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ON SANOV 4TH-COMPOUNDS OF A GROUP

MARTIN L. NEWELL

Dedicated to the memory of my teacher Professor Reinhold Baer

1. Introduction

In his elegant inductive proof that every finitely generated group of exponent 4 is finite, Sanov used the following construction.

Let M be a group and let u be an involution in M. We form a group $S_u(M,a)$ by means of the relations $a^2 = u$ and $(ma)^4 = 1$ for every $m \in M$. When $u = 1$, we write $S_0(M,a)$ for the corresponding group.

We call $S_u(M,a)$ a Sanov compound and there is one for every conjugacy class of involutions in M. Sanov proved that for finite M of order m, every Sanov compound $S_u(M,a)$ has finite order at most m^{m+1}. (See, for example, [2, Theorem 18.3.1] or [3, Theorem 14.2.4].) Here we establish some general results concerning $S_u(M,a)$. For example, if M is infinite cyclic, then $S_0(M,a)$ is the extension of a countable elementary abelian 2-group by the infinite dihedral group. If M is cylic of order 3, then $S_0(M,a)$ is isomorphic to S_4. For $M = A_4$, $S_0(M,a)$ has order $2^9 \cdot 3$, while $S_u(M,a)$ has order $2^6 \cdot 3$ for $u = (1,2)(3,4)$.

For computational purposes one uses a presentation for M via generators and relations. Then one adds the extra relations defining $S_u(M,a)$. These extra relations usually induce further relations in M. Thus, while M itself may not be a subgroup of $S_u(M,a)$, there exists a normal subgroup K_u of M such that $S_u(M,a)$ is isomorphic to $S_{\overline{u}}(\overline{M},a)$, where $\overline{M} = M/K_u$ belongs to $S_{\overline{u}}(\overline{M},a)$. For example, when M is a dihedral group of order $2n$, with n odd, $S_0(M,a) = S_0(C_2,a)$ is dihedral of order 8 and $K_0 = M'$, the commutator subgroup of M. We also show that for M finite, simple and non-abelian, $S_u(M,a) = S_0(1,a)$ is cyclic of order 2. Originally these investigations were prompted by a remark of M. Newman who asked if every Sanov compound of a 2-group M is itself a 2-group. We give a positive answer to this question, and a bound for the order. In a later paper we will examine the compounds of soluble groups and present further information on the groups M/K_u.

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2. Elementary properties of a compound $S_u(M, a)$

LEMMA 1. *Let x, y and t belong to M. Let $\alpha = at$. Then:*

(1) $x^{\alpha+1}$ *is inverted by* α^2.
(2) $(x^{\alpha+1})^y = (y^{\alpha+1})^{-1}(yx)^{\alpha+1}[x, y]$.
(3) $x^{\alpha+1}$ *commutes with* $y^{\alpha^{-1}+1}$ *when* $[y, x] = 1$.
(4) $[x^\alpha, x] = 1$, *when* x *is inverted by* α^2.
(5) $[x^\alpha, y] = [x, y^\alpha]$, *when* x, y *and* xy^{-1} *are inverted by* α^2.

Proof. By hypothesis $(x\alpha^{-1})^4 = 1 = xx^\alpha x^{\alpha^2} x^{\alpha^3} = x^{(1+\alpha)(1+\alpha^2)}$. Hence (1) and (4) are immediate consequences. Since

$$x^{(\alpha+1)y} = y^{-1}x^\alpha xy = (y^\alpha y)^{-1}(yx)^\alpha yx[x, y],$$

we get property (2).

Let $[y, x] = 1$. Then $x^{y\alpha+1} = x^{\alpha+1}$ is inverted by α^2 and $(y\alpha)^2 = yy^{\alpha^{-1}}\alpha^2$. Therefore $yy^{\alpha-1}$ and $(y^{\alpha^{-1}}y)^{-1}$ commutes with $x^{\alpha+1}$. This proves (3).

Finally (5) follows from the fact that x^α, y^α, and $x^\alpha y^{-\alpha}$ commute, respectively, with x, y, and xy^{-1}, and $(xy^{-1})^\alpha xy^{-1} = x^\alpha x[x, y^\alpha]y^{-\alpha}y^{-1}$. □

LEMMA 2. *Let $t, a^2 \in M$ and put $z = [a^2, t]$. Then:*

(1) $z = t^{a^{-1}+1}t^{a+1}$.
(2) $[z, z^a] = 1$.
(3) $z^{at} = z^{t^{-1}a}$.
(4) $(t^{a+1})^2 = zz^{at}$.

Proof. $(ta^{-1})^4 = 1$ implies that $tt^a t^{a^2} t^{a^{-1}} = 1$. Hence $t^{-a^2} = t^{a^{-1}}tt^a$ and (1) follows. Since $z^{a^2} = z^{-1}$, (2) is a consequence of Lemma 1(4). Also, by Lemma 1(3), z commutes with $t^{a^{-1}+1}$ and since $at = (t^{a^{-1}}t)t^{-1}a$, property (3) follows. Finally, $z^{at} = (t^a t)(t^{a+1})^{at} = (t^a t)(t^{a-1}t)^{-1}$ and $zz^{at} = (t^a t)^2$. This completes the proof. □

3. Examples

EXAMPLE 1. (a) Let $M = \langle t \rangle$ be cyclic, put $S = S_0(M, a)$ and let $T_k = (t^k)^{a+1}$ for every integer $k \neq 0$. Then, by Lemma 2, each T_k is an involution and the group $T = \langle T_k \rangle$ is an elementary abelian 2-group, by Lemma 1(3). It is normalized by t, since $T_k^t = T_1^{-1}T_{k+1}$, by Lemma 1(2). It is normalized by a, since $T_k^a = (t^k)^{a^2+a} = T_{-k}$. Then S/T is a dihedral group. When t has order m, the normal subgroup T of S has order 2^{m-1} and S has order $2^m m$.

(b) Suppose t has order $2m$ and let $u = t^m$. Then a^2 is central in $S_u(M)$ and $S_u(M)/\langle a^2 \rangle$ is isomorphic to a subgroup of $S_0(M/\langle a^2 \rangle, a)$.

Example 2. (a) Let M be a dihedral group, $M = \langle s, r\rangle$ for involutions s and r. Let $t = sr$ and $S = S_0(M, a)$. Then $1 = (at^{-1})^4 = aa^ta^ta^{t^3}t^{-4}$. Also the involutions a,s generate a dihedral group of order 8, since $(as)^4 = 1$.

In particular, a^s commutes with a. The same is true for a^{st^k} for every integer k. Thus a^s commutes with a, a^t, a^{t^2}, a^{t^3} and consequently with t^4. But then a commutes with t^4. Now $(t^4)^{a+1} = t^8$ is an involution by Lemma 2(4) and hence $t^{16} = 1$. So for $M = D_\infty$ we have $S_0(M, a) = S_0(\langle s\rangle, a)$ is dihedral of order 8. The same is true for a dihedral group M or order $2n$, n odd.

(b) Let $a^2 = s \in M$ and let $S = S_s(M, a)$. Then $t^{a^2} = t^{-1}$ and $[t^a, t] = 1$ in S, by Lemma 1(4). The abelian group $A = \langle t, t^a\rangle$ is normal in S and S/A is cyclic of order 4.

Example 3. The symmetric group $M = S_4$ has essentially three compounds, where $u = 1, (1, 2)(3, 4)$ and $(1, 2)$, respectively. The first $S_0(M, a)$ is isomorphic to D_8, the dihedral group of order 8. So is the second, while the third compound has order 36 and is isomorphic to $C_3 \times C_3$ extended by C_4, with a^2 acting by inversion. We already noted that S_4 is the compound of C_3. Thus for a given group M, by iterating the process one can develop a tree of compounds. For $M = 1$, the associated tree is an interesting family of 2-groups. We will see later that for $M = S_n, n > 4$, the only possible Sanov compounds are C_2, C_4 and D_8.

We now consider the Sanov compounds of nilpotent groups.

Theorem 1. *Let M be a nilpotent group. Let $a^2 = u$ be an involution in M. Then $S_u(M, a)$ is soluble.*

If M is finite of order m, then $S_u(M, a)$ is finite of order dividing $2^m m$.

Proof. Let $s \neq 1$ be an element of $Z(M)$. Then $[a^2, s] = 1$ and s^{a+1} is an involution, by Lemma 2(4). Let $A = \langle s^{a+1} : 1 \neq s \in Z(M)\rangle$. By Lemma 1(2) and Lemma 1(3) it follows that A is an elementary abelian 2-group and is normalized by $Z(M)$.

Let $y \in M$. Then y^{a+1} commutes with $s^{a^{-1}+1} = s^{a+1}$, by Lemma 1(3). Hence $(s^{a+1})^y$ centralizes A for all $y \in M$ by Lemma 1(2). It follows that $B = \langle A^M\rangle$ is an elementary abelian 2-group, which is normalized by M. Furthermore, $ya = yy^{a^{-1}}ay^{-1}$ and thus $s^{(a+1)ya} = s^{(1+a)y^{-1}}$ belongs to B. Therefore B is a normal subgroup of $S = S_u(M, a)$. Since a inverts s in S/B, it follows that for $C = \langle Z(M)^S\rangle$, the group C/B is abelian. Also the group S/C is isomorphic to a subgroup of $S_{\overline{u}}(M/Z(M), a)$ where $\overline{u} = uZ(m)$. By induction on the nilpotency class, we conclude that S/C and hence S is soluble. If M is finite of order m, let $Z(M)$ have order c. Then $M/Z(M)$ has order $m' = m/c$, $|A|$ divides 2^{c-1}, and $|B|$ divides $|A|^{m'}$, since $Z(M)$

normalizes A. Finally, $|C|$ divides $|Z(M)||B| = 2^{(c-1)m'}c$. By induction $|S/C|$ divides $2^{m'}m'$, and $|S|$ divides $2^{cm'}m'c = 2^m m$. This completes the proof. □

Corollary 1.

(1) *Every Sanov compound of a finite 2-group is a finite 2-group.*
(2) *Every Sanov compound of a nilpotent group of class d is soluble with derived length at most $2d$.*

4. Properties of a Sanov involution

Let $a^2 = u \in M$. When performing calculations, we will for simplicity identify the elements in M with their images in $S_u(M, a)$.

Theorem 2. *Let $M = \langle a^2, H\rangle$, where $H = \langle x, y : [x, y] = 1\rangle$ and $a^2 \neq 1$. Let $T_h = h^{a+1}$ in $S = S_u(M, a)$. Then:*

(1) *$[T_x, T_y]$ is inverted by a and commutes with x and y in S.*
(2) *$[a^2, x]$ commutes with $[a^2, x]^{ay}$ in S.*

Proof. Let $z_x = [a^2, x]$. Then $z_x = T_x T^a_{x^{-1}}$ by Lemma 2(1), since $T^a_{h^{-1}} = T^{-a^{-1}}_{h^{-1}} = h^{a^{-1}+1}$. Also, for $h, k \in H$ it follows from Lemma 1(2) and (3) that T_k commutes with T^a_h and $T^h_k = T^{-1}_h T_{hk}$, and $ah = (h^{a^{-1}}h)h^{-1}a$ implies that $T^{ah}_k = T^{h^{-1}a}_k$. Now z^{1+ay}_x is inverted by $(ay)^2 = a^2 T_y$. But

$$z^{1+ay}_x = T_x T^a_{x^{-1}} T^{ay}_x T^{-y}_{x^{-1}} = (T_x T^{-y}_{x^{-1}})(T_{x^{-1}} T^{y^{-1}}_x)^a$$

and

$$(T_x T^{-y}_{x^{-1}})^{1+(ay)^2} = (v^{-a})^{1+(ay)^2},$$

where $v = T_{x^{-1}} T^{y^{-1}}_x$. Now $(T_x T^{-y}_{x^{-1}})^{1+(ay)^2}$ equals

$$(T_x T^{-1}_{yx^{-1}} T_y)^{1+a^2 T_y} = (T_x T^{-1}_{yx^{-1}} T_y)(T^{-1}_x T_{yx^{-1}} T^{-1}_y)^{T_y} = \left[T^{-1}_x, T_{yx^{-1}}\right],$$

while

$$(T^{-y^{-1}}_x T^{-1}_{x^{-1}})^{a(1+a^2 T_y)} = (T^{-y^{-1}}_x T^{-1}_{x^{-1}})^{(1+a^2)a}.$$

Therefore

$$\left[T^{-1}_x, T_{yx^{-1}}\right] = (T^{-y^{-1}}_x T^{-1}_{x^{-1}})^{(1+a^2)a}.$$

From this we deduce that $T = \langle T_h : h \in H\rangle$ is nilpotent of class 2.

Expanding

$$(T^{-y^{-1}}_x T^{-1}_{x^{-1}})^{1+a^2} = T^{-y^{-1}}_x T^{-1}_{x^{-1}} T^{-y^{-1}a^2}_x T_{x^{-1}},$$

using

$$T^{-y^{-1}a^2}_x = T^{y^{-a^2}}_x = T^{y^{-1}[y^{-1},a^2]}_x = T^{y^{-1}}_x \left[T^{y^{-1}}_x, T^{-1}_{y^{-1}}\right] = (T_x\,[T_x, T_y])^{y^{-1}},$$

we get

$$\left[T_x^{y^{-1}}, T_{x^{-1}}\right]\left[T_x^{y^{-1}}, T_{y^{-1}}^{-1}\right] = \left[T_x^{y^{-1}}, T_{y^{-1}}^{-1}T_{x^{-1}}\right] = \left[T_x, T_{yx^{-1}}\right]^{y^{-1}}.$$

It follows that

$$\left[T_x^{-1}, T_{yx^{-1}}\right] = \left[T_x, T_{yx^{-1}}\right]^{-1} = (T_x^{-y^{-1}}T_{x^{-1}}^{-1})^{(1+a^2)a} = \left[T_x, T_{yx^{-1}}\right]^{y^{-1}a}$$

for all $x, y \in H$. Thus

$$[T_x, T_y]^{-1} = [T_x, T_y]^{x^{-1}y^{-1}a}$$

and

$$[T_x, T_y]^{-ay} = [T_x, T_y]^{x^{-1}} = [T_x, T_y]^{-y^{-1}a},$$

since $[T_x, T_y]$ commutes with a^2. Therefore

$$\left[T_{x^{-1}}, T_{x^{-1}y}\right]^{-1} = \left[T_{y^{-1}x}, T_{y^{-1}}\right]^a \text{ for all } x, y \in H.$$

Hence

$$[T_h, T_k]^{-1} = [T_k, T_h] = [T_{k^{-1}}, T_{hk^{-1}}]^a \text{ for all } h, k \in H,$$

and

$$\left[T_{x^{-1}}, T_{x^{-1}y}\right] = [T_x, T_y]^a.$$

Thus

$$[T_x, T_y]^{x^{-1}} = \left[T_{x^{-1}}, T_{x^{-1}y}\right]^{-1} = [T_x, T_y]^{-a} = [T_x, T_y]^{-y^{-1}a}.$$

Therefore $[T_x, T_y]$ commutes with y and so with x by symmetry and it is inverted by a. This concludes the proof of (1).

Since

$$z_x z_x^{ay} z_x^{(ay)^2} z_x^{y^{-1}a^{-1}} = 1$$

and

$$z_x^{(ay)^2} = z_x^{a^2 T_y} = z_x^{-T_y} = (z_x [T_x, T_y])^{-1},$$

it follows that

$$z_x z_x^{ay} z_x^{-1} z_x^{-ay} [T_x, T_y]^{-1} [T_x, T_y]^{-ay} = 1,$$

and therefore

$$[z_x, z_x^{ay}] = 1.$$

This proves (2). □

Corollary 2. *Let $u = a^2 \neq 1$ and $t \in M$. Then $\langle a^2, t\rangle$ is central by metabelian in $S_u(M, a)$.*

Proof. Let $T_i = (t^i)^{a+1}$ and let $z_i = \left[a^2, t^i\right]$. Then $z_i = T_i T_{-i}^a$ and

$$[z_i, z_j] = [T_i, T_j]\,[T_{-i}, T_{-j}]^a = [T_i, T_j]\,[T_{-i}, T_{-j}]^{-1}.$$

Therefore the group $Z = \langle z_i : i \text{ an integer}\rangle$ is nilpotent of class 2. Further, z_1 commutes with z_1^a and z_1^{at}, by Theorem 3. Since

$$z_1^{at} = z_1^{t^{-1}a} = z_{-1}^{-a},$$

we have that z_1^a and z_{-1}^{-a} commute with z_1. Because

$$z_1^{at^{-i}} = z_1^{t^i a} = z_i^{-a} z_{i+1}^a$$

commutes with z_1, we conclude by induction that z_j^a commutes with z_1 for every integer j. Further, since $(z_i)^t = z_1^{-1} z_{i+1}$, a similar induction yields that z_i commutes with z_j^a for all integers i and j. Since $[z_i, z_j]$ is inverted by a, it commutes with a^2. Because $[T_1, T_j]$ commutes with t for all j, we deduce by induction that $[T_i, T_j]$ commutes with t, for all integers i, j. Hence $[z_i, z_j]$ is central in $\langle a^2, t\rangle$ and $\langle a^2, t\rangle$ is central by metabelian. □

Theorem 3. *Every Sanov-compound of a non-abelian, finite simple group is cyclic of order 2.*

Proof. Let $a^2 = u(\neq 1) \in M$. By the Theorem in [4], there exists an element t such that $M = \langle a^2, t\rangle$. By Corollary 2, this has trivial image in $S_u(M, a)$ and hence $S_u(M, a)$ is cyclic of order 2.

Consider $S_0(M, a)$ with $a^2 = 1$. Let x be an involution in M and let $y \in M$. Then $\langle x^y, x\rangle$ is dihedral and by Example 2(a) it is either trivial or a 2-group. In particular, $[y, x]$ has order dividing 16. Thus x is a left-engel element of M and by [1] it is contained in the Fitting subgroup of M. Thus M is trivial in $S_0(M, a)$ and this group is cyclic of order 2. □

Corollary 3. *Let $M = S_n$ be the symmetric group with $n > 4$. Then $S_u(M, a)$ is isomorphic to D_8, C_4 or C_2.*

Proof. The Sanov compound of the trivial group is C_2. We consider then the remaining cases.

Let $a^2 = u$. If u is even, $S_u(M, a)$ contains $S_u(A_n, a)$; also $u \in A_n \subseteq K_u$ and

$$S_u(M, a) \simeq S_0(C_2, a) \simeq D_8.$$

For $a^2 = (1, 2), S_n = \langle a^2, t\rangle$, where t is an n-cycle, but this group is not central by metabelian. In this case we can assume that $t \in K_u$ and $S_u(M, a) \simeq C_4$. Now let $a^2 = (1, 2)(3, 4)(5, 6)v$, where v is an even involution. Then $\langle a^2, t_0\rangle$ is not central by metabelian, where $t_0 = (1, 2, 3, 4, 5)$. So we may assume that every 5-cycle is in K_u and that $A_n \subseteq K_u$. Then $a^2 \equiv (1, 2) \bmod A_n$ and the compound is C_4. □

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Department of Mathematics, National University of Ireland Galway, Galway, Ireland
E-mail address: martin.newell@nuigalway.ie