Ebenen anhand ihrer Kollineationen*, Math. Z. **191** (1986), 117–136. MR **87d:**51027

[33] M. Hall, *Projective planes*, Trans. Amer. Math. Soc. **54** (1943), 229–277. MR 5,72c

[34] ———, *The theory of groups*, The Macmillan Co., New York, N.Y., 1959. MR 21#1996

[35] D. R. Hughes, *A class of non-Desarguesian projective planes*, Canad. J. Math. **9** (1957), 378–388. MR 19,444f

[36] D. R. Hughes and F. C. Piper, *Projective planes*, Graduate Texts in Mathematics, vol. 6, Springer-Verlag, New York-Berlin, 1973. MR 48#12278

[37] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Mathematical Series, vol. 4, Princeton University Press, Princeton, N.J., 1948. MR 3,312b

[38] N. L. Johnson, *Derivation in infinite planes*, Pacific J. Math. **43** (1972), 387–402. MR 48#7107

[39] ———, *A characterization of generalized Hall planes*, Bull. Austral. Math. Soc. **6** (1972), 61–67. MR 45#5861

[40] ———, *Subplane covered nets*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 222, Marcel Dekker, Inc., New York, 2000. MR **2001h:**51010

[41] H. Kestelman, *Automorphisms of the field of complex numbers*, Proc. London Math. Soc. (2) **53** (1951), 1–12. MR 12,812b

[42] H. Klein, N. Knarr, and R. Löwen, *Four-dimensional compact projective planes admitting an affine Hughes group*, Results Math. **38** (2000), 270–306. MR **2001h:**51022

[43] C. A. M. Klever, *On transitive Baer subplanes of generalized Hall planes*, Geom. Dedicata **9** (1980), 347–373. MR **81m:**51015

[44] N. Knarr, *Translation planes*, Lecture Notes in Mathematics, vol. 1611, Springer-Verlag, Berlin, 1995. MR **98e:**51019

[45] L. Kramer, *The point and line space of a compact projective plane are homeomorphic*, Math. Z. **234** (2000), 83–102. MR **2001c:**55009

[46] R. Löwen, *Affine Hughes groups acting on* 4*-dimensional compact projective planes*, Forum Math. **10** (1998), 435–451. MR **99f:**51026

[47] ———, *Nonexistence of disjoint compact Baer subplanes*, Arch. Math. (Basel) **73** (1999), 235–240. MR **2000g:**51012

[48] H. Lüneburg, *Characterizations of the generalized Hughes planes*, Canad. J. Math. **28** (1976), 376–402. MR 53#3874

[49] ———, *Translation planes*, Springer-Verlag, Berlin-New York, 1980. MR **83h:**51008

[50] H. Mäurer, *Die affine Projektivitätengruppe der Hallebenen* [*The affine group of projectivities of the Hall planes*], Aequationes Math. **32** (1987), 271–273. MR **88g:**51004

[51] T. A. McKee, *A formal analogy between Baer subplanes and their complements*, J. Geometry **6** (1975), 97–104. MR 52#9072

[52] F. R. Moulton, *A simple non-Desarguesian plane geometry*, Trans. Amer. Math. Soc. **3** (1902), 192–195. MR 1 500 595

[53] T. G. Ostrom, *Semi-translation planes*, Trans. Amer. Math. Soc. **111** (1964), 1–18. MR 28#2472

[54] ———, *A characterization of the Hughes planes*, Canad. J. Math. **17** (1965), 916–922. MR 32#392

[55] ———, *Vector spaces and construction of finite projective planes*, Arch. Math. (Basel) **19** (1968), 1–25. MR 37#2081

[56] T. G. Ostrom and A. Wagner, *On projective and affine planes with transitive collineation groups*, Math. Z. **71** (1959), 186–199. MR 22#1843

[57] A. R. Pears, *Dimension theory of general spaces*, Cambridge University Press, Cambridge, England-New York-Melbourne, 1975. MR 52#15405

[58] B. Priwitzer and H. Salzmann, *Large automorphism groups of* 16*-dimensional planes are Lie groups*, J. Lie Theory **8** (1998), 83–93. MR **99f:**51027

[59] O. Prohaska, *Endliche ableitbare affine Ebenen*, Geom. Dedicata **1** (1972), 6–17. MR 46#4357

[60] A. Rahilly, *On tangentially transitive projective planes*, Geom. Dedicata **13** (1982), 173–194. MR **84d:**51012

[61] L. A. Rosati, *Unicità e autodualità dei piani di Hughes*, Rend. Sem. Mat. Univ. Padova **30** (1960), 316–327. MR 24#A3553

[62] H. Salzmann, *Topological Planes*, Advances in Math. **2** (1967), 1–60. MR 36#3201

[63] ———, *Baer-Unterebenen* 4*-dimensionaler Ebenen*, Arch. Math. (Basel) **23** (1972), 337–341. MR 47#7588

[64] ———, *Reelle Kollineationen der komplexen projektiven Ebene*, Geom. Dedicata **1** (1973), 344–348. MR 47#5711

[65] ———, *Baer-Kollineationsgruppen der klassischen projektiven Ebenen* [*Baer collineation groups of the classical projective planes*], Arch. Math. (Basel) **38** (1982), 374–377. MR **84b:**51027

[66] ———, *Compact* 8*-dimensional projective planes*, Forum Math. **2** (1990), 15–34. MR **90k:**51028

[67] ———, *Characterization of* 16*-dimensional Hughes planes*, Arch. Math. (Basel) **71** (1998), 249–256. MR **99i:**51013

[68] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. Stroppel, *Compact projective planes*, de Gruyter Expositions in Mathematics, vol. 21, Walter de Gruyter & Co., Berlin, 1995. MR **97b:**51009

[69] B. Schnor, *Involutions in the group of automorphisms of an algebraically closed field*, J. Algebra **152** (1992), 520–524. MR **93k:**12010

[70] P. A. Smith, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415. MR 23#A2880

[71] T. Soundararajan, *On the cardinality of the group of automorphisms of algebraically closed extension fields*, Acta Math. Vietnam. **16** (1991), 147–153. MR **93f:**12009

[72] E. H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966. MR 35#1007

[73] E. Steinitz, *Algebraische Theorie der Körper*, J. Reine Angew. Math. **137** (1910), 167–309; Chelsea Publishing Co., New York, N.Y., 1950 (ed. R. Baer and H. Hasse). MR 12,238d

[74] M. Sved, *Baer subspaces in the n-dimensional projective space*, Combinatorial mathematics, X (Adelaide, 1982), Lecture Notes in Math., vol. 1036, Springer, Berlin, 1983, pp. 375–391. MR **85i:**51004

[75] T. Szönyi, *Blocking sets in Desarguesian affine and projective planes*, Finite Fields Appl. **3** (1997), 187–202. MR **98h:**51019

[76] J. Ueberberg, *Projective planes and dihedral groups*, Discrete Math. **174** (1997), 337–345. MR **98i:**51005

[77] J. Tits, *Sur les groupes doublement transitifs continus: correction et compléments*, Comment. Math. Helv. **30** (1956), 234–240. MR 17,1109b

[78] H. Zassenhaus, *Über endliche Fastkörper*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 187–220.

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

E-mail address: `helmut.salzmann@uni-tuebingen.de`