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THREE-STAR PERMUTATION GROUPS

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To the memory of Reinhold and Marianne Baer

Abstract. A permutation group is a three-star group if it induces a non-trivial group on each 3-element subset of points. Our main results are that a primitive three-star group is generously transitive and that a finite primitive three-star group has rank at most 3, that is, a stabiliser has at most 3 orbits. We also describe the structure of an arbitrary (non-primitive) three-star group and give a collection of examples. In particular, we sketch a construction of infinite primitive three-star groups of arbitrarily high rank.

1. Introduction

A permutation group G acting on a set Ω will be said to be a *three-star* group if it has the following property: for every 3-subset Θ of Ω the permutation group G^Θ induced on Θ by its setwise stabiliser $G_{\{\Theta\}}$ is non-trivial. Praeger and Schneider [5] came across this condition in a study of overgroups of finite permutation groups that have a transitive minimal normal subgroup.

To exclude trivialities we assume throughout that $|\Omega| \geqslant 3$. In [4] a group G was defined to be *generously k-transitive* if $G^\Theta = \mathrm{Sym}(\Theta)$ for all $(k+1)$-subsets Θ of Ω and *almost generously k-transitive* if $G^\Theta \geqslant \mathrm{Alt}(\Theta)$ for all $(k+1)$-subsets Θ of Ω. In particular, an almost generously 2-transitive group is a three-star group. It was shown in [4] that an almost generously 2-transitive group is (as the terminology suggests) doubly transitive. So strong a conclusion cannot be expected with the weaker hypothesis treated here. Nevertheless, we find that the three-star condition is quite strong. Our main theorems are that a primitive three-star group is generously transitive and that a finite primitive three-star group has rank at most 3—that is to say, a stabiliser has at most 3 orbits in Ω. The proofs of these facts are given in Section 2 below. In Section 3 we consider the structure of an arbitrary (non-primitive) three-star group and describe a range of examples. In particular, we sketch a construction of infinite primitive three-star groups of arbitrarily high rank.

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2. Primitive three-star groups

In this section we focus on primitive three-star groups. This is, of course, a significant restriction. However, there is quite a strong sense in which the study of arbitrary three-star groups may be reduced to the study of primitive ones. We will return to this point in Section 3 below.

Some general theory of permutation groups is needed for the statement and proof of our results. Recall (see, for example, [2, §3.2]) that for a group G acting on a set Ω the orbitals are the orbits of G in $\Omega \times \Omega$. When G is transitive these are in one-one correspondence with the suborbits, that is to say, the orbits of a stabiliser G_α for $\alpha \in \Omega$. An orbital Γ corresponds to the suborbit $\Gamma(\alpha)$, where $\Gamma(\alpha) := \{\omega \in \Omega \mid (\alpha, \omega) \in \Gamma\}$; the so-called trivial orbital $\{(\omega, \omega) \mid \omega \in \Omega\}$ corresponds to the trivial suborbit $\{\alpha\}$. The number of orbitals (or of suborbits) is known as the rank of G. Associated with an orbital Γ is its paired orbital Γ^* defined by $\Gamma^* := \{(\omega_1, \omega_2) \mid (\omega_2, \omega_1) \in \Gamma\}$. The orbital Γ is said to be self-paired if $\Gamma = \Gamma^*$. This is the case if and only if for any $(\omega_1, \omega_2) \in \Gamma$ there is a permutation in G that transposes ω_1 and ω_2; therefore G is generously transitive if and only if all orbitals are self-paired. For subsets Γ, Δ of Ω^2 we define

$$\Gamma \circ \Delta := \{(\omega_1, \omega_2) \in \Omega^2 \mid (\exists\, \omega \in \Omega) \colon\, (\omega_1, \omega) \in \Gamma \text{ and } (\omega, \omega_2) \in \Delta\}.$$

If Γ, Δ are orbitals then $\Gamma \circ \Delta$ will be a union of orbitals. Note that $(\Gamma \circ \Delta)^* = \Delta^* \circ \Gamma^*$ and that $\Gamma \circ (\Delta \circ \Phi) = (\Gamma \circ \Delta) \circ \Phi$.

Theorem 2.1. *With one exception a primitive three-star group is generously transitive. The exception is the alternating group* Alt(3).

Proof. Let G be a primitive three-star group acting on the set Ω, and suppose that G is not generously transitive. Let Γ be a non-self-paired orbital. We claim that $\Gamma \circ \Gamma = \Gamma^*$. Choose $(\alpha, \gamma) \in \Gamma \circ \Gamma$. By definition there exists $\beta \in \Omega$ such that $(\alpha, \beta) \in \Gamma$ and $(\beta, \gamma) \in \Gamma$. Now $\alpha \neq \gamma$ (Γ is not self-paired), and so α, β, γ are distinct. Let $\Theta := \{\alpha, \beta, \gamma\}$ and $T := G^\Theta$. Since Γ is not self-paired, T contains neither of the transpositions $(\alpha\,\beta)$, $(\beta\,\gamma)$. Nor does it contain $(\alpha\,\gamma)$ since α, γ lie in different orbits of the stabiliser G_β. By assumption, however, $T \neq \{1\}$, and therefore $(\alpha\,\beta\,\gamma) \in T$. It follows that $(\gamma, \alpha) \in \Gamma$, whence $\Gamma \circ \Gamma = \Gamma^*$. Then also $\Gamma^* \circ \Gamma^* = \Gamma$.

Now define $\Delta := \Gamma \circ \Gamma^*$. Then $\Delta = \Gamma \circ \Gamma \circ \Gamma = \Gamma^* \circ \Gamma$, and so $\Delta \circ \Gamma = \Gamma \circ \Delta = \Gamma$. As a binary relation Δ is reflexive and symmetric. It is also transitive because $\Delta \circ \Delta = \Delta \circ \Gamma \circ \Gamma^* = \Gamma \circ \Gamma^* = \Delta$. Thus Δ is a G-invariant equivalence relation on Ω. Since G is primitive Δ is either the universal relation U or the trivial relation E (equality). However, $U \circ \Gamma = U \neq \Gamma$, and so $\Delta \neq U$. Therefore $\Delta = E$. Let $\gamma, \gamma' \in \Gamma^*(\alpha)$. Then $\gamma' \in \Delta(\gamma)$, whence $\gamma = \gamma'$. Thus Γ^* has subdegree 1. Similarly of course Γ has subdegree 1. It follows immediately that $G = \mathrm{Alt}(3)$.

THEOREM 2.2. *A finite primitive three-star group has rank at most* 3.

Proof. Suppose that Ω and G are finite and that G acts as a primitive three-star group on Ω. Clearly we may assume that $|\Omega| > 3$, so that, by what has just been proved, all orbitals are self-paired. An edge (α, β) of the complete graph with vertex-set Ω will be said to be of colour Γ (where Γ is an orbital) if $(\alpha, \beta) \in \Gamma$. Let Γ, Δ be distinct orbitals and let $\alpha \in \Omega$. The three-star condition implies that no triangle in Ω can have edges of three different colours, and so all edges between points in $\Gamma(\alpha)$ and points in $\Delta(\alpha)$ are coloured Γ or Δ. Suppose that all such edges had the same colour, say Γ. If $\beta \in \Gamma(\alpha)$ and $\gamma \in \Delta(\beta)$ then $\gamma \notin \Delta(\alpha)$ and so the third edge (α, γ) of the triangle $\{\alpha, \beta, \gamma\}$ must have colour Γ. Thus $\Gamma(\alpha)$ would be a union of components of the graph (Ω, Δ), and this is impossible since G is primitive. Therefore there are edges of both colours Γ and Δ between $\Gamma(\alpha)$ and $\Delta(\alpha)$. Thus for any ordered pair (Γ, Δ) of colours there are triangles with edges coloured Γ, Γ, Δ. In particular, every orbital graph has diameter 2, and for every Γ there are edges of every colour, except possibly Γ itself, within $\Gamma(\alpha)$.

We continue to focus on a point α of Ω and distinct orbitals Γ, Δ. Let Φ denote the merger of all the colours other than Γ and Δ: that is, (Ω, Φ) is the graph whose edge-set consists of all edges of the complete graph with colours different from Γ and Δ. Let $\gamma_1, \gamma_2 \in \Gamma(\alpha)$ and suppose that the edge (γ_1, γ_2) is coloured Φ. For any $\delta \in \Delta(\alpha)$ the edges (γ_1, δ) and (γ_2, δ) have colour Γ or Δ. Since the triangle $(\gamma_1, \gamma_2, \delta)$ cannot have three differently coloured edges, the colours of (γ_1, δ) and (γ_2, δ) must be the same. It follows that if $\Gamma_1, \ldots, \Gamma_c$ are the components of the Φ-graph with vertex-set $\Gamma(\alpha)$, and if $\delta \in \Delta(\alpha)$, then all edges from vertices in Γ_i to δ have the same colour. Interchanging the roles played by Γ and Δ, we see that if $\Delta_1, \ldots, \Delta_d$ are the components of the Φ-graph with vertex-set $\Delta(\alpha)$ then all edges between a component Γ_i and a component Δ_j have the same colour.

Suppose the Φ-graph with vertex-set $\Gamma(\alpha)$ were connected. Then all edges between points of $\Gamma(\alpha)$ and a given point $\delta \in \Delta(\alpha)$ would be the same colour. Since G_α acts transitively on $\Delta(\alpha)$ it would follow that all edges between points of $\Gamma(\alpha)$ and points of $\Delta(\alpha)$ would be the same colour. This is not the case (see above) and therefore the Φ-graph with vertex-set $\Gamma(\alpha)$ is not connected, that is, $c > 1$. Similarly, the Φ-graph with vertex-set $\Delta(\alpha)$ is not connected, that is, $d > 1$.

If there is a Δ-coloured edge (γ_1, γ_2) with $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ then we shall say that Δ *dominates* Γ. Suppose for the moment that this is the case. If $\gamma_1' \in \Gamma_1$ and $(\gamma_1, \gamma_1') \in \Phi$ then the edge (γ_1', γ_2) must also be coloured Δ. It follows that all edges from points of Γ_1 to γ_2 are coloured Δ, and then that all edges between points of Γ_1 and points of Γ_2 are coloured Δ. Thus if Δ dominates Γ then the Δ-components of $\Gamma(\alpha)$ are proper unions of Φ-components Γ_i; if Δ does not dominate Γ then of course the Φ-components

Γ_i are $(\Phi \cup \Delta)$-components in $\Gamma(\alpha)$. If Δ dominates Γ then for every other orbital Δ' the Δ-components of $\Gamma(\alpha)$ are proper unions of Δ'-components, and therefore Δ' cannot dominate Γ since the Δ'-components of $\Gamma(\alpha)$ are then not unions of Δ-components. Clearly therefore, for an orbital Γ, at most one orbital Δ can dominate Γ.

Since G_α acts transitively on $\Gamma(\alpha)$ all the Φ-components Γ_i of $\Gamma(\alpha)$ have the same size, say a. Similarly, all the Φ-components Δ_j of $\Delta(\alpha)$ have the same size, say b. Suppose that $a \leqslant b$. Let $\gamma \in \Gamma_1$ and consider the set $\Delta(\gamma)$. We know that $\Delta(\gamma) \subseteq \Gamma(\alpha) \cup \Delta(\alpha)$, and $\Delta(\gamma) \cap \Delta(\alpha)$ is a union of some but not all of the Φ-components Δ_j of $\Delta(\alpha)$. Let $n_\Delta := |\Delta(\alpha)|$. Then $|\Delta(\gamma) \cap \Delta(\alpha)| \leqslant n_\Delta - b$ and so $|\Delta(\gamma) \cap \Gamma(\alpha)| \geqslant b$. It follows that $\Delta(\gamma) \cap \Gamma(\alpha)$ cannot be contained in the Φ-component Γ_1, and so Δ dominates Γ. Of course if $b \leqslant a$ then we find that Γ dominates Δ. Thus, of any two orbitals, one dominates the other.

Now let r be the rank of G and let $k := r-1$. By what has just been proved there are at least $\binom{k}{2}$ ordered pairs (Γ, Δ) of non-trivial orbitals in which Δ dominates Γ. On the other hand, for each Γ there is at most one orbital Δ that dominates Γ and therefore there are at most k such pairs. Thus $\binom{k}{2} \leqslant k$ and so $k \leqslant 3$.

Suppose that $k = 3$. Let Γ, Δ, Φ be the non-trivial orbitals and let a_Γ be the size of the Φ-components in $\Gamma(\alpha)$, a_Δ the size of the Γ-components in $\Delta(\alpha)$, and a_Φ the size of the Δ-components in $\Phi(\alpha)$. Let n_Φ be the valency of the graph (Ω, Φ), so that $n_\Phi = |\Phi(\alpha)|$. Consider $\Phi(\omega)$, where $\omega \in \Gamma(\alpha)$. If Γ_1 is the Φ-component of $\Gamma(\alpha)$ containing ω then $\Phi(\omega) = \big(\Phi(\omega) \cap \Gamma_1\big) \cup \big(\Phi(\omega) \cap \Phi(\alpha)\big)$. Now $\Phi(\omega) \cap \Gamma_1 \subseteq \Gamma_1 \setminus \{\omega\}$ and so $|\Phi(\omega) \cap \Gamma_1| \leqslant a_\Gamma - 1$. Also, $\Phi(\omega) \cap \Phi(\alpha)$ is a union of some but not all of the Δ-components in $\Phi(\alpha)$, and so $|\Phi(\omega) \cap \Phi(\alpha)| \leqslant n_\Phi - a_\Phi$. Therefore $n_\Phi \leqslant (a_\Gamma - 1) + (n_\Phi - a_\Phi)$ and so $a_\Phi \leqslant a_\Gamma - 1$. Similarly, considering $\Gamma(\omega)$ for $\omega \in \Delta(\alpha)$ we find that $a_\Gamma \leqslant a_\Delta - 1$ and considering $\Delta(\omega)$ for $\omega \in \Phi(\alpha)$ we find that $a_\Delta \leqslant a_\Phi - 1$. These inequalities imply that $a_\Phi \leqslant a_\Phi - 3$, which is absurd. It follows that $k \leqslant 2$ and so the rank of G is at most 3, as our theorem states.

3. Commentary

There is quite a strong sense in which the study of arbitrary three-star groups may be reduced to that of primitive three-star groups. First, we have the following:

Observation 3.1. *If G is an intransitive three-star group then it has exactly two orbits Ω_1 and Ω_2. Moreover, G acts as a three-star group on each of Ω_1, Ω_2, and, as G-spaces, Ω_1, Ω_2 are strongly orthogonal in the sense that for $\omega_1 \in \Omega_1$ the stabiliser G_{ω_1} is generously transitive on Ω_2 and for $\omega_2 \in \Omega_2$ the stabiliser G_{ω_2} is generously transitive on Ω_1.*

Proof. If there were three or more orbits then there would be a triple consisting of points from different orbits, and its stabiliser would act trivially on it, contrary to assumption. Thus, given that G is intransitive, there are just two orbits Ω_1, Ω_2. The fact that G acts as a three-star group on each of Ω_1, Ω_2 is clear. Consider any point $\omega_1 \in \Omega_1$ and any pair $\{\alpha, \beta\}$ of points from Ω_2. Since the stabiliser of the triple $\{\omega_1, \alpha, \beta\}$ is non-trivial G contains a permutation fixing ω_1 and interchanging α, β. Therefore G_{ω_1} is generously transitive on Ω_2. And of course, similarly, for $\omega_2 \in \Omega_2$, G_{ω_2} is generously transitive on Ω_1.

Observation 3.2. *Suppose that G is a three-star group which is transitive but imprimitive on Ω. Let ρ be a non-trivial proper G-congruence on Ω, let Γ be a ρ-class in Ω, let $\Delta := \Omega/\rho$, let $C := G^{\Gamma}$, the group induced on Γ by its setwise stabiliser in G, and let $D := G^{\Delta}$. Then C is a three-star group on Γ and D is a three-star group on Δ. Moreover, C is generously transitive on Γ.*

Conversely, if C is a generously transitive three-star group on the set Γ, and D is a three-star group on the set Δ, then the wreath product C Wr D is a three-star group in its natural imprimitive representation on $\Gamma \times \Delta$.

Since the proof is routine we leave it to the interested reader. Note that here we should permit the possibility that $|\Gamma| = 2$ and $C = \mathrm{Sym}(\Gamma)$ or that $|\Delta| = 2$ and $D = \mathrm{Sym}(\Delta)$.

We have not sought to compile a systematic catalogue of primitive three-star groups, but we do not think that would be a very difficult project. There are several interesting families of examples. As has already been observed, any almost generously 2-transitive group is a three-star group. Many of the finite 2-transitive groups are almost generously 2-transitive; the only ones that are not are those contained in affine groups $\mathrm{A\Gamma L}(d, q)$ for $q \geqslant 5$ and the almost simple groups whose socle is a Suzuki group $\mathrm{Sz}(q)$, where $q = 2^{2m+1}$ and $m \geqslant 1$, or a Ree group $\mathrm{Ree}(q)$ where $q = 3^{2m+1}$ and $m \geqslant 1$. It is not hard to see that the Suzuki and Ree groups are not three-star groups. Some of the affine groups that are not almost generously 2-transitive are three-star groups, however.

Example 3.3. The affine groups $\mathrm{AGL}(d, 5)$ are three-star groups.

Proof. Let Θ be a triple of points of the affine space $\mathrm{AG}(d, 5)$ and let $G := \mathrm{AGL}(d, 5)$. If Θ consists of non-collinear points then $G^{\Theta} = \mathrm{Sym}(\Theta)$ and so certainly $G^{\Theta} \neq \{1\}$. If Θ is a collinear triple then, as is not hard to see, it is equivalent under affine transformations to the triple $\{0, 1, 4\}$ or to the triple $\{0, 2, 3\}$ in an affine line in $\mathrm{AG}(d, 5)$. Both of these triples admit involutions, so $G^{\Theta} \neq \{1\}$.

There are several families of primitive three-star groups of rank 3.

EXAMPLE 3.4. Let $G := \mathrm{Sym}(m)$ where $m \geqslant 3$, and let $\Omega := m^{\{2\}}$, the set of pairs from $\{1, \ldots, m\}$. In its natural action on Ω, G is a primitive three-star group of rank 3.

Proof. That G is primitive on Ω is well known and easy to prove. Define

$$\begin{aligned}
\Theta_1 &:= \big\{\{1,2\},\ \{1,3\},\ \{2,\ 3\}\big\}\,,\\
\Theta_2 &:= \big\{\{1,2\},\ \{1,3\},\ \{1,\ 4\}\big\}\,,\\
\Theta_3 &:= \big\{\{1,2\},\ \{2,3\},\ \{3,\ 4\}\big\}\,,\\
\Theta_4 &:= \big\{\{1,2\},\ \{2,3\},\ \{4,\ 5\}\big\}\,,\\
\Theta_5 &:= \big\{\{1,2\},\ \{3,4\},\ \{5,\ 6\}\big\}\,.
\end{aligned}$$

Any triple of unordered pairs is equivalent to one of these five, and for each of these five it is easy to see that $G^{\Theta} \neq \{1\}$.

EXAMPLE 3.5. Let H be a group acting generously 2-transitively on a set Γ of size $\geqslant 3$. If $G := H \,\mathrm{Wr}\, \mathrm{Sym}(2)$ and $\Omega := \Gamma^2$, then G is a primitive three-star group of rank 3.

Proof. As in the previous example, that G is primitive on Ω is well known and easy to prove. Let α_1, α_2, α_3 be distinct points of Γ and define

$$\begin{aligned}
\Theta_1 &:= \big\{(\alpha_1,\alpha_1),\ (\alpha_2,\alpha_2),\ (\alpha_3,\alpha_3)\big\}\,,\\
\Theta_2 &:= \big\{(\alpha_1,\alpha_1),\ (\alpha_2,\alpha_2),\ (\alpha_2,\alpha_3)\big\}\,,\\
\Theta_3 &:= \big\{(\alpha_1,\alpha_1),\ (\alpha_1,\alpha_2),\ (\alpha_1,\alpha_3)\big\}\,,\\
\Theta_4 &:= \big\{(\alpha_1,\alpha_1),\ (\alpha_1,\alpha_2),\ (\alpha_2,\alpha_1)\big\}\,.
\end{aligned}$$

Any triple of ordered pairs is equivalent to one of these four, and for each of these four it is easy to see that $G^{\Theta} \neq \{1\}$.

EXAMPLE 3.6. Let Q be a non-degenerate quadratic form on the vector space Ω of dimension $2m$ over the field $\mathbb{F}_2$ and let $G := \mathrm{AO}(2m, 2)$, the group generated by translations and orthogonal transformations of Ω with respect to Q. Then G is a primitive three-star group of rank 3.

Proof. Triples Θ are triangles in the affine space Ω with side-lengths $\{0, 0, 0\}$, $\{0, 0, 1\}$, $\{0, 1, 1\}$, or $\{1, 1, 1\}$. In each case $G^{\Theta} \neq \{1\}$.

EXAMPLE 3.7. Let Q be a non-degenerate quadratic form on the vector space Ω of dimension d over the field $\mathbb{F}_3$ and let $G := \mathrm{AGO}(d, 3)$, the group generated by translations and transformations of Ω that preserve Q up to scalar multiplication. Then G is a primitive three-star group of rank 3.

Proof. Triples Θ are of the following kinds. First, there are triples $\{\alpha, \beta, \gamma\}$ forming a line of the affine space Ω. For these $G^{\Theta} = \mathrm{Sym}(\Theta)$. Secondly, there are triangles in the affine space Ω. Triangles can have side-lengths a, b, c, each of which can be 0, 1 or 2 (in $\mathbb{F}_3$). It is easy to see that if two side-lengths are the same then $G^{\Theta} \neq \{1\}$. If the side-lengths are all different then the triangle is equivalent to $\{0, u, w\}$, where $Q(u) = 1$, $Q(w) = 2$ and $Q(u-w) = 0$. Now there is a linear transformation $T \in \mathrm{GO}(\Omega)$ for which $Q(Tv) = 2Q(v)$ for all $v \in \Omega$ and which interchanges u and w. Thus in all cases $G^{\Theta} \neq \{1\}$.

The situation is different for infinite permutation groups. Although Theorem 2.1 does not require finiteness of G or Ω, so that an infinite primitive three-star group is generously transitive, Theorem 2.2 fails without the finiteness assumption.

Observation 3.8. *There are infinite primitive three-star groups of arbitrary rank.*

Proof. We confine ourselves to a sketch of the construction. It is based on the theory of C-relations and C-sets propounded in [1]. Let (Ω, C) be the C-set whose chains are isomorphic to $(\mathbb{Z}, \leqslant)$ and whose branching number is s (the value of s is irrelevant, as it happens). The construction of such a C-set is described on page 43 of [1]—take Q_0 there to be $\mathbb{Z}$ with a least element adjoined. In slightly different terms, Ω may be taken to be the set of doubly infinite sequences $(q_i)_{i \in \mathbb{Z}}$, where $q_i \in \{0, 1, \ldots, s-1\}$, and which are of finite support in the sense that there exists $n \in \mathbb{N}$ such that $q_i = 0$ when $|i| > n$. Let W be the wreath power $\mathrm{Wr}\,\big(\mathrm{Sym}(s)\big)^{\mathbb{Z}}$ defined by Philip Hall in [3] as a subgroup of $\mathrm{Sym}(\Omega)$. Let $m \geqslant 1$. The infinite cyclic group Z acts by translation through m on $\mathbb{Z}$, that is, with its generator acting as $i \mapsto i + m$. This extends in a natural way to an action of Z on Ω, and then Z, as subgroup of $\mathrm{Sym}(\Omega)$, normalises W. Let $G := W.Z \leqslant \mathrm{Sym}(\Omega)$. It is not hard to see that the only W-invariant equivalence relations on Ω are the relations ρ_r $(r \in \mathbb{Z})$ defined by

$$(q_i) \equiv (q_i') :\Leftrightarrow q_i = q_i' \text{ for all } i \geqslant r\,.$$

Since these are not Z-invariant G acts primitively on Ω. Also, the stabiliser G_0 of the 0-sequence is $U.Z$, where $U := \mathrm{Wr}\,\big(\mathrm{Sym}(k-1)\big)^{\mathbb{Z}}$. For any other sequence (q_i) define $m((q_i)) := \max\{i \mid q_i \neq 0\}$. It is not hard to calculate that non-zero sequences (q_i), (q_i') are in the same G_0-orbit if and only if $m((q_i)) \equiv m((q_i')) \pmod m$. Thus G has rank $m+1$. To see that G is a three-star group consider three distinct elements α, β, γ of Ω and let $\Theta := \{\alpha, \beta, \gamma\}$. We may suppose that α is the 0-sequence, $\beta = (q_i)$ and $\gamma = (q_i')$. It is not hard to calculate the following: if $m((q_i)) < m((q_i'))$ then the setwise stabiliser in G of Θ contains (and in fact is generated by) the transposition $(\alpha\,\beta)$; if $m((q_i)) > m((q_i'))$ then the setwise stabiliser in G of Θ contains the transposition $(\alpha\,\gamma)$; if $m((q_i)) = m((q'_i)) = j$ and $q_j = q'_j$ then the setwise stabiliser in G of Θ contains the transposition $(\beta\,\gamma)$; if $m((q_i)) = m((q'_i)) = j$ and $q_j \neq q'_j$ then $G^{\Theta} = \mathrm{Sym}(\Theta)$.

To produce a primitive three-star group with infinite rank κ one replaces $(\mathbb{Z}, \leqslant)$ with a suitable linearly ordered set $(Q, \leqslant)$. All that is required is that $(Q, \leqslant)$ should admit an infinite cyclic group Z of automorphisms whose orbits are co-initial and co-final in Q (that is, bounded neither below nor above in Q) and that Z should have κ orbits in Q.

Final Note. The notion of three-star group has an obvious generalisation to that of k-star group for any $k \geqslant 2$. It is not hard to see that the infinite groups described in the proof of Observation 3.8 are k-star groups for every finite k. For $k > 3$ we know little about finite primitive k-star groups but we believe them to be rather rare. As it happens, however, Example 3.4 is a four-star group and a five-star group.

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