Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 327–328
S 0019-2082

A NON-EXTENDABLE ABSTRACT KERNEL

L. G. KOVÁCS

To the memory of Reinhold Baer

ABSTRACT. A modification is suggested for a 1934 example of Reinhold Baer which shows that not all homomorphisms $C \to (\operatorname{Aut} A)/(\operatorname{Inn} A)$ arise from conjugation action in group extensions $1 \to A \to B \to C \to 1$.

Obstructions to group extensions were first observed in Reinhold Baer's classic paper [1]. He noted there that not all abstract kernels (or, in modern terminology, couplings) are extendable, that is, not all group homomorphisms $C \to \operatorname{Out} A$ into outer automorphism class groups $\operatorname{Out} A = (\operatorname{Aut} A)/(\operatorname{Inn} A)$ arise from conjugation action in group extensions

$$1 \to A \to B \to C \to 1.$$

Searching the literature for an explicit finite example still leads back to his paper. However, the example that he wrote down seems to need a little modification.

What Baer wrote was based on the following criterion. A finite abelian group D of exponent m is the central factor group of some other group if and only if D contains a direct product of two cyclic groups of order m. In effect, he asked: if A is a finite nilpotent group of class 2 and D is an abelian subgroup of $\operatorname{Aut} A$ including all inner automorphisms and acting trivially on the centre Z of A, why should D have to pass this test? If A and D are chosen so as to fail it, the inclusion of $C = D/(\operatorname{Inn} A)$ in $\operatorname{Out} A$ cannot be realized by any extension B, because the centre of such a B would have to be Z, and one would have to have $B/Z \cong D$. For instance, take A to be the group defined on the generators a, b, c by the relations

$$a^{n^2} = b^{n^2} = c^{n^2} = 1, \quad ba = ab, \quad ca = ac, \quad cb = a^n bc,$$

and let D be generated by $\operatorname{Inn} A$ and the automorphism α which maps a to a, b to ab, and c to c. Then D is abelian, of order n^3 and exponent n^2, so we have our example.

Received September 14, 2001.
2000 *Mathematics Subject Classification.* Primary 20E22. Secondary 20J06.

The trouble with this is that D does not act trivially on Z: clearly, α moves b^n. Indeed, the inclusion of C in $\operatorname{Out} A$ is realized by the extension B obtained by adjoining to A an element d such that $d^n = c$, $da = ad$, and $ba = abd$. Moreover, an easy answer to the rhetorical question above is: because the exponent of such a D must be the same as the exponent of $\operatorname{Inn} A$. [If δ ($\in D$) maps x ($\in A$) to xz_x (with $z_x \in Z$), then $x^n \in Z$ and so $z_x^n = 1$; hence $\delta^n = 1$.]

To set matters right, change the action of α on c: let it still map a to a and b to ab, but let it now map c to c^{n+1}. The required extension B would still have to be generated by A and an element d which conjugates A according to α, with d^n an element of A fixed by α and inducing the inner automorphism α^n. The elements inducing α^n are precisely the elements of the coset of c modulo Z, but now none of these is fixed by α, so no such B can exist.

This seems to be 'the simplest' example. As it is also very close to what Baer wrote down, it may well have been the example he intended.

It may be worth emphasizing that the phenomenon occurs even with C cyclic (though of course not if C is infinite cyclic). Another simple example, justified like that above, shows that one can even keep both C and Z to order 2: take A as the group defined on a, b, c by

$$a^{16} = b^2 = c^2 = 1, \quad ba = a^{-1}b, \quad ca = a^9 c, \quad cb = bc,$$

and let α map a to a^3, b to b, and c to $a^8 c$.

References

[1] R. Baer, *Erweiterungen von Gruppen und ihren Isomorphismen*, Math. Z. **38** (1934), 375–416.

Australian National University, Canberra ACT 0200, Australia
E-mail address: kovacs@math.anu.edu.au