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DIAGONAL LIMITS OF FINITE ALTERNATING GROUPS: CONFINED SUBGROUPS, IDEALS, AND POSITIVE DEFINITE FUNCTIONS

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In memory of Reinhold Baer on the occasion of the hundredth anniversary of his birth.

Abstract. A non-finitary group G is said to be an $\mathrm{L}\mathfrak{A}$-*group* if it is a direct limit of finite alternating groups $G_i = \mathrm{Alt}(\Omega_i)$ $(i \in I)$ such that each G_i has only trivial or natural orbits on the sets Ω_j $(j > i)$. We determine the confined subgroups of $\mathrm{L}\mathfrak{A}$-groups and relate them naturally to the ideals in the group algebra $\mathbb{K}G$ over any field $\mathbb{K}$ of characteristic zero. Moreover, we show that the non-trivial ideals in $\mathbb{C}G$ can be related to normalized positive definite class functions $f\colon G \to \mathbb{C}$ if and only if the number of G_i-orbits in Ω_j $(j > i)$ is asymptotically a linear function of $|\Omega_j|$ for all i.

1. Introduction

In recent years many authors have studied the structure of simple locally finite groups, that is, simple groups which are direct limits of finite groups. These investigations were pursued in several directions; a good account of this area of research can be found in B. Hartley's survey article [2]. An interesting subject is the study of the lattice of (two-sided) ideals in group algebras of simple locally finite groups. Over fields of characteristic zero, the ideals are closely related to the asymptotic character theory of finite groups, as well as to certain issues arising in the theory of $\mathbb{C}^*$-algebras. For more information we refer the reader to A. E. Zalesskiĭ's survey article [11].

One of the central problems in Zalesskiĭ's survey is to determine those simple locally finite groups G, for which the complex group algebra $\mathbb{C}G$ (or more generally $\mathbb{K}G$, where $\mathbb{K}$ denotes a field of characteristic zero) is *augmentation simple*, that is, admits only the three *trivial* ideals: 0, $\mathbb{C}G$, and the augmentation ideal. This problem has now been solved with the contribution of several authors – see [6] for the last step. On the other hand, when the ideal

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lattice of $\mathbb{K}G$ is non-trivial, its structure is known in just a few cases (see [10], [3], [6]).

When G is a locally finite group and $\mathbb{K}$ is a field of characteristic zero, the existence of non-trivial ideals in $\mathbb{K}G$ is reflected in the subgroup lattice of G. A subgroup X of G is said to be *confined* in G if there exists a finite subgroup $F \leq G$ such that $F \cap X^g \neq 1$ for every $g \in G$. In their remarkable paper [4], Hartley and Zalesskiĭ proved that in the presence of non-trivial ideals there are proper confined subgroups in G. It follows from [5] and [6] that the converse also holds. Nevertheless it is still unclear how to relate a particular proper confined subgroup to a non-trivial ideal. A reasonable conjecture is that, if X is a proper confined subgroup in G, and if Ω the set of right cosets of X in G, then the annihilator in $\mathbb{K}G$ of the permutation module $\mathbb{K}\Omega$ is a non-trivial ideal. This holds, for example, when G is a finitary transvection group [3].

In [8] and [5], the confined subgroups in *finitary* simple locally finite groups were determined. In the present paper, we shall describe the confined subgroups in diagonal limits of finite alternating groups (L$\mathfrak{A}$-*groups* for short). To be precise, an L$\mathfrak{A}$-*group* shall be a non-finitary group G which is a direct limit of finite alternating groups $G_i = \mathrm{Alt}(\Omega_i)$ $(i \in I)$ such that each G_i has only trivial or natural orbits on the sets Ω_j $(j > i)$. A non-trivial transitive permutation representation of G on a set Ω is said to be *natural*, if every G_i has only trivial or natural orbits on Ω. Our classification of confined subgroups is as follows.

Theorem A. *Let X be a proper subgroup of the* L$\mathfrak{A}$*-group G.*

(a) *If X is the intersection of finitely many point stabilizers of natural permutation representations of G, then every overgroup of X in G is confined in G.*
(b) *If X is confined in G with respect to the finite subgroup F, then X contains a simple normal subgroup N, which is an intersection of point stabilizers as in part* (a)*, and whose index in X is finite and bounded by a function of $|F|$.*

As with finitary simple locally finite groups, the confined subgroups of L$\mathfrak{A}$-groups are closely related to the action on certain structures which are naturally associated to the groups. When we first approached our problem, we were under the impression that it should be possible to find, for any given confined subgroup X in the L$\mathfrak{A}$-group G, a natural permutation representation of G in such a way that the normal subgroup N in part (b) becomes the pointwise stabilizer of a finite subset in that representation. As it turns out, this is not always possible. Example 3.1 will illustrate this somewhat unpleasant behaviour.

It follows from Theorem A that every L$\mathfrak{A}$-group of cardinality $\aleph$ has $2^{\aleph}$ confined subgroups (Theorem 3.3), a behaviour which we met already in the context of finitary simple locally finite groups. In view of the link between confined subgroups and ideals, this fact may come as a surprise, since the group algebras of $\mathrm{L}\mathfrak{A}$-groups have just countably many ideals [10] (which form a descending chain of length $\omega+1$). Nevertheless, along the lines of the above mentioned conjecture, we can still relate confined subgroups of the $\mathrm{L}\mathfrak{A}$-group G naturally to ideals in $\mathbb{K}G$.

Theorem B. *Let G be an $\mathrm{L}\mathfrak{A}$-group, and let $\mathbb{K}$ be a field of characteristic zero. Suppose that the confined subgroup X in G contains a simple normal subgroup of finite index which is the intersection of d pairwise distinct point stabilizers of natural permutation representations of G. Then $\bigcap_{g\in G}(X^g-1)\mathbb{K}G$ is the d^{th} non-trivial ideal in the descending chain of ideals in $\mathbb{K}G$.*

We shall finally consider the question of when non-trivial ideals in the complex group algebra $\mathbb{C}G$ of an $\mathrm{L}\mathfrak{A}$-group G are definable in terms of normalized positive definite class functions $f\colon G\to\mathbb{C}$. The way in which an ideal $J(f)$ is obtained from such a function f is detailed in Section 5. In [11, Section 5], Zalesskiĭ gave two examples of $\mathrm{L}\mathfrak{A}$-groups G for which the largest non-trivial ideal in $\mathbb{C}G$ is not of the form $J(f)$. These examples now illustrate the following general result.

Theorem C. *Let G be an $\mathrm{L}\mathfrak{A}$-group, which is the diagonal limit of subgroups $G_i=\mathrm{Alt}(\Omega_i)$ $(i\in I)$.*

(a) *If some non-trivial ideal in $\mathbb{C}G$ is of the form $J(f)$ for a normalized positive definite class function $f\colon G\to\mathbb{C}$, then the number of non-trivial G_i-orbits in Ω_j $(j>i)$ is asymptotically a linear function of $|\Omega_j|$ for every $i\in I$.*

(b) *Conversely, if for some $i\in I$ the number of non-trivial G_i-orbits in Ω_j $(j>i)$ is asymptotically a linear function of $|\Omega_j|$, then every non-trivial ideal in $\mathbb{C}G$ is of the form $J(f)$ for some normalized positive definite class function $f\colon G\to\mathbb{C}$.*

Quite surprisingly, Theorem C singles out a particular class of diagonal limits of finite alternating groups. Amongst them are the *strictly* diagonal limits of finite alternating groups G_i, that is, $\mathrm{L}\mathfrak{A}$-groups where each G_i fixes no point in any Ω_j $(j>i)$. Note that the number of non-trivial G_i-orbits in Ω_j $(j>i)$ is always bounded by the linear function $|\Omega_j|/|\Omega_i|$. Therefore the linear growth asserted in Theorem C is maximal. Sublinear growth is equivalent to saying that the ratio $|\mathrm{fix}_{\Omega_j}G_i|/|\Omega_j|$ tends to 1 for $j\gg i$ (Proposition 5.1). We shall close by showing that every $\mathrm{L}\mathfrak{A}$-group has at least $2^{\aleph_0}$ indecomposable normalized positive definite class functions (Theorem 5.2). A detailed description of these *extremal characters* will be the topic of a subsequent paper by the authors.

2. Classification of confined subgroups

Let $\mathbb{K}$ be a field of characteristic zero, and let G be a direct limit of finite alternating groups. The group algebra $\mathbb{K}G$ has non-trivial ideals if and only if G is an $\mathrm{L}\mathfrak{A}$-group or an alternating group [10]. Since the alternating groups have been dealt with in [8], we shall only consider $\mathrm{L}\mathfrak{A}$-groups in the sequel.

Throughout this section, we shall use the following notation. G will be a non-finitary group which has a local system of finite simple alternating groups $G_i = \mathrm{Alt}(\Omega_i)$ $(i \in I)$ such that each G_i has only trivial or natural orbits on the sets Ω_j $(j > i)$. Moreover X will denote a proper confined subgroup in G with respect to the finite subgroup F. By enlarging F and passing to a subsystem of the local system we may always assume that F is a finite alternating group of some degree n, which is contained in every G_i and which has only trivial or natural orbits on each of the sets Ω_i. This implies, in particular, that $X_i = X \cap G_i$ is confined in G_i with respect to F for every i. We may also assume that I contains a unique minimal index which we denote by 0. In general F will be a proper subgroup of G_0.

Construction 2.1. The group G has natural permutation representations.

Proof. For each i, we choose a G_0-map $\eta_{0i}\colon \Omega_0 \longrightarrow \Omega_i$. Since every G_i acts transitively on Ω_i, there are unique G_i-maps $\eta_{ij}\colon \Omega_i \longrightarrow \Omega_j$ for all $j > i$ such that $\eta_{0i}\eta_{ij} = \eta_{0j}$. The direct limit of the sets Ω_i and maps η_{ij} is a natural G-set. □

Remark 2.2. Every finite subgroup of G has infinitely many non-trivial orbits on every non-trivial G-set Ω.

Proof. Because the finitary symmetric group on a set Ω is a normal subgroup in $\mathrm{Sym}(\Omega)$, the non-finitary simple group G cannot contain a non-trivial element with finite support. □

Proposition 2.3. *The confined subgroup X of G has a unique infinite orbit Δ on every natural G-set Ω, and $|\Omega \setminus \Delta| \leq n-3$.*

Proof. Let $\Theta = \{\omega_1, \dots, \omega_n\}$ be one of the natural F-orbits in Ω. Assume by way of contradiction that X has at least n orbits $\Delta_1, \dots, \Delta_n$ in Ω. Since G acts highly transitively on Ω, there is an element $g \in G$ such that $\omega_k g \in \Delta_k$ for $1 \leq k \leq n$. Every $x \in F^g \cap X$ must fix each $\omega_k g$. Since every non-trivial F^g-orbit in Ω is isomorphic to Θg, it follows that x acts trivially on Ω. Thus $F^g \cap X = 1$, a contradiction. This contradiction shows that X has at most $n-1$ orbits in Ω. Because Ω is infinite, one of them must also be infinite.

Let Δ be an infinite X-orbit, and let $\Gamma = \Omega \setminus \Delta$. Assume that Γ contains at least $n-2$ elements. By Remark 2.2 F has infinitely many non-trivial orbits on Ω. Choose $n-2$ distinct natural F-orbits $\Theta_1, \ldots, \Theta_{n-2}$ and points $\theta_k \in \Theta_k$ in such a way that each θ_k has a different stabilizer S_k in F. Since G acts highly transitively on Ω, there is an element $g \in G$ such that $\Theta_k g \cap \Gamma = \{\theta_k\}$ for all k. It follows that $F^g \cap X \subseteq S_1 \cap \cdots \cap S_{n-2} = 1$, a contradiction. Hence $|\Gamma| \leq n-3$. □

Proposition 2.4. *In the situation of Proposition 2.3, the subgroup X acts primitively on Δ.*

Proof. Assume that X has a proper system of imprimitivity in its action on the infinite orbit Δ. Suppose first that the system is finite. In this case each block is infinite, and there are at least two blocks Δ_1 and Δ_2. Choose pairwise distinct natural F-orbits Θ_f and Λ_f $(1 \neq f \in F)$ in Δ. Pick elements $\alpha_f, \beta_f \in \Theta_f$ and $\sigma_f, \tau_f \in \Lambda_f$ in such a way that

- their point stabilizers satisfy $F_{\alpha_f} = F_{\sigma_f}$ and $F_{\beta_f} = F_{\tau_f}$ for $1 \neq f \in F$, and
- $\alpha_f f = \beta_f$ and $\sigma_f f = \tau_f$ for $1 \neq f \in F$.

By high transitivity we can now find some $g \in G$ which fixes the finite set $\Omega \setminus \Delta$ and satisfies

$$(\alpha_f)g^{-1} \in \Delta_1,\ (\beta_f)g^{-1} \in \Delta_2,\ \text{ and } (\Lambda_f)g^{-1} \subseteq \Delta_1 \text{ for every } f \in F \setminus 1.$$

Consider an element $1 \neq f \in F \cap X^g$. It moves α_f to β_f, hence $\Delta_1 g$ onto $\Delta_2 g$. On the other hand f must leave $\Delta_1 g$ fixed because σ_f and τ_f belong to $\Delta_1 g$. But this is a contradiction since $\Delta_1 g$ and $\Delta_2 g$ are blocks under the action of X^g on Δg. It follows that the system of imprimitivity must be infinite.

Now we know that each block contains at least two points. Recall that n denotes the natural degree of the finite alternating group F. We consider n^2 distinct X-blocks $\Delta_{k\ell}$ $(k, \ell = 1, \ldots, n)$. In each $\Delta_{k\ell}$ we pick two distinct points $\theta_{k\ell}$ and $\lambda_{k\ell}$. Then we define the sets $\Theta_s = \{\theta_{s1}, \ldots, \theta_{sn}\}$ for every s, and $\Lambda_r = \{\lambda_{1r}, \ldots, \lambda_{nr}\}$ for every r. Using high transitivity we obtain $g \in G$ such that the sets Θ_s and Λ_r become natural F^g-orbits whose similarity is explained by the following F^g-maps:

$\Theta_{s_1} \sim \Theta_{s_2}$ via $\theta_{s_1\ell} \mapsto \theta_{s_2\ell}$ for all ℓ,
$\Lambda_{r_1} \sim \Lambda_{r_2}$ via $\lambda_{kr_1} \mapsto \lambda_{kr_2}$ for all k, and
$\Theta_s \sim \Lambda_r$ via $\theta_{sj} \mapsto \lambda_{jr}$ for all j.

Consider a non-trivial element $x \in F^g \cap X$. There is an index j such that $\theta_{jj}x \neq \theta_{jj}$. As λ_{jj} and θ_{jj} lie in the same block, x must move λ_{jj} too. However, $\theta_{jj}x = \theta_{j\ell}$ for some $\ell \neq j$ while $\lambda_{jj}x = \lambda_{kj}$ for some $k \neq j$. The first equation implies $\Delta_{jj}x = \Delta_{j\ell}$ while the second equation implies $\Delta_{jj}x = \Delta_{kj}$. This final contradiction shows that X cannot have a system of imprimitivity in its action on Δ. □

It is useful to note that Remark 2.2 and Propositions 2.3 and 2.4 have a *local* version. The reader can easily convince himself that the above proofs work for the subgroups G_i in place of G provided that they are large enough. This leads to the following conclusion.

PROPOSITION 2.5. *There exists $i \in I$, depending on X and F, such that for every $j > i$, the subgroup $X_j = X \cap G_j$ has a unique largest orbit Δ_j in Ω_j, and the complement $\Gamma_j = \Omega_j \setminus \Delta_j$ has size at most $n-3$. Moreover X_j acts primitively on Δ_j.* □

The next result will actually be proved from the above local version.

PROPOSITION 2.6. *In the situation of Proposition 2.5, a large enough choice of the index i ensures that X_j contains the pointwise stabilizer in G_j of Γ_j for all $j > i$.*

Proof. We know that $|\Gamma_j| \leq n-3$. A large choice of i allows us to assume that $|\Delta_j| > 16\,n^4$, and that F has at least $n!\,n$ natural orbits $\Lambda_{k,\ell} = \{\lambda_{k,\ell,1}, \ldots, \lambda_{k,\ell,n}\}$ $(k = 1, \ldots, n!,\ \ell = 1, \ldots, n)$ on Δ_j for every $j > i$. Let $\Lambda_k = \Lambda_{k,1} \cup \cdots \cup \Lambda_{k,n}$. For each k there is an element $g_k \in G_j$ which fixes $\Omega_j \setminus \Lambda_k$ pointwise and sends each $\lambda_{k,\ell,m}$ to $\lambda_{k,m,\ell}$. The first condition implies that f and f^{g_k} act in the same way on $\Omega_j \setminus \Lambda_k$ for every $f \in F$, while the second condition ensures that $F^{g_r} \cap F^{g_s}$ is trivial whenever $r \neq s$.

Because X_j is confined in G_j with respect to F, there exists a non-trivial element $x_k \in F^{g_k} \cap X_j$ for each k. By the Pigeon Hole Principle there are some $f \in F$ and two indices $r \neq s$ with $x_r = f^{g_r}$ and $x_s = f^{g_s}$. In particular x_r and x_s coincide with f on $\Omega_j \setminus (\Lambda_r \cup \Lambda_s)$. But then $y = x_r x_s^{-1}$ is a non-trivial element in X_j whose support is contained in $\Lambda_r \cup \Lambda_s \subset \Delta_j$. It follows that the image of y in $\overline{X_j} = X_j/(X_j)_{(\Delta_j)}$ is non-trivial. By Proposition 2.5, $\overline{X_j}$ is a primitive permutation group on Δ_j, and $\overline{y}$ is an element of degree at most $2n^2 < \sqrt{|\Delta_j|}/2$ in $\overline{X_j}$. By [1, Theorems 5.3A and 5.4A] the alternating group on Δ_j is contained in $\overline{X_j}$.

Let H_j be the preimage of this alternating group in X_j, and consider the alternating group $A_j = (G_j)_{(\Gamma_j)}$. The product $H_j A_j$ is a subgroup because Δ_j is X_j-invariant and A_j is a normal subgroup of the setwise stabilizer $(G_j)_{\{\Delta_j\}}$. Clearly $[A_j, (X_j)_{(\Delta_j)}] = 1$, so that we can consider the quotient $\overline{H_j A_j}$, which has the same order as $\overline{H_j}$. This shows that $A_j \leq X_j$, as desired. □

In the sequel we shall assume that the minimal index $0 \in I$ is chosen so large that the conclusions of Propositions 2.5 and 2.6 hold for every G_i. Moreover, after passing to a suitable subsystem of the local system in G, we may assume that there is a fixed $\nu \leq n-3$ such that $|\Gamma_i| = \nu$ for all i. We now study a particularly important special case of the above situation.

Proposition 2.7. *Suppose that each X_i $(i \in I)$ is the pointwise stabilizer of a set Γ_i of fixed size ν. Then the confined subgroup X is an intersection of ν point stabilizers of natural permutation representations of G.*

Proof. If $\nu = 0$, then $X = G$, and the claim holds. Suppose now that $\nu \geq 1$. We say that two points in some G_0-sets are 0-equivalent if their stabilizers in G_0 coincide. Note that no two distinct elements of Γ_0 are 0-equivalent, because Γ_0 lies in the natural G_0-orbit Ω_0.

Assume by way of contradiction that some Γ_i contains a non-empty subset Θ which is fixed by G_0. Then $X_0 = X_i \cap G_0$ consists of the elements in G_0 which fix precisely those points in Ω_0 which are 0-equivalent to some point in $\Gamma_i \setminus \Theta$. In particular, X_0 is the pointwise stabilizer in G_0 of a subset of Ω_0 of size $< \nu$. This contradiction shows that every Γ_i is contained in a union of natural G_0-orbits. The same kind of argument shows that no Γ_i contains two 0-equivalent points.

Assume next that some Γ_i contains a point γ which is not 0-equivalent to any point in Γ_0. Then $X_0 = X_i \cap G_0$ fixes not only Γ_0 pointwise, but also the point in Ω_0 which is 0-equivalent to γ. In particular, X_0 is the pointwise stabilizer in G_0 of a subset of Ω_0 of size $> \nu$, again a contradiction. Altogether it follows that there are bijections $\epsilon_{0i}\colon \Gamma_0 \longrightarrow \Gamma_i$ for all i which preserve 0-equivalence. Clearly, $\epsilon_{0j} = \epsilon_{0i}\epsilon_{ij}$ whenever $i < j$.

Pick a point $\gamma \in \Gamma_0$. Since every G_i acts transitively on Ω_i, there are unique G_i-maps $\eta^{\gamma}_{ij}\colon \Omega_i \longrightarrow \Omega_j$ $(i<j)$ such that $\gamma\eta^{\gamma}_{0i} = \gamma\epsilon_{0i}$ and $\eta^{\gamma}_{ik} = \eta^{\gamma}_{ij}\eta^{\gamma}_{jk}$ for all $i<j<k$. The direct limit Σ_γ of the sets Ω_i and maps η^{γ}_{ij} is a natural G-set, and γ can be identified with a certain point in Σ_γ. Let G_γ denote the stabilizer in G of this point. Of course, $X \leq \bigcap_{\gamma\in\Gamma_0} G_\gamma$. On the other hand, $G_i \cap \bigcap_{\gamma\in\Gamma_0} G_\gamma = (G_i)_{(\Gamma_i)} = X_i \leq X$ for all i. Hence X is equal to the intersection of point stabilizers G_γ $(\gamma \in \Gamma_0)$. □

Proposition 2.8. *The arbitrary confined subgroup X contains a simple normal subgroup N of index at most $\nu!$, whose intersection N_i with G_i is the pointwise stabilizer in G_i of Γ_i for all i. Moreover, N is confined in G.*

Proof. By Proposition 2.6 every X_i contains the simple normal subgroup $N_i = (G_i)_{(\Gamma_i)}$. For $j > i$, the intersection $N_i \cap N_j$ is either trivial or equal to N_i. Since we can assume that $\nu < |\Omega_i \setminus \Gamma_i|$, the group N_i is too large to embed into X_j/N_j. Hence $N_i \leq N_j$ for all $j > i$, and $N = \bigcup_{i\in I} N_i$ is a simple normal subgroup of index at most $\nu!$ in X. The discussion of 0-equivalence in the proof of Proposition 2.7 shows that $N_j \cap G_i = N_i$ whenever $i < j$. Hence $N \cap G_i = N_i$ for all i.

Finally we shall show that N is confined with respect to G_0. To this end, consider an element g in some G_i. Since $|\Gamma_i| = \nu \ll |\Omega_0|$, there are at least three points in the natural G_0-orbit Ω_0 whose stabilizers are different from the stabilizer of every point in $\Gamma_i g$. The 3-cycle z on these points lies in G_0 and acts trivially on $\Gamma_i g$. Therefore z is a non-trivial element in $G_0 \cap N_i^g$. $\square$

Proof of Theorem A. The argument used to show the last statement of the above proposition can be applied accordingly to establish part (a). Suppose next that X is confined in G with respect to F. Consider the subgroup $N \leq X$ as in Proposition 2.8. Part (b) of Theorem A is a consequence of Proposition 2.7, applied to N in place of X. $\square$

3. Some examples

We shall now give an example to illustrate that certain intersections of point stabilizers of natural permutation representations of an $\mathrm{L}\mathfrak{A}$-group cannot be expressed as pointwise stabilizers of a finite set of points in a better natural permutation representation.

Example 3.1. We shall construct recursively an ascending chain of finite alternating groups $G_i = \mathrm{Alt}(\Omega_i)$ $(i \in \mathbb{N})$ as follows. Let $G_0 = \mathrm{Alt}(\Omega_0)$ be any alternating group of degree at least five. Given G_i, let Ω_{i+1} be the disjoint union of two copies Ω_{i1} and Ω_{i2} of the G_i-set Ω_i, and embed G_i diagonally into G_{i+1}. The union G of the ascending chain of groups G_i is an $\mathrm{L}\mathfrak{A}$-group.

Pick two distinct points α_0 and β_0 in Ω_0. Given $\alpha_i, \beta_i \in \Omega_i$, let $\alpha_{i+1} \in \Omega_{i1}$ resp. $\beta_{i+1} \in \Omega_{i2}$ be the corresponding points under the above G_i-similarities $\Omega_{i1} \sim \Omega_i \sim \Omega_{i2}$. Finally, let $\Gamma_i = \{\alpha_i, \beta_i\}$ for all i. Clearly, $(G_i)_{(\Gamma_{i+1})} = (G_i)_{(\Gamma_i)}$ for all i. Therefore $X = \bigcup_{i \geq 0} (G_i)_{(\Gamma_i)}$ is a subgroup in G. It follows as in the proof of Proposition 2.8 that X is confined in G. Note that X is simple as it is the union of the finite alternating groups $(G_i)_{(\Gamma_i)}$.

Now let Σ be any natural G-set. Fix some $\sigma \in \Sigma$ and consider the finite sets $\Sigma_i = \sigma G_i$ $(i \in \mathbb{N})$. Every Σ_i is of course similar to Ω_i as G_i-set. Assume that $X = G_{(\Delta)}$ for some finite subset $\Delta \subseteq \Sigma$. By the transitivity of G on Σ there exists $i \geq 0$ with $\Delta \subseteq \Sigma_i$. In particular, $\Delta \subseteq \mathrm{fix}_{\Sigma_i}(X \cap G_i)$. From the G_i-similarity between Σ_i and Ω_i it is clear that Δ consists of the two elements $\delta_1, \delta_2 \in \Sigma_i$ which correspond to α_i and β_i in Ω_i. The G_{i+1}-similarity between Σ_{i+1} and Ω_{i+1} yields that δ_1 and δ_2 lie in distinct G_i-orbits in $\Sigma_{i+1} \subset \Sigma$. This holds for all but finitely many i. We conclude that δ_1 and δ_2 lie in distinct G-orbits in Σ. But this contradicts the transitivity of G on Σ. $\square$

Another interesting fact is that a proper confined subgroup in an $\mathrm{L}\mathfrak{A}$-group G might act transitively on some natural G-set.

Example 3.2. Let G be an $\mathrm{L}\mathfrak{A}$-group as constructed in Example 3.1. Choose a finite subset $\Gamma_0 \subseteq \Omega_0$. Given Γ_i, let $\Gamma_{i+1} \subseteq \Omega_{i2}$ be the corresponding subset under the above G_i-similarity $\Omega_i \sim \Omega_{i2}$. Now $X = \bigcup_{i \geq 0} (G_i)_{(\Gamma_i)}$ is a proper confined subgroup in G. In fact, X is the pointwise stabilizer of two points in the natural G-set which is the direct limit of the G_i-sets Ω_i with respect to the G_i-maps $\Omega_i \longrightarrow \Omega_{i2} \subset \Omega_{i+1}$. However, X acts transitively on the natural G-set which is the direct limit of the G_i-sets Ω_i with respect to the G_i-maps $\Omega_i \longrightarrow \Omega_{i1} \subset \Omega_{i+1}$. $\square$

We conclude our discussion of confined subgroups in $\mathrm{L}\mathfrak{A}$-groups by counting their number.

THEOREM 3.3. *Every $\mathrm{L}\mathfrak{A}$-group G of cardinality $\aleph$ has $2^{\aleph}$ confined subgroups.*

Proof. Let G be the union of the direct limit of finite groups $G_i = \mathrm{Alt}(\Omega_i)$ $(i \in I)$ with diagonal embeddings. Consider Construction 2.1. Fix some $\omega_0 \in \Omega_0$, and let $\omega_i = \omega_0\eta_{0i} \in \Omega_i$ for all i. Since every G_i acts transitively on Ω_i, the permutation representation of G on the direct limit Ω of the sets Ω_i and maps η_{ij} is completely determined by the tuple $(\omega_i)_{i\in I}$, and every choice of a tuple gives rise to a natural permutation representation of G.

Let X be the point stabilizer in G of the point in Ω corresponding to $(\omega_i)_{i\in I}$. Then there exists some $i \in I$ such that $X \cap G_j = (G_j)_{(\omega_j)}$ for all $j > i$. By choice of the tuple we have $(G_j)_{(\omega_j)} \cap G_k = (G_k)_{(\omega_k)}$ for all $k < j$, whence $X \cap G_k = (G_k)_{(\omega_k)}$ for all k. It follows that any tuple which differs from $(\omega_i)_{i\in I}$ gives rise to a different point stabilizer in G. Since there are $2^{\aleph}$ possible tuples, we obtain $2^{\aleph}$ point stabilizers. Each of them is confined by Theorem A. $\square$

4. Connection between confined subgroups and ideals

In this section, $\mathbb{K}$ will always denote a field of characteristic zero, and G will always be an $\mathrm{L}\mathfrak{A}$-group with a local system of finite alternating subgroups G_i $(i\in I)$, embedded diagonally into each other. In order to prove Theorem B, we need to recall a few facts from the representation theory of finite alternating groups (cf. [10]).

To every irreducible $\mathbb{K}\,\mathrm{Alt}(n)$-representation φ we can associate a Young diagram which is completely described by a finite non-increasing sequence $(\ell_1, \ell_2, \ldots, \ell_k)$ of positive integers with sum n. Here ℓ_i is the number of cells in the i-th row of the Young diagram. The *depth* $\delta(\varphi)$ of the representation is defined as $\delta(\varphi) = \ell_2 + \cdots + \ell_k$. The trivial representation is the unique irreducible representation of depth zero. The Young diagram of the unique irreducible representation η_n of depth one is described by the sequence $(n–1, 1)$. We denote the $\mathbb{K}\,\mathrm{Alt}(n)$-module affording η_n by E_n.

LEMMA 4.1 ([10, Lemma 5]). *Let $d > 1$. Then the set of all irreducible components of the d-fold inner tensor power of the representation η_n of $\mathrm{Alt}(n)$ coincides with $\{\, \psi\in\mathrm{Irr}_{\mathbb{K}}(\mathrm{Alt}(n)) \mid \delta(\psi)\le d \,\}$.* $\square$

LEMMA 4.2 ([10, Lemma 9]). *Let $k > 4$. If the group $\mathrm{Alt}(k)$ is embedded diagonally in $\mathrm{Alt}(n)$ and $\mathrm{Alt}(k)$ has at least $d+1$ non-trivial orbits on the natural $\mathrm{Alt}(n)$-set, then every irreducible representation of $\mathrm{Alt}(k)$ of depth $\leq d$ is a constituent of every irreducible representation of $\mathrm{Alt}(n)$ of depth d.* $\square$

A set $\Phi = \{\Phi_i \mid i \in I\}$ is said to be an *inductive system* if

- each Φ_i is a set of irreducible $\mathbb{K}G_i$-representations (resp. $\mathbb{K}G_i$-modules), and
- whenever $i < j$, then Φ_i consists precisely of the irreducible $\mathbb{K}G_i$-constituents of the representations (resp. modules) in Φ_j.

By [11, Prop. 1.2] the inductive systems are in one-to-one correspondence with the ideals in the group algebra $\mathbb{K}G$. Here an ideal J corresponds to the inductive system $\Phi = \{\Phi_i \mid i \in I\}$, where Φ_i consists of the irreducible $\mathbb{K}G_i$-modules occurring in $\mathbb{K}G_i/(J \cap \mathbb{K}G_i)$. By [10] the ideals in $\mathbb{K}G$ form a chain $\mathbb{K}G \supset J_0 \supset J_1 \supset \cdots \supset \bigcap_{d\in\mathbb{N}} J_d = 0$, and we denote the inductive system corresponding to J_d by $\Phi(d)$. It was shown in [10] that $\Phi_i(d) = \{\psi \in \mathrm{Irr}_{\mathbb{K}}(G_i) \mid \delta(\psi) \leq d\}$.

If V is any $\mathbb{K}G$-module, we can define Φ_i to be the set of all irreducible $\mathbb{K}G_i$-submodules of V. Then $\Phi = \{\Phi_i \mid i \in I\}$ is an inductive system, which we call the inductive system associated to V. The corresponding ideal in $\mathbb{K}G$ is the annihilator of V.

LEMMA 4.3. *Let $\Omega_1, \ldots, \Omega_d$ be natural G-sets. Then $\Phi(d)$ is the inductive system associated to the $\mathbb{K}G$-module $V = \mathbb{K}(\Omega_1 \times \cdots \times \Omega_d) = \mathbb{K}\Omega_1 \otimes \cdots \otimes \mathbb{K}\Omega_d$.*

Proof. Let $G_i = \mathrm{Alt}(n_i)$. Clearly V can be decomposed into a direct sum of submodules of the form $L_1 \otimes \cdots \otimes L_d$ where each L_k is an irreducible $\mathbb{K}G_i$-submodule of $\mathbb{K}\Omega_k$. But every non-trivial irreducible $\mathbb{K}G_i$-submodule of $\mathbb{K}\Omega_k$ is isomorphic to E_{n_i}. Since V contains a $\mathbb{K}G_i$-submodule where all the L_k are isomorphic to E_{n_i}, Lemma 4.1 can be applied to get the claim. $\square$

PROPOSITION 4.4. *Let $\Omega_1, \ldots, \Omega_d$ be natural G-sets such that the stabilizers in G of certain points $\omega_k \in \Omega_k$ $(k = 1, \ldots, d)$ are pairwise different. Let Δ be the G-orbit of $(\omega_1, \ldots, \omega_d)$ in $\Omega_1 \times \cdots \times \Omega_d$. Then $\Phi(d)$ is the inductive system associated to the $\mathbb{K}G$-module $\mathbb{K}\Delta$.*

Proof. Consider a tuple $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Omega_1 \times \cdots \times \Omega_d$ with the property $G_{\alpha_1} = G_{\alpha_2}$. Then the map $\alpha g \longmapsto (\alpha_2, \ldots, \alpha_d)g$ embeds the G-orbit αG into the G-set $\Omega_2 \times \cdots \times \Omega_d$. By Lemma 4.3, the inductive system associated to $\mathbb{K}(\alpha G)$ is therefore contained in $\Phi(d-1)$, and in particular strictly smaller than $\Phi(d)$. The same conclusion holds for any tuple α with two entries which have the same point stabilizer in G.

Suppose next that the tuple $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Omega_1 \times \cdots \times \Omega_d$ satisfies $G_{\alpha_k} \neq G_{\alpha_\ell}$ for all $k \neq \ell$. Choose $i \in I$ so large that $\alpha_k \in \omega_k G_i$ for $k = 1, \ldots, d$. The natural G_i-set Ω_i contains points $\alpha'_1, \ldots, \alpha'_d, \omega'_1, \ldots, \omega'_d$ satisfying $(G_i)_{\alpha_k} = (G_i)_{\alpha'_k}$ and $(G_i)_{\omega_k} = (G_i)_{\omega'_k}$ for all k. By high transitivity there is some $g \in G_i$ which maps $(\omega'_1, \ldots, \omega'_d)$ onto $(\alpha'_1, \ldots, \alpha'_d)$, and hence also $(\omega_1, \ldots, \omega_d)$ onto $(\alpha_1, \ldots, \alpha_d)$. This shows that Δ contains *every* tuple α satisfying $G_{\alpha_k} \neq G_{\alpha_\ell}$ for all $k \neq \ell$.

Consider any G_j of large enough degree. By the above reasoning and Lemma 4.3, every irreducible $\mathbb{K}G_j$-representation of depth d must be be a constituent of $\mathbb{K}\Delta$. Now Lemma 4.2 ensures that for every $i < j$ the module $\mathbb{K}\Delta$ contains every irreducible $\mathbb{K}G_i$-representation of depth at most d. On the other hand, Lemma 4.3 yields that there are no further irreducible representations. □

Proposition 4.5. *If the confined subgroup X in G is the intersection of d pairwise distinct point stabilizers of natural permutation representations of G, then $\bigcap_{g \in G}(X^g - 1)\mathbb{K}G$ is the d^{th} non-trivial ideal in the descending chain of ideals in $\mathbb{K}G$.*

Proof. Let $\Omega_1, \ldots, \Omega_d$ be natural G-sets containing points $\omega_k \in \Omega_k$ such that $X = \bigcap_{i=1}^d G_{\omega_i}$. Let $\Delta = \omega G$ where $\omega = (\omega_1, \ldots, \omega_d) \in \Omega_1 \times \cdots \times \Omega_d$. We have $\operatorname{ann}_{\mathbb{K}G}(\omega g^{-1}) = (X^g - 1)\mathbb{K}G$ for every $g \in G$. By Proposition 4.4, $\Phi(d)$ is the inductive system associated to $\mathbb{K}\Delta$. Hence $J_d = \operatorname{ann}_{\mathbb{K}G}(\mathbb{K}\Delta) = \bigcap_{g \in G} \operatorname{ann}_{\mathbb{K}G}(\omega g^{-1}) = \bigcap_{g \in G}(X^g - 1)\mathbb{K}G$. □

Proof of Theorem B. Let N denote the simple normal subgroup of finite index in X. Let $G_i = \operatorname{Alt}(\Omega_i)$ where $|\Omega_i| = n_i$, and let $X_i = X \cap G_i$, and $N_i = N \cap G_i$ for every $i \in I$. By Propositions 2.5 and 2.8 we may assume that there exists a subset Γ_i of size d in each Ω_i such that $(G_i)_{(\Gamma_i)} = N_i \leq X_i \leq S_i = (G_i)_{\{\Gamma_i\}}$. Consider the action of G by right translation on the set $[X \backslash G]$ of right cosets of X in G. As above, $\operatorname{ann}_{\mathbb{K}G}(\mathbb{K}[X \backslash G]) = \bigcap_{g \in G}(X^g - 1)\mathbb{K}G$. Hence it suffices to show that the inductive system $\Phi = \{\Phi_i \mid i \in I\}$ associated to $\mathbb{K}[X \backslash G]$ coincides with $\Phi(d)$.

Proposition 4.5, applied to N in place of X, yields $\Phi \subseteq \Phi(d)$. Conversely, by Lemma 4.2, it suffices to show that every Φ_i contains an irreducible representation of G_i of depth d. However, S_i is the intersection of G_i with the Young subgroup $\operatorname{Sym}(\Omega_i \backslash \Gamma_i) \times \operatorname{Sym}(\Gamma_i)$ of $\operatorname{Sym}(\Omega_i)$, and so the irreducible representation φ_i of G_i with Young diagram $(n_i - d, d)$ occurs as a constituent of the permutation representation of G_i on the coset space $\mathbb{K}[S_i \backslash G_i]$. Any right transversal of S_i in G_i is part of right transversals of X_i in G_i and of X in G. This shows that $\varphi_i \in \Phi_i$, as desired. □

5. Positive definite functions

In this section we shall consider the *complex* group algebra $\mathbb{C}G$ of an L$\mathfrak{A}$-group G. Let $\mathcal{F}(G)$ denote the set of all normalized positive definite complex class functions on G, that is, the set of all functions $f\colon G \to \mathbb{C}$ satisfying

- $f(1) = 1$ and $f(ghg^{-1}) = f(h)$ for all $g, h \in G$, and
- $\sum_{k=1}^{n} \sum_{\ell=1}^{n} c_k^* f(g_k^{-1} g_\ell)\, c_\ell \;\geq\; 0$ for all $g_1, \ldots, g_n \in G$ and all $c_1, \ldots, c_n \in \mathbb{C}$.

Here, c^* denotes the complex conjugate of $c \in \mathbb{C}$. Examples of such functions are the so-called *trivial* ones, namely the convex combinations of the constant function 1 and the normalized function which sends every non-trivial element in G to 0. If $f \in \mathcal{F}(G)$, then $f|_{G_i} = \sum_\chi c_\chi \chi$, where the c_χ are non-negative real numbers, and where χ ranges over the irreducible complex characters of G_i. In this situation the set $\Phi(f) = \{\Phi_i(f) \mid i \in I\}$ with $\Phi_i(f) = \{\chi \in \mathrm{Irr}_{\mathbb{C}}(G_i) \mid c_\chi > 0\}$ is an inductive system, and the corresponding ideal is of the form

$$J(f) \;=\; \left\{ \sum_{k=1}^{n} c_k g_k \in \mathbb{C}G \;\middle|\; \sum_{k=1}^{n} \sum_{\ell=1}^{n} \; c_k^* f(g_k^{-1} g_\ell)\, c_\ell \;=\; 0 \right\}$$

(see [11], Corollary 5.11 and Proposition 5.16). Note again that there are only a few possibilities for $\Phi(f)$ here. Either every $\Phi_i(f)$ is empty, or every $\Phi_i(f)$ contains all complex characters of G_i, or there exists some d such that every $\Phi_i(f)$ consists of all complex characters of G_i of depth at most d.

Proof of Theorem C. Suppose that there exists a positive integer d such that $J_d = J(f)$ for some $f \in \mathcal{F}(G)$. Without loss we may assume that $|\Omega_i| > 2d+1$. Consider some $j > i$ and let $n_j = |\Omega_j|$. Clearly, $f_j = f|_{G_j}$ is an $\mathbb{R}$-linear combination of all irreducible complex characters of G_j of depths at most d. If α is an irreducible complex character of G_j with Young diagram $(\ell_1, \ldots, \ell_k)$, then the permutation character π_α obtained from the action of G_j on the right cosets of the Young subgroup $\mathrm{Alt}(n_j) \cap (\mathrm{Sym}(\ell_1) \times \cdots \times \mathrm{Sym}(\ell_k))$ is the sum of α and a $\mathbb{Z}$-linear combination of irreducible characters preceding α in the dominance order (see [7, Cor. 2.4.7]). Therefore f_j is also an $\mathbb{R}$-linear combination of the π_α:

$$f_j = \sum_{\delta(\alpha) \leq d} c_\alpha \pi_\alpha \quad \text{for certain} \quad c_\alpha \in \mathbb{R}.$$

Let s_j denote the number of non-trivial G_i-orbits in Ω_j. Let M_α be the G_j-module with character π_α. We call this module a *Young module* for G_j. Considered as G_i-module, M_α is a $\mathbb{Z}$-linear combination of Young modules for G_i. We want to calculate the multiplicity with which M_τ occurs in this combination, where τ is the irreducible complex character of G_i with Young diagram $(n_i - d, 1, \ldots, 1)$. This multiplicity is an integral polynomial in s_j, depending on the distribution into G_i-orbits of the points of the partition of

Ω_j with respect to the Young diagram $(\ell_1, \ldots, \ell_k)$ of α. In fact, since M_α can be viewed as the permutation module arising from the action on all $(k-1)$-tuples of ℓ_r-sets $(2 \le r \le k)$, the multiplicity in question is only non-zero when $\delta(\alpha) = d$ and the points corresponding to $(\ell_2, \ldots, \ell_k)$ lie in pairwise different non-trivial G_i-orbits. Therefore we get

$$\begin{aligned}(\pi_\alpha|_{G_i}, \pi_\tau) &= \binom{s_j}{\ell_2}\binom{s_j-\ell_2}{\ell_3}\cdots\binom{s_j-\ell_2-\cdots-\ell_{k-1}}{\ell_k} \\ &\doteq \frac{s_j^d}{\ell_2!\ldots\ell_k!} = \frac{s_j^d}{\ell_\alpha}\end{aligned}$$

whenever $\delta(\alpha) = d$. Now $f_j|_{G_i} = f_i$ implies that

$$c_\tau \;\doteq\; s_j^d \left(\sum_{\delta(\alpha)=d} \frac{c_\alpha}{\ell_\alpha} \right).$$

On the other hand, the function f is normalized, that is,

$$1 \;=\; f_j(1) \;=\; \sum_{\delta(\alpha)\le d} c_\alpha \pi_\alpha(1).$$

The degrees $\pi_\alpha(1)$ are integral polynomials in n_j, and the highest power of n_j occurs again when α has depth d. In this case we have

$$\pi_\alpha(1) \;=\; \binom{n_j}{\ell_2}\binom{n_j-\ell_2}{\ell_3}\cdots\binom{n_j-\ell_2-\cdots-\ell_{k-1}}{\ell_k} \;\doteq\; \frac{n_j^d}{\ell_\alpha}.$$

It follows that

$$1 \;=\; f_j(1) \;\doteq\; n_j^d \left(\sum_{\delta(\alpha)=d} \frac{c_\alpha}{\ell_\alpha} \right) \;\doteq\; \frac{n_j^d}{s_j^d}\, c_\tau .$$

Because c_τ is fixed, we conclude that s_j must asymptotically be a linear function of n_j, as desired.

Suppose conversely that s_j is asymptotically linear in n_j, i.e., that there exists a constant $a > 0$ satisfying $an_j < s_j \le n_j/n_i$ for large j. We shall show then that $J_1 = J(f)$ for some $f \in \mathcal{F}(G)$. Let ι_{n_j} (resp. η_{n_j}) denote the irreducible characters of G_j of depths 0 (resp. 1). Choose

$$f_j \;=\; \left(1 - \frac{an_j}{s_j}\right)\iota_{n_j} + \frac{a}{s_j}\,\pi_{\eta_{n_j}} \;=\; \left(1 - \frac{a(n_j-1)}{s_j}\right)\iota_{n_j} + \frac{a(n_j-1)}{s_j}\,\frac{\eta_{n_j}}{n_j-1}.$$

Since f_j is a convex combination of normalized irreducible characters, [11, Lemma 5.4] gives $f_j \in \mathcal{F}(G_j)$ for large j. Moreover,

$$\pi_{\eta_{n_k}} \;=\; \frac{s_k}{s_j}\,\pi_{\eta_{n_j}} + (n_k - \frac{s_k}{s_j} n_j)\,\iota_{n_j} \qquad \text{whenever} \quad k \ge j \gg i.$$

We conclude that

$$\begin{aligned} f_k|_{G_j} &= \left(1-\frac{an_k}{s_k}+\frac{a}{s_k}\left(n_k-\frac{s_k}{s_j}n_j\right)\right)\iota_{n_j}+\frac{a}{s_k}\frac{s_k}{s_j}\,\eta_{n_j} \\ &= \left(1-\frac{an_j}{s_j}\right)\iota_{n_j}+\frac{a}{s_j}\,\pi_{\eta_{n_j}} \quad = \quad f_j \qquad \text{whenever} \quad k\geq j\gg i. \end{aligned}$$

Therefore $f\in\mathcal{F}(G)$ is well defined via $f|_{G_j}=f_j$ for all $j\gg i$. Because the expansion of f_j involves all irreducible characters of depth ≤ 1, it is clear that $J(f)=J_1$.

For any $d>1$, the restriction to any G_j of the d^{th} power f^d of f is a convex combination of normalized characters of the d^{th} cartesian power G_j^d of G_j, whence $f^d\in\mathcal{F}(G^d)$ by [11, Lemma 5.4]. We identify G with the diagonal subgroup of G^d. Then $f^d|_G\in\mathcal{F}(G)$. Since J_d is the d^{th} tensor power of J_1, i. e., since the representations in $\Phi_i(d)$ are the d-fold tensor products of the representations in $\Phi_i(1)$, it follows that $J_d=J(f^d|_G)$. □

The orbit growth in Theorem C can also be expressed in terms of growth of fixed point sets.

Proposition 5.1. *The number of non-trivial G_i-orbits in Ω_j $(j>i)$ is asymptotically a sublinear function of $|\Omega_j|$ if and only if the quotient $|\mathrm{fix}_{\Omega_j}G_i|/|\Omega_j|$ tends to 1 for $j\gg i$.*

Proof. With the notation developed in the proof of Theorem C, we clearly have

$$\frac{|\mathrm{fix}_{\Omega_j}G_i|}{|\Omega_j|} \quad = \quad 1-\frac{|\mathrm{supp}_{\Omega_j}G_i|}{|\Omega_j|} \quad = \quad 1-\frac{s_j}{n_j}\,n_i.$$

Hence $|\mathrm{fix}_{\Omega_j}G_i|/|\Omega_j|$ tends to 1 if and only if s_j/n_j becomes arbitrarily small. □

Finally we shall show that for every $\mathrm{L}\mathfrak{A}$-group G the set $\mathcal{F}(G)$ contains many *indecomposable* functions, that is, functions which are not proper convex combination of functions from $\mathcal{F}(G)$. In the case of $\mathrm{L}\mathfrak{A}$-groups G with sublinear orbit growth the inductive systems associated with these indecomposable functions contains all complex characters of the finite alternating subgroups G_i.

Theorem 5.2. *Every $\mathrm{L}\mathfrak{A}$-group G admits at least $2^{\aleph_0}$ indecomposable normalized positive definite class functions $f\colon G\longrightarrow\mathbb{C}$.*

Proof. Again, G is the diagonal limit of finite alternating subgroups $G_j=\mathrm{Alt}(\Omega_j)$ $(j\in I)$, and we may assume without loss that there exists a minimal index $i\in I$. Moreover, s_j (resp. t_j) denotes the number of natural (resp. trivial) G_i-orbits in Ω_j. Every G_j embeds naturally into a countably infinite alternating group A_j. Moreover, the diagonal embeddings $G_j\to G_k$ $(j<k)$

extend naturally to diagonal embeddings $A_j \to A_k$, where A_j has s_j natural and t_j trivial orbits on the natural A_k-set. Then G is contained in the diagonal limit A of the countably infinite alternating groups A_j $(j \in I)$.

Let $\mathcal{E}(G)$ be the set of all indecomposable normalized positive definite class functions $G \to \mathbb{C}$. Fix any real number $\alpha \in]\,0,1\,[$, and define $f_j^\alpha \colon A_j \longrightarrow \mathbb{R}$ $(j \in I)$ via

$$f_j^\alpha(g) \;=\; \prod_{\nu=2}^{r} (\alpha^\nu)^{\gamma_\nu/s_j} \qquad \text{whenever } g \in A_j \text{ has cycle type } (\gamma_1, \ldots, \gamma_r) \text{ as a permutation of } \Omega_j.$$

An application of [9, Satz 3] (with α^{1/s_j} in place of α_1) and [9, Satz 6] shows that $f_j^\alpha \in \mathcal{E}(A_j)$. Moreover, f_k^α extends f_j^α whenever $k > j$. Therefore, the union f^α of the f_j^α $(j \in I)$ is in $\mathcal{E}(A)$.

Suppose that $c_1 f^{\alpha_1}|_G + \cdots + c_\ell f^{\alpha_\ell}|_G = 0$ for pairwise distinct $\alpha_1, \ldots, \alpha_\ell \in]\,0,1\,[$ and certain $c_\nu \in \mathbb{C}$. Choose $j \in I$ so large that $|\Omega_j| > 2\ell$. Let $\beta_\mu = \alpha_\mu^{2/s_j}$ for all μ. Let g_ν be a $(2\nu+1)$-cycle in G_j for $1 \le \nu \le \ell$. Then

$$f^{\alpha_\mu}(g_\nu) \;=\; \alpha_\mu^{1/s_j} \beta_\mu^\nu \qquad \text{for all } \mu, \nu,$$

whence

$$\begin{pmatrix} c_1 \alpha_1^{1/s_j} & \ldots\ldots & c_\ell\, \alpha_\ell^{1/s_j} \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_1^2 & \ldots & \beta_1^\ell \\ \beta_2 & \beta_2^2 & \ldots & \beta_2^\ell \\ \vdots & \vdots & & \vdots \\ \beta_\ell & \beta_\ell^2 & \ldots & \beta_\ell^\ell \end{pmatrix} \;=\; 0\,.$$

The Vandermonde argument implies that $c_\nu = 0$ for all ν. We conclude that the functions $f^\alpha|_G \in \mathcal{F}(G)$ $(0 < \alpha < 1)$ are $\mathbb{C}$-independent. Because every function in $\mathcal{F}(G)$ is a convex combination of functions from $\mathcal{E}(G)$, we must have $|\mathcal{E}(G)| \ge 2^{\aleph_0}$. $\square$

It is uncertain whether the functions f^α constructed above are always indecomposable. However, we can show that $f^\alpha \in \mathcal{E}(G)$ whenever G has sublinear orbit growth.

Proposition 5.3. *Let G be an $\mathrm{l}\mathfrak{A}$-group with sublinear orbit growth, and form $A \ge G$ as in the proof of Theorem 5.2. Then G contains a conjugate of every finite subgroup in A. In particular, every $f \in \mathcal{F}(G)$ extends to a unique $\hat{f} \in \mathcal{F}(A)$, and the map $f \mapsto \hat{f}$ preserves decomposability.*

Proof. Since the functions in $\mathcal{F}(G)$ are class functions, it suffices to show that G contains a conjugate of every finite subgroup F in A. By choosing the minimal index i large enough we may assume that $F \le A_i$. Let Λ_j be the support of F on the natural A_j-set. Then $|\Lambda_j|/|\Omega_j| = |\Lambda_i|\, s_j/|\Omega_j|$ tends to zero for large j. Hence there exists some j such that $|\Omega_j| > |\Lambda_j|$. Because

A_j is highly transitive, we can choose $g \in A_j$ such that $\Lambda_j g \subset \Omega_j$. Then $F^g \leq G_j$. $\square$

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