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THE RESIDUAL FINITENESS OF ASCENDING HNN-EXTENSIONS OF CERTAIN SOLUBLE GROUPS

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To the memory of Reinhold Baer on the 100th anniversary of his birth

Abstract. If G is a group with an injective endomorphism ϕ, then the HNN-extension $G_\phi = \langle G, t : t^{-1}gt = g\phi \text{ for all } g \in G\rangle$ is called the ascending HNN-extension of G determined by ϕ. We prove that G_ϕ is residually finite when G is either finitely generated abelian-by-polycyclic-by-finite or reduced soluble-by-finite minimax. We also provide an example of a 3-generator residually finite soluble group G of derived length 3 with a non-residually-finite ascending HNN-extension.

1. Introduction

Let G be a group with an injective endomorphism ϕ, and consider the HNN-extension

$$G_\phi = \langle G, t : t^{-1}gt = g\phi \text{ for all } g \in G\rangle.$$

The problems of the Hopficity and the residual finiteness of G_ϕ, for various classes of finitely generated groups G, have been the subject of renewed interest in recent years (see [2] and [4] for background and comprehensive bibliographies). When G is polycyclic-by-finite, Baumslag and Bieri [1] have shown that the ascending HNN-extension G_ϕ is residually finite (a new proof of this was recently given by Hsu and Wise [4] using entirely different techniques). In this paper, we show that this result can be extended to larger classes of soluble groups, but not to all. To be precise, our main results are as follows.

Theorem 1.1. *Let G be a finitely-generated abelian-by-polycyclic-by-finite group, and let ϕ be an injective endomorphism of G. Then G_ϕ is residually finite.*

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The residual finiteness of the base group G was proved by Roseblade [8]. The following shows that the result does not generalize to all finitely generated residually finite soluble groups.

THEOREM 1.2. *There exists a 3-generator residually-finite soluble group G of derived length 3, and a monomorphism $\phi : G \to G$, such that the ascending HNN-extension G_ϕ is not residually finite.*

In the positive direction, we have the following result (which generalizes the result for a polycyclic base group in a direction different from Theorem 1.1).

THEOREM 1.3. *Let G be a finitely generated, residually finite, soluble-by-finite group of finite Prüfer rank. Then G_ϕ is residually finite for every injective endomorphism ϕ of G.*

This is a consequence of the following more general result. We thank Derek Robinson for suggesting the result and supplying its proof.

THEOREM 1.4. *Let G_ϕ be an ascending HNN-extension of a reduced soluble-by-finite minimax group G. Then G_ϕ is again a reduced and soluble-by-finite minimax group, and as such it is residually finite.*

Recall that "reduced" means that there are no non-trivial radicable subgroups and is equivalent to residually finite for soluble groups of finite rank (see [5, p. 170]). Since a finitely generated soluble group with finite Prüfer rank is minimax [6], Theorem 1.4 implies Theorem 1.3. On the other hand, one cannot extend the theorem to groups of finite Prüfer rank because of the simple example $G_\phi = \langle G, t \mid g^t = 2g,\ g \in G\rangle$, where $G = \{m/n \mid m, n \in \mathbb{Z},\ n \text{ odd}\}$.

Unlike the groups in Theorems 1.1 and 1.4, the counterexample in Theorem 1.2 is not polycyclic-by-finite modulo its Fitting subgroup. This prompts the following question.

QUESTION. Is every ascending HNN-extension of a finitely generated, residually finite, nilpotent-by-polycyclic-by-finite group G residually finite?

An affirmative answer would, in particular, imply that every ascending HNN-extension G_ϕ of a finitely generated soluble linear group G is residually finite.

The paper is organized as follows. In Section 2, necessary and sufficient conditions for the residual finiteness of a general HNN-extension G_ϕ are stated, and the relationship between the normal subgroups of G and those of G_ϕ is elucidated. Theorems 1.1–1.4 are proved in Section 3.

2. Generalities

Given a pair (G, ϕ) consisting of a group G and an injective endomorphism ϕ of G, define the following:

$$\begin{aligned}
X(G,\phi) &= \{N \triangleleft G \,:\, N\phi \subseteq N\},\\
Y(G,\phi) &= \{N \triangleleft G \,:\, N\phi = N \cap G\phi\},\\
Y_f(G,\phi) &= \{N \in Y(G,\phi) \,:\, |G:N| < \infty\},\\
R(G,\phi) &= \bigcap_{P \in Y_f(G,\phi)} P.
\end{aligned}$$

Clearly $X(G, \phi)$ and $Y(G, \phi)$ are closed under arbitrary intersections. In particular, $R(G, \phi) \in Y(G, \phi)$.

Theorem 2.1. *The HNN-extension $G_\phi = \langle G, t \,:\, t^{-1}gt = g\phi$ for all $g \in G\rangle$ is residually finite if and only if $R(G, \phi) = 1$.*

Proof. It is well-known (and trivial to verify) that $P \in Y_f(G, \phi)$ if and only if $P = G \cap Q$ for some $Q \triangleleft_f G_\phi$. The result follows from this, and the simple fact that HNN-extensions with a finite base group are residually finite. □

If $N \in X(G, \phi)$, then ϕ_N denotes the endomorphism induced on G/N by ϕ. Clearly, ϕ_N is injective if and only if $N \in Y(G, \phi)$. The next two results are formal exercises.

Lemma 2.2. *Let $N \in Y(G, \phi)$. Then:*

(a) *$X(G/N, \phi_N) = \{M/N \,:\, M \in X(G, \phi)$ and $M \supseteq N\}$, and for $* =$ blank or f, we have $Y_*(G/N, \phi_N) = \{M/N \,:\, M \in Y_*(G, \phi)$ and $M \supseteq N\}$.*

(b) *$R(G/N, \phi_N) \supseteq R(G, \phi)N/N$. In particular, if $(G/N)_{\phi_N}$ is residually finite, then $R(G, \phi) \subseteq N$.*

(c) *If $R(G, \phi) \neq 1$ and $N \subseteq R(G, \phi)$, then $R(G/N, \phi_N) = R(G, \phi)/N$.* □

Lemma 2.3. *For any $M \triangleleft G$, define a sequence of subgroups M_i by $M_i\phi^i = M \cap G\phi^i$ for all $i \geq 0$. Then:*

(a) *$M_i \triangleleft G$ for all i.*

(b) *$M_{i+1}\phi = M_i \cap G\phi$ for all i.*

(c) *If $M \subseteq N \in Y(G, \phi)$, then $M_i \subseteq N$ for all i.* □

Lemma 2.4. *Let $M \in X(G, \phi)$, and set $\widetilde{M} = \bigcup_{i \geq 0} M_i$. Then:*

(a) *$M_i \subseteq M_{i+1}$ for all i, so $\widetilde{M}$ is a normal subgroup of G.*

(b) *$\widetilde{M} \in Y(G, \phi)$.*

(c) *If $L \leqslant G$ with $L\phi = L \cap G\phi$, then $M \subseteq L$ implies that $\widetilde{M} \subseteq L$.*

(d) *$\widetilde{M} = M_j$ for some j if and only if $\widetilde{M}\phi^j \subseteq M$.*

Proof. By definition $M_1\phi = M \cap G\phi \supseteq M_0\phi$, so $M_1 \supseteq M_0$. If $M_{i+1} \supseteq M_i$ for some i, then by Lemma 2.3(b), $M_{i+2}\phi = M_{i+1} \cap G\phi \supseteq M_i \cap G\phi = M_{i+1}\phi$, so $M_{i+2} \supseteq M_{i+1}$ as well. Thus, (a) follows by induction on i. As for (b), we have $\widetilde{M}\phi = \bigcup_{i\geq 1} M_i\phi = \bigcup_{i\geq 1}(M_{i-1} \cap G\phi) = \widetilde{M} \cap G\phi$.

For (c), it is obvious that $L\phi^i = L \cap G\phi^i$ for all i, so $M_i\phi^i = M \cap G\phi^i \subseteq L \cap G\phi^i = L\phi^i$, whence $M_i \subseteq L$ for all i.

Finally for (d), begin by observing that $M_i\phi^{i+1} = M_i\phi^i\phi = (M \cap G\phi^i)\phi = M\phi \cap G\phi^{i+1} = M\phi \cap M \cap G\phi^{i+1} = M\phi \cap M_{i+1}\phi^{i+1}$, so $M_i\phi^i = M \cap M_{i+1}\phi^i$. Thus, if $\widetilde{M} = M_j = M_{j+1}$, then $\widetilde{M}\phi^j = M_j\phi^j = M \cap M_{j+1}\phi^j = M \cap \widetilde{M}\phi^j$, so $\widetilde{M}\phi^j \subseteq M$. Conversely, if $\widetilde{M}\phi^j \subseteq M$, then $\widetilde{M}\phi^j = \widetilde{M}\phi^j \cap G\phi^j \subseteq M \cap G\phi^j = M_j\phi^j$, so $\widetilde{M} \subseteq M_j \subseteq \widetilde{M}$. □

LEMMA 2.5. *Let G be a finitely generated group.*

(a) *If $M \triangleleft_f G$, then there exists $j \geq 0$ such that $N = \bigcap_{i\geq 0} M_i = \bigcap_{i=0}^{j} M_i$. In particular, $N \triangleleft_f G$ and $N\phi \subseteq N$.*
(b) *Suppose that G is also residually finite. If $Q \triangleleft G$, $Q\phi \subseteq Q \subseteq R(G,\phi)$, and Q is finite, then $Q = 1$.*

Proof. (a) We have $|G : M_i| = |G\phi^i : M_i\phi^i| = |G\phi^i : M \cap G\phi^i| \leqslant |G : M|$. As G has only a finite number of subgroups of index at most $|G : M|$, the set $\{M_i : i \geq 0\}$ is finite, so N is a finite intersection and so in particular has finite index in G. By Lemma 2.3(b), we have $N \cap G\phi = \bigcap_{i\geq 0} M_i \cap G\phi = \bigcap_{i\geq 0} M_{i+1}\phi \supseteq N\phi$, as required.

(b) Of course $Q\phi \subseteq Q$ implies that $Q\phi = Q$, because Q is finite. As G is residually finite, there exists $M \triangleleft_f G$ such that $M \cap Q = 1$, and by part (a) we may assume that $M\phi \subseteq M$. Then $\widetilde{M} \in Y_f(G,\phi)$, so $Q \subseteq R(G,\phi) \subseteq \widetilde{M}$ by definition. On the other hand, the chain $M = M_0 \subseteq M_1 \subseteq \cdots$ is finite, so $\widetilde{M} = M_j$ for some j, whence $\widetilde{M}\phi^j \subseteq M$ by the previous lemma. But then $Q = Q\phi^j \subseteq \widetilde{M}\phi^j \subseteq M$, so $Q = M \cap Q = 1$, as required. □

From time to time, we have to deal with subgroups $N \triangleleft G$ such that $N \cap G\phi \subseteq N\phi$, i.e., the reverse inclusion to that defining the elements of $X(G,\phi)$ (such subgroups, for example, include $\zeta_1(G)$, the centre of G). We have the following.

LEMMA 2.6. *Let $M \triangleleft G$ such that $M \cap G\phi \subseteq M\phi$. Then:*

(a) *$M \supseteq M_1 \supseteq M_2 \supseteq \cdots$, and the subgroup $\widehat{M} = \bigcap_{i\geq 0} M_i$ is the unique maximal element of $Y(G,\phi)$ contained in M.*
(b) *Assume that G/M is polycyclic-by-finite, with a torsion-free subgroup of finite index n. Then there exists $j \geq 0$ such that $M_j \in Y(G,\phi)$ and $|M/M_j| \leqslant n$. Moreover, G/M_j is isomorphic to a subgroup of*

G/M. Furthermore, there exists $H \in Y_f(G,\phi)$ such that H/M_j is torsion-free (so in particular $H \cap M = M_j$).

(c) *If G/M is finite, then $M \in Y(G,\phi)$.*

(d) *Let H be a subgroup of finite index in G such that $H\phi = H \cap G\phi$. Then the normal core $H_G \in Y_f(G,\phi)$. In particular, $R(G,\phi) = R(H,\phi)$.*

Proof. By definition, $M_1\phi = M \cap G\phi \subseteq M\phi$, so $M_1 \subseteq M_0$. If $M_{i+1} \subseteq M_i$ for some i, then $M_{i+2}\phi = M_{i+1} \cap G\phi \subseteq M_i \cap G\phi = M_{i+1}\phi$, so $M_{i+2} \subseteq M_{i+1}$. This proves the first statement of (a). Also, $\widehat{M} \cap G\phi = \cap_{i\geq 0}(M_i \cap G\phi) = \cap_{i\geq 0} M_{i+1}\phi = \widehat{M}\phi$. Conversely, if $N \in Y(G,\phi)$ and $N \subseteq M$, then $M_i\phi^i = M \cap G\phi^i \supseteq N \cap G\phi^i = N\phi^i$, so $N \subseteq \widehat{M}$.

Next, by part (a) we have $M \cap G\phi^i = M_i\phi^i \subseteq M\phi^i$, so $M \cap M\phi^i = M \cap G\phi^i$ for all $i \geq 0$. Therefore, $G/M_i \cong G\phi^i/M_i\phi^i \hookrightarrow G/M$. On the other hand, $M_i \subseteq M$, so G/M is a homomorphic image of G/M_i. It follows that G/M and G/M_i have the same Hirsch length, and so M/M_i, being the kernel of the map $G/M_i \twoheadrightarrow G/M$, is finite. In particular, $|M/M_i| \leqslant n$ for all i. This, and the fact that $M \supseteq M_1 \supseteq M_2 \supseteq \cdots$, evidently imply the existence of $j \geq 0$ such that $M_{j+1} = M_j$, which is equivalent to $M_j \in Y(G,\phi)$. The other properties of M_j have already been established in the course of the proof.

We now turn to the existence of H. Write $P = M_j \in Y(G,\phi)$. The polycyclic-by-finite group G/P contains a torsion-free subgroup T/P of finite index. In particular, $T \cap M = P$ as M/P is finite. For any i, we have $(T_i \cap M)\phi^i = T_i\phi^i \cap M\phi^i = T \cap G\phi^i \cap M\phi^i = T \cap M\phi^i \supseteq T \cap M \cap G\phi^i = P \cap G\phi^i = P\phi^i$, so $P \subseteq \bigcap_{i\geq 0} T_i = N$, say. By Lemma 2.5(b), we have $N\phi \subseteq N$ and $N \lhd_f G$. By construction, N/P is torsion-free. Thus, the set $S = \{N/P : N \in X(G,\phi),\ P \subseteq N \lhd_f G,\ N/P \text{ is torsion-free}\}$ is non-empty. Let H/P be a maximal element of the above set (i.e., one with minimal index in G/P). It is sufficient to show that $H \in Y(G,\phi)$. As $H \lhd_f G$, we have $\widetilde{H} = H_k$ for some $k \geq 0$, and so $\widetilde{H}\phi^k \subseteq H$ by Lemma 2.4(d). Then $(\widetilde{H}\phi^k)P/P$ is torsion-free, being a subgroup of H/P. But $(\widetilde{H}\phi^k)P/P \cong \widetilde{H}\phi^k/(\widetilde{H}\phi^k \cap P)$, and $\widetilde{H}\phi^k \cap P = \widetilde{H} \cap G\phi^k \cap P = \widetilde{H}\phi^k \cap P\phi^k = (\widetilde{H} \cap P)\phi^k = P\phi^k$. Therefore, $(\widetilde{H}\phi^k)P/P \cong \widetilde{H}\phi^k/P\phi^k \cong \widetilde{H}/P$. Thus, $\widetilde{H}/P$ is torsion-free, so $\widetilde{H}$ belongs to S. The maximal choice of H now implies that $H = \widetilde{H} \in Y(G,\phi)$, as claimed.

The proof of (c) is much simpler, because $|G : M_1| = |G\phi : M_1\phi| \leqslant |G : M| \leqslant |G : M_1|$, so again $M = M_1$.

For (d), let $M = H_G = \bigcap_{x\in G} H^x \lhd_f G$. Then $M\phi = \bigcap_{x\in G}(H\phi)^{x\phi} = \bigcap_{x\in G}(H \cap G\phi)^{x\phi} = \bigcap_{y\in G\phi} H^y \cap G\phi \supseteq M \cap G\phi$. Therefore, $M \in Y_f(G,\phi)$ by part (c). In particular, if $P \in Y_f(H,\phi)$, then $P\phi = P \cap H\phi = P \cap H \cap G\phi = P \cap G\phi$. By the first part of (d), we have $P_G \in Y_f(G,\phi)$, so $R(G,\phi) \subseteq P$. This is for all such P, so $R(G,\phi) \subseteq R(H,\phi)$. Conversely, if $N \in Y_f(G,\phi)$, then $N \cap H \in Y_f(H,\phi)$, so $R(H,\phi) \subseteq H \cap N$. Since this holds for all N, we have $R(H,\phi) \subseteq R(G,\phi)$, as required. ∎

Lemma 2.7. *Let F be a non-trivial nilpotent group of class c, and let ϕ be an injective endomorphism of F. Then $\widetilde{\gamma_c(F)} \subseteq \widehat{\zeta_1(F)}$.*

Proof. There is nothing to prove if $c = 1$, so assume that $c \geq 2$. Write $A = \gamma_c(F)$ and $Z = \zeta_1(F)$. Then $A \subseteq Z$, so $A_i \subseteq Z_i$ for all i, and plainly $A\phi \subseteq A$. Fix i, and consider any $j \geq i$. Then $A_i \subseteq A_j \subseteq Z_j$, so $A_i \subseteq \bigcap_{j\geq i} Z_j = \widehat{Z}$. Thus $\widetilde{A} \subseteq \widehat{Z}$, as required. □

3. Proof of the Main Theorems

For the proof of Theorem 1.1, we require the case $J = \mathbb{Z}$ of the following result of Roseblade ([7, Corollary C5, p. 321]):

Suppose J is a commutative Hilbert domain and G is polycyclic-by-finite. Suppose M is a finitely generated JG-module and ϕ an endomorphism of it. If $M\phi$ is contained in every maximal ϕ-invariant JG-submodule of M, then $M\phi^n = 0$ for some n.

Proof of Theorem 1.1. Recall that finitely generated abelian-by-polycyclic-by-finite groups satisfy the maximal condition on normal subgroups (see [3]). Let G be a minimal counterexample so that for some injective endomorphism ϕ of G, the group G_ϕ is not residually finite, but every ascending HNN extension of each proper quotient of G is residually finite. It then follows that $R(G, \phi)$ is the unique minimal non-trivial member of the set $Y(G, \phi)$. Let $F = \mathrm{Fitt}(G)$ be the product of all the nilpotent normal subgroups of G. Then F is nilpotent of class c for some $c \geq 1$. Since G/F is polycyclic-by-finite, parts (b) and (d) of Lemma 2.6 apply with F playing the role of M in that lemma. In particular, G may be replaced by a subgroup $H \in Y_f(G, \phi)$, and we may assume that $F \in Y(G, \phi)$. Let $B = \widetilde{\gamma_c(F)}$. Then $B \subseteq \widehat{\zeta_1(F)} \subseteq \zeta_1(F)$ by Lemma 2.7, so $A = R(G, \phi) \subseteq \zeta_1(F)$.

Now let $N \subset A$ be any maximal element of $X(G, \phi)$. Then $N \neq N_1$ and $N_1 \in X(G, \phi)$, so $N_1 = A$. By definition, this means that $A\phi = N_1\phi = N \cap A\phi$, so $A\phi \subseteq N$. By the above-quoted result of Roseblade, this implies that $A\phi^n = 1$ for some integer n, which is impossible as $A \neq 1$ and ϕ is injective. □

Proof of Theorem 1.2. Let A be the additive subgroup of $\mathbb{Q}$ consisting of all the rational numbers with odd denominator (in lowest form), and let $B = Dr_{i\in Z} A_i$ be the direct product of countably many copies of A indexed by the integers. A typical element $b \in B$ is written as $b = (b_i) = (b_i)_i$, the outside subscript indicating the position of the entry b_i (where necessary).

Let $\mu : B \to B$ be the map defined by $(b\mu)_i = (2i + 1)b_i$ for all i and all $b \in B$. Note that $\mu \in \mathrm{Aut}(B)$. Next, let $\nu : B \to B$ be the shift automorphism defined by $(b\nu)_i = b_{i-1}$ for all i. Let $H = \langle \mu, \nu \rangle \leqslant \mathrm{Aut}(B)$ and $G = B \rtimes H$. We begin by establishing some properties of H and G.

1. $H = \langle\mu\rangle \wr \langle\nu\rangle \cong C_\infty \wr C_\infty$*:* For any $b \in B$ and any integer k, we have $b\mu\nu^{-k}\mu\nu^k = ((2i+1)b_i)_i\nu^{-k}\mu\nu^k = ((2(i+k)+1)b_i)_i\mu\nu^k = ((2i+1)(2(i+k)+1)b_i)_i\nu^k = ((2(i-k)+1)(2i+1)b_i)_i$, while $b\nu^{-k}\mu\nu^k\mu = (b_{i+k})_i\mu\nu^k\mu = ((2i+1)b_{i+k})_i\nu^k\mu = ((2(i-k)+1)b_i)_i\mu = ((2(i-k)+1)(2i+1)b_i)_i$. In other words, the normal closure $\langle\mu^H\rangle$ is abelian. The verification that H is the indicated wreath product is now trivial.

2. G is a 3*-generator group:* Let $c \in B$ be the element with $c_0 = 1$ and $c_i = 0$ for all $i \neq 0$. Then for any odd integer $2k+1$, we have $c\nu^k\mu^{-1}\nu^{-k} = (c_i/(2(i+k)+1))_i$, which has $1/(2k+1)$ in the 0-th component, and 0 elsewhere. Thus, the normal closure $\langle c^G\rangle$ contains A_0, and as ν is the shift map, we must have $B = \langle c^G\rangle$. Thus $G = \langle c, u, v\rangle$, where conjugation by u (respectively v) induces μ (resp. ν) on B.

3. G is soluble of length 3*:* This is clear from Point 1.

4. G is residually finite: For any positive integer k, the subgroup $2^kB \lhd G$, and $B/2^kB$ is the direct product of countably many cyclic groups of order 2^k. Moreover, $\bigcap_{k\geq 0} 2^kB = 1$. In order to show that G is residually finite, it suffices to show that $G/2^kB$ is residually finite for every positive integer k. Let $m = 2^{k-1}$. Then for any $b \in B$, so $b\mu^{-1}\nu^{-m}\mu\nu^m = ((\frac{2(i-m)+1}{2i+1})b_i)_i$. But $2m = 2^k$, so $2(i-m)+1 \equiv 2i+1 \pmod{2^k}$, and therefore $b\mu^{-1}\nu^{-m}\mu\nu^m \equiv b \pmod{2^kB}$. In other words, $[u, v^m] \in C = C_G(B/2^kB)$. Thus, $C \cap H \lhd G$ and contains $[u, v^m]$. Now $H/(C\cap H)$ is polycyclic of Hirsch length at most $m+1$, so the group $W = G/2^kB(C\cap H)$ is finitely generated and abelian-by-polycyclic-by-finite, and so residually finite. In particular, as $B/2^kB$ embeds into W, every element of $B/2^kB$ can be excluded from some normal subgroup of finite index in W, and hence in $G/2^kB$. Finally, any $g \in G \setminus B$ has non-trivial image in $G/B = H$, and so can be excluded from a normal subgroup of finite index because H is residually finite. (Note that G is, in fact, residually a finite 2-group.)

Now define $\phi : G \to G$ by $(bh)\phi = (2b)h$ for all $b \in B, h \in H$. This is plainly injective, and is a homomorphism of G because it is the identity on H, and the action of H on B commutes with the action of ϕ on B. In the HNN-extension $G_\phi = \langle G, t : t^{-1}gt = g\phi \text{ for all } g \in G\rangle$, the subgroup $\bigcup_{n\in\mathbb{Z}} B^{t^n}$ is the direct product of countably many copies of $(\mathbb{Q}, +)$, which is divisible and hence not residually finite. In particular, G_ϕ is not residually finite, as required. □

Proof of Theorem 1.4. Let $H = \langle t, G \mid g^t = g^\phi,\ g \in G\rangle$, where G is a reduced soluble-by-finite minimax group and ϕ is an injective endomorphism of G. Write $\bar{G} = \bigcup_{i=0,1,2,\ldots} G_{(i)}$ where $G_{(i)} = G^{t^{-i}}$. Of course $G = G_{(0)} \leqslant G_{(1)} \leqslant \cdots$.

(a) *We may assume that G is soluble.* For, there exists $m > 0$ such that $S = G^m$ is soluble, G/S is finite, and $S\phi \leqslant S$. Let $N = \widetilde{S}$. Clearly $N = S_j$

for some j, so $N\phi^j \subseteq S$ by 2.4(d), and hence N is soluble. Finally, $\langle t, N\rangle$ is an ascending HNN-extension of finite index in H.

(b) *H is soluble with finite Prüfer rank.* Clearly $\bar{G}$ is soluble, being the union of a chain of subgroups isomorphic with G. Hence H is soluble. Suppose that G has rank r. Then any finitely generated subgroup of $\bar{G}$ is contained in some $G_{(i)}$ and hence can be generated by r elements. Therefore $\bar{G}$ has finite rank, and hence so does H.

(c) *H is a minimax group.* First note that $|G : G\phi|$ is finite. For otherwise there would be the infinite descending chain $G > G\phi > G\phi^2 \dots$ in which each $|G\phi^i : G\phi^{i+1}|$ is infinite. This implies that G does not satisfy the weak minimal condition, thus contradicting an easy result of Zaicev (see [5, Section 10.3]). Write $m = |G : G\phi|$. Then $|G_{(j+1)} : G_{(j)}| = m$ for all j. Let A be an abelian subgroup of H. If $a \in A$, then $a \in G_{(j)}$ for some j and it follows that $a^{(m!)^j} \in A \cap G$. Hence $A/A \cap G$ is a π-group where π is the set of all primes not exceeding m. Since $A/A \cap G$ has finite rank and π is finite, it follows that $A/A \cap G$ satisfies the minimal condition, and so A is minimax.

(d) *H is reduced and hence residually finite.* For if H were not reduced, then it would have a subgroup of Prüfer type p^∞, which would have to be contained in $\bar{G}$. But in G, and hence in $\bar{G}$, torsion elements have bounded order, a contradiction. □

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