Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 317–326
S 0019-2082

SOME SUBGROUPS DEFINED BY IDENTITIES

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Meinem verehrten Lehrer Reinhold Baer zum hundertsten Geburtstag gewidmet

Abstract. The subgroups studied in this paper are generalizations of the subgroup $R_2(G) = \{x \in G | [x, g, g] = 1,\ \forall g \in G\}$ of right 2-Engel elements of G. It is shown that they are actually partial margins and their embedding in G is investigated.

1. Introduction

Let G be a group and $f(x_0, x_1, \ldots, x_m)$ a word in $x_0, \ldots, x_m$. Define a subset $B_{(f)}$ of G by

$$B_{(f)}(G) = \{x \in G \mid f(x, g_1, \ldots, g_m) = 1,\ \forall g_1, \ldots, g_m \in G\}.$$

In general $B_{(f)}(G)$ is not a subgroup, but it is always a characteristic set. The terms $Z_m(G)$ of the upper central series are familiar examples, and they are subgroups. On the other hand, $f(x_0) = x_0^2$ and $f(x_0, x_1) = [x_0, x_1, x_0]$ are simple examples where $B_{(f)}(G)$ is not a subgroup. The set $R_n(G)$ of right n-Engel elements of G is defined by

$$R_n(G) = \{x \in G \mid [x, {}_n g] = 1,\ \forall g \in G\},$$

so it is $B_{(f)}(G)$ for the word $f(x_0, x_1) = [x_{0,n} x_1]$. Here commutators are denoted by $[x, y] = x^{-1}y^{-1}xy = [x, {}_1 y], [x_1, \ldots, x_n, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$, and $[x, {}_{n+1} y] = [[x, {}_n y], y]$.

For $n = 1$ this is a subgroup, namely $R_1(G) = Z_1(G)$. For $n = 2$ again $R_2(G)$ is a subgroup [4], but for $n = 3$, an example by I.D. Macdonald [8] shows that $R_3(G)$ is in general not a subgroup. More recently, Nickel [10] has shown that for any integer $n \geq 3$ there is a group G with $R_n(G)$ not a subgroup.

There are other ways of associating subsets of G with a given word $f(x_0, x_1, \ldots, x_m)$. The margin $F^*(G)$ introduced by P. Hall [2] and the partial margins $F_i^*(G)$ investigated by L.C. Kappe [3] are examples.

Received August 30, 2002.
2000 *Mathematics Subject Classification.* 20F45, 20F12.

Definition 1.1. For a given word $f(x_0, \ldots, x_m)$ define the i-th partial margin of G as

$$F_i^*(G) = \{x \in G \mid f(a_0, \ldots, xa_{i-1}, \ldots, a_m) = f(a_0, \ldots, a_{i-1}, \ldots, a_m),\\ \forall a_0, \ldots, a_m \in G\}.$$

The margin $F^*(G)$ is then the intersection of all the $F_i^*(G)$.

Unlike the sets $B_{(f)}(G)$, the margin and the partial margins are always characteristic subgroups. If the word f satisfies $f(1, x_1, \ldots, x_m) = 1$, then for $x \in F_1^*(G)$ and $a_0 = 1$ it follows from

$$f(x, a_1, \ldots, a_m) = f(1, a_1, \ldots, a_m) = 1$$

that $F_1^*(G) \subseteq B_{(f)}(G)$. The subgroups $B_n(G)$ studied in this paper are generalizations of $R_2(G)$. To simplify notation for $B_{(f)}(G)$ and the first partial margin $F_1^*(G)$ for the word

$$f(x_0, x_1, \ldots, x_{n+1}) = [x_0, x_1, \ldots, x_{n+1}, x_1]$$

we give the following definition.

Definition 1.2. For a positive integer n let

$$B_n(G) = \{x \in G \mid [x, g, a_1, \ldots, a_n, g] = 1,\ \forall g, a_1, \ldots, a_n \in G\}$$
$$C_n(G) = \{x \in G \mid [xa_0, g, a_1, \ldots, a_n, g] = [a_0, g, a_1, \ldots, a_n, g],\\ \forall g, a_0, \ldots, a_n \in G\}.$$

As observed above, $C_n(G) \subseteq B_n(G)$. It will be shown that $C_n(G) = B_n(G)$. Thus $B_n(G)$ is a characteristic subgroup of G. The remaining questions concern the structure of $B_n(G)$ and the embedding of $B_n(G)$ in G. The structure of $B_n(G)$ has already been determined by I.D. Macdonald [6], [7]: $B_n(G)$ is nilpotent of class $n + 2$ at most.

2. Preliminaries

Since $R_2(G) = \{x \in G \mid [x, g, g] = 1,\ \forall g \in G\}$ is both a tool and a model for the investigation of $B_n(G)$, the relevant facts are summarized in the next theorem.

Theorem 2.1. *Let G be a group. Then:*

(2.1.1) *$R_2(G)$ is a characteristic subgroup of G, and $R_2(G)$ is the first partial margin of $[x_0, x_1, x_1]$.*

(2.1.2) *For $x, y \in R_2(G)$ and $a, b, c \in G$ we have:*

- (a) *The normal closure x^G of x is abelian.*
- (b) $[x, a, b] = [x, b, a]^{-1}$,
- (c) $[x, [a, b]] = [x, a, b]^2$,
- (d) $[x, [a, b, c]] = 1$,
- (e) $[[x, a], [b, c]] = 1$,
- (f) $[x, a, b, c]^2 = 1$,
- (g) $[x, y, a]^3 = 1$,
- (h) $[x, y, a, b] = 1$.

(2.1.3) *$Z_2(G) \subseteq R_2(G)$, and if $[R_2(G), {}_3G]$ has no elements of order 2, then $R_2(G) \subseteq Z_3(G)$.*

(2.1.4) *For every positive integer m there exists a finite group G with $R_2(G) \nsubseteq Z_m(G)$.*

Proof. The fact that $R_2(G)$ is a subgroup is proven in [4] and that it is the first partial margin of $[x_0, x_1, x_1]$ is due to Teague [11]. The identities (a) through (d) are from [4]. Concerning (f), it was noted in [9] that $[x, a, b, c]^2 = 1$, improving on $[x, a, b, c]^4 = [x, [a, b, c]] = 1$. Furthermore, $[[x, a], [b, c]] = [x, a, b, c]^2 = 1$ gives (e), since $[x, g] \in R_2(G)$ by (2.1.1). Identities (g) and (h) generalize Levi's results on 2-Engel groups [5]. We have $[x, y, a] = [x, a, y]^{-1} = [y, [x, a]] = [y, x, a]^2 = [[x, y]^{-1}, a]^2 = [x, y, a]^{-2}$ from (b), (c), and (a). Hence $[x, y, a]^3 = 1$, proving (g). Further, $1 = [[x, y, a]^3, b] = [x, y, a, b]^3$ combined with (f) yields (h).

For (2.1.3) note that $[x, a, b, c] \in [R_2(G), {}_3G]$. The result then follows from (f) of (2.1.2).

Finally, (2.1.4) is due to Gruenberg [1]. Let G be the wreath product of a group of order 2 and a finite elementary abelian 2-group H. If the base group of G is denoted by N, then both N and G/N have exponent 2 and $N \nsubseteq Z_m(G)$ for sufficiently large H. For $x \in N$ and $g \in G, g^2 \in N$, and N abelian of exponent 2 gives $1 = [x, g^2] = [x, g][x, g, g][x, g] = [x, g]^2[x, g, g] = [x, g, g]$, so $N \subseteq R_2(G)$ and $R_2(G) \nsubseteq Z_m(G)$. □

In the next lemma and throughout the rest of the paper we will use the following familiar commutator expansion formulas without further reference:

$$[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z];$$
$$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z];$$
$$[x, y]^z = [x^z, y^z] = [x, y][x, y, z].$$

Lemma 2.2. *If $[b, g_1, \ldots, g_{m-1}, g_m, c] = 1$ for fixed $b, g_1, \ldots, g_{m-1}, c \in G$ and all $g_m \in G$, then $[[b, g_1, \ldots, g_{m-1}, g_m]^G, c^G] = 1$.*

Proof. Commutator expansion gives

$$1 = [[b, g_1, \dots, g_{m-1}, g_m d], c] = [[b, g_1, \dots, g_{m-1}, d][b, g_1, \dots, g_{m-1}, g_m]^d, c]$$
$$= [[b, g_1, \dots, g_{m-1}, g_m]^d, c]. \qquad \square$$

LEMMA 2.3. *Let $x \in B_n(G)$. Then for all $g, a_1, \dots, a_n, w_0, \dots, w_n \in G$ we have*

$$[[x, g, a_1, \dots, a_n]^G, g^G] = 1, \tag{2.3.1}$$
$$[[x, g, a_1, \dots, a_n]^G, x^G] = 1, \tag{2.3.2}$$
$$[\dots[x, g]^{w_0}, a_1]^{w_1}, \dots, a_n]^{w_n}, g] = 1. \tag{2.3.3}$$

Proof. For $x \in B_n(x)$ we have $[x, g, a_1, \dots, a_n, g] = 1$. Thus (2.3.1) follows directly from Lemma 2.2 for $m = n + 1, b = x, g_1 = c = g$ and $g_2 = a_1, \dots, g_m = a_n$. For (2.3.2) note that

$$1 = [x, xg, a_1, \dots, a_n, xg] = [x, g, a_1, \dots, a_n, g][x, g, a_1, \dots, a_n, x]^g$$
$$= [x, g, a_1, \dots, a_n, x]^g,$$

and (2.3.2) follows from Lemma 2.2.

To prove (2.3.3), note that

$$[\dots[x, g]^{w_0}, a_1]^{w_1}, \dots, a_n]^{w_n} = [[x, g], a_1^{v_1}, \dots, a_n^{v_n}]^{v_{n+1}}$$

for suitable $v_1, \dots, v_{n+1} \in G$, and observe that (2.3.1) holds for all $a_i \in G$. Thus (2.3.3) follows. $\square$

For $f \in G$, define $[f, G] = \langle [f, h] \mid h \in G \rangle$. Then $[f, h]^k = [f, k]^{-1}[f, hk]$ for $f, h, k \in G$ shows that $[f, G]$ is a normal subgroup of G. If N is normal, define inductively $[N, {}_{i+1}G] = [[N, {}_i G], G]$ and note that $[N, G_i] \subseteq [N, {}_i G]$, where G_i is the i-th term of the lower central series. For $x \in B_n(G)$ and $N = [x, g]^G$ we have $[N, {}_i G] = \langle [x, g, g_1, \dots, g_i] \mid g_1, \dots, g_i \in G \rangle$, and so (2.3.1) says that $[N, {}_n G, g] = 1$. In the following lemma a simplification is given for some terms that occur in commutator expansions.

LEMMA 2.4. *If $x \in B_n(G), v_1, \dots, v_n \in [x, g]^G$ and $a, b, a_1, \dots, a_n \in G$, then*

$$[\dots[a, b, a_1]^{v_1}, \ \dots, a_n]^{v_n}, g] = [[a, b, a_1, \dots, a_n], g].$$

Proof. Set $N = [x, g]^G$ and observe that $[a, b, a_1]^{v_1} = [a, b, a_1][a, b, a_1, v_1] \equiv [a, b, a_1]$ modulo $[N, {}_3 G]$, since $[v_1, [a, b, a_1]] \in [N, G_3] \subseteq [N, {}_3 G]$. Assume inductively that $y^v \equiv y$ modulo $[N, {}_{k+2} G]$ for $y \in G_{k+2}$ and $v \in N$. Then $y^v = zy$ for some $z \in [N, {}_{k+2} G]$ and

$$[y^v, h] = [zy, h] = [z, h]^y [y, h] \equiv [y, h] \text{ modulo } [N, {}_{k+3} G].$$

Since $[[N, {}_n G], g] = 1$ by (2.3.1), this proves the lemma. $\square$

3. Basic results for $B_n(G)$

The goal of this section is to prove the following results for $B_n(G)$.

THEOREM 3.1. *For all positive integers n and a group G we have:*

(3.1.1) $B_n(G) = C_n(G)$ *and hence* $B_n(G)$ *is a characteristic subgroup of* G.

(3.1.2) $R_2(G) \subseteq B_1(G)$ *and* $B_n(G) \subseteq B_{n+1}(G)$.

(3.1.3) $[x, g, a_1, \ldots, a_n, h, h] = 1$ *for* $x \in B_n(G)$ *and all* $g, a_1, \ldots, a_n, h \in G$, *i.e.,* $[x, g, a_1, \ldots, a_n] \in R_2(G)$.

(3.1.4) $[x, g, g, a_1, \ldots, a_n, h] = 1$ *for* $x \in B_n(G)$ *and all* $g, a_1, \ldots, a_n, h \in G$, *i.e.,* $xZ_{n+1}(G)/Z_{n+1}(G) \subseteq R_2(G/Z_{n+1}(G))$.

(3.1.5) $[x, g, a_1, \ldots, a_n, b, c, d]^2 = 1$ *for* $x \in B_n(G)$, $g, a_1, \ldots, a_n, b, c, d \in G$.

(3.1.6) $[x, g, a_1, \ldots, a_n, h] = [x, h, a_1, \ldots, a_n, g]^{-1}$ *for* $x \in B_n(G)$, $g, a_1, \ldots, a_n,\ h \in G$.

Proof. We have $[xa, g] = [x, g]^a[a, g]$ and by induction

$$[xa, g, a_1, \ldots, a_n] = [[[x, g]^{w_0}, a_1]^{w_1}, \ldots, a_n]^{w_n}[a, g, a_1, \ldots, a_n]$$

for suitable $w_0, w_1, \ldots, w_n \in G$. Since $x \in B_n(G)$, the first factor on the right commutes with g by (2.3.3). Hence $[xa, g, a_1, \ldots, a_n, g] = [a, g, a_1, \ldots, a_n, g]$, i.e., $B_n(G) \subseteq C_n(G)$, and (3.1.1) follows since $C_n(G) \subseteq B_n(G)$ was noted before.

To prove (3.1.2), let $x \in R_2(G)$. Since $R_2(G)$ is normal in G, also $[x, g] \in R_2(G)$ and from (b) of (2.1.2) we have

$$[[x, g], a, g] = [[x, g], g, a]^{-1} = [1, a]^{-1} = 1,$$

proving $R_2(G) \subseteq B_1(G)$. For $x \in B_n(G)$ we have $[x, g, a_1, \ldots, a_n, a_{n+1}] \in [x, g, a_1, \ldots, a_n]^G$, so (2.3.2.) yields $[x, g, a_1, \ldots, a_n, a_{n+1}, g] = 1$ and $B_n(G) \subseteq B_{n+1}(G)$. Commutator expansion of $[x, gh, a_1, \ldots, a_n]$ yields

$$[x, gh, a_1, \ldots, a_n] = [[x, h][x, g]^h, a_1, \ldots, a_n] = y_1y_2,$$
$$y_1 = [\ldots[x, h, a_1]^{w_1}, \ldots, a_n]^{w_n},$$
$$y_2 = [[x, g]^h, a_1, \ldots, a_n]$$

for suitable $w_1, \ldots, w_n \in G$. By (2.3.3) we have $[y_1, h] = 1$ and $[y_2, g] = 1$. Then the commutator expansion of $1 = [x, gh, a_1, \ldots, a_n, gh]$ gives

$$1 = [y_1, gh]^{y_2}[y_2, gh] = [y_1, h]^{y_2}[y_1, g]^{hy_2}[y_2, h][y_2, g]^h.$$

Hence $1 = [y_1, g]^{hy_2}[y_2, h]$.

Commuting with h and observing that $[[y_1,g]^{hy_2},h] = 1$ by (2.3.3), we obtain $[y_2,h,h] = 1$. The substitution of a_i^h for a_i finally gives

$$1 = [[x,g]^h, a_1^h, \ldots, a_n^h, h, h] = [x, g, a_1, \ldots, a_n, h, h]^h,$$

proving (3.1.3).

To prove (3.1.4), substitute $[x,g]$ for x in $1 = [y_1,g]^{hy_2}[y_2,h]$ and note that $[y_1,g] = 1$ by (2.3.3). Thus $1 = [y_2,h] = [[x,g,g]^h, a_1, \ldots, a_n, h]$ for all $a_i \in G$, proving (3.1.4).

Next, (3.1.5) follows from (3.1.3) and (f) of (2.1.2). Finally, for (3.1.6), commutator expansion of $1 = [x, gh, a_1, \ldots, a_n, gh]$, as in the proof of (3.1.3), leads to $1 = [y_1,g]^{hy_2}[y_2,h]$, where $y_1 = [\ldots[x,h,a_1]^{w_1}, \ldots, a_n]^{w_n}$ with

$$w_1 = [x,g]^h, w_2 = [w_1,a_1], \ldots, w_n = [w_{n-1}, a_{n-1}],$$

which are all elements of $[x,g]^G$. Thus Lemma 2.4 implies that $[y_1,g] = [x,h,a_1,\ldots,a_n,g]$. We have $[y_1,g]^{hy_2} = [y_1,g]$ by (2.3.1) and (2.3.2), since $y_2 = [[x,g]^h, a_1, \ldots, a_n] \in x^G$. To simplify $[y_2,h]$, write $[x,g]^h = [x,g][x,g,h]$ and expand

$$[y_2,h] = [\ldots[x,g,a_1]^{v_1}, \ldots, a_n]^{v_n}, h]^{v_{n+1}}[x,g,h,a_1,\ldots,a_n,h],$$

where

$$v_1 = [x,g,h], v_2 = [v_1,a_1], \ldots, v_n = [v_{n-1},a_{n-1}], v_{n+1} = [v_n,a_n].$$

Here $[x,g,h,a_1,\ldots,a_n,h] = 1$, since $[x,g] \in B_n(G)$ and

$$[\ldots[x,g,a_1]^{v_1}, \ldots, a_n]^{v_n}, h]^{v_{n+1}} = [x,g,a_1,\ldots,a_n,h]^{v_{n+1}}$$

by Lemma 2.4 and (2.3.2), since $v_1,\ldots,v_n \in [x,g,h]^G$ and $v_{n+1} \in x^G$. Altogether we have

$$1 = [y_1,g]^{hy_2}[y_2,h] = [x,h,a_1,\ldots,a_n,g][x,g,a_1,\ldots,a_n,h],$$

proving (3.1.6). □

4. The embedding of $B_n(G)$

The following simple observation leads to an estimate of the embedding of $B_n(G)$ in the upper central series. From (3.1.5) we have

$$[x,g,a_1,\ldots,a_n,b,c,d]^2 = 1$$

for $x \in B_n(G)$. So, if $R_2(G)$ or $B_n(G)$ have no elements of order 2, then $B_n(G) \subseteq Z_{n+4}(G)$. We will show next that this can be improved for even n.

Lemma 4.1. *Let N be a normal subgroup of G and $i \geq 1$. If $y_1, y_2 \in [N, {}_iG], a \in G$ and $y \equiv y_1y_2$ mod $[N, {}_{i+2}G]$, then $[y,a] \equiv [y_1,a][y_2,a]$ mod $[N, {}_{i+3}G]$.*

Proof. By assumption $y = zy_1y_2$ with $z \in [N, {}_{i+2}G]$ and $y_1, y_2 \in [N, {}_iG]$. Then $[z, a]^{y_1y_2} \in [N, {}_{i+3}G]$ and $[y_1, a, y_2] \in [N, {}_{i+1}G, G'] \subseteq [N, {}_{i+3}G]$ so that $[y, a] = [z, a]^{y_1y_2}[y_1, a][y_1, a, y_2][y_2, a] \equiv [y_1, a][y_2, a] \bmod [N, {}_{i+3}G]$. □

Lemma 4.2. *For $x \in B_n(G)$ and $g, t, h, a_1, \dots, a_n \in G$ we have*

$$[x, t, h, a_1, \dots, a_n, g] \equiv [x, g, a_1, \dots, a_n, t, h]^{-1} \mod [B_n(G),_{n+4} G].$$

Proof. This result is obtained from (3.1.6) by substituting th for h and commutator expansion. Let $N = x^G$. Then $[x, th] = [x, h][x, t][x, t, h]$ with $[x, h], [x, t], [x, t, h] \in [N, G]$. Apply Lemma 4.1 to obtain

$$[x, th, a_1] \equiv [x, h, a_1][x, t, a_1][x, t, h, a_1] \mod [N, {}_4G]$$

and by induction,

$$[x, th, a_1, \dots, a_n, g] \equiv [x, h, a_1 \dots, a_n, g][x, t, a_1, \dots, a_n, g][x, t, h, a_1, \dots, a_n, g]$$

modulo $[N, {}_{n+4}G]$. We also have

$$[x, g, a_1, \dots, a_n, th] = [x, g, a_1, \dots, a_n, h][x, g, a_1, \dots, a_n, t][x, g, a_1, \dots, a_n, t, h].$$

All these commutators commute by (2.3.2), and (3.1.5) gives

$$[x, y, a_1, \dots, a_n, g] = [x, g, a_1, \dots, a_n, y]^{-1}$$

for $y = th, t$ and h. Together this yields

$$[x, t, h, a_1, \dots, a_n, g] \equiv [x, g, a_1, \dots, a_n, t, h]^{-1} \bmod [B_n(G), {}_{n+4}G],$$

the desired result. □

That some restrictions on elements of order 2 are needed for our estimates follows from $R_2(G) \subseteq B_n(G)$ and (2.1.4).

Theorem 4.3. *Let G be a group with $[B_n(G), {}_{n+4}G]$ having no elements of order 2. Then:*

(4.3.1) $B_n(G) \subseteq Z_{n+4}(G)$.

(4.3.2) $[x, t, h, a_1, \dots, a_n, g] = [x, g, a_1, \dots, a_n, t, h]^{-1}$.

(4.3.3) *If n is even and $[B_n(G), {}_{n+3}G]$ has no elements of order 2, then $B_n(G) \subseteq Z_{n+3}(G)$.*

Proof. From (3.1.5) we have $[x, g, a_1, \dots, a_n, b, c, d]^2 = 1$ and $[x, g, a_1, \dots, a_n, b, c, d] \in [B_n(G), {}_{n+4}G]$. The assumption gives $[x, g, a_1, \dots, a_n, b, c, d] = 1$ and so (4.3.1) holds. For (4.3.2) note that the elements $[x, g, a_1, \dots, a_n, b, c, d]$ generate $[B_n(G), {}_{n+4}G]$; hence $[B_n(G), {}_{n+4}G] = 1$ and (4.3.2) follows by Lemma 4.2.

Finally, since $[B_n(G), {}_{n+4}G] \subseteq [B_n(G), {}_{n+3}G]$ and by assumption $[B_n(G), {}_{n+3}G]$ has no elements of order 2, we have from (4.3.2) that

$$[x, t, h, a_1, \dots, a_n, g] = [x, g, a_1, \dots, a_n, t, h]^{-1}.$$

So repeated application gives

$$[x, g, a_1, \ldots, a_n, t, h] = [x, g, a_1, \ldots, a_n, t, h]^{(-1)^{n+3}},$$

since the permutation of the arguments is a cycle of length $n + 3$. For n even we have $[x, g, a_1, \ldots, a_n, t, h]^2 = 1$ and $[x, g, a_1, \ldots, a_n, t, h] = 1$, since $[x, g, a_1, \ldots, a_n, t, h] \in [B_n(G), {}_{n+3}G]$, which has no elements of order 2 by assumption, proving (4.3.3). □

5. An example

From Theorem 4.3 we see that both $B_1(G)$ and $B_2(G)$ are contained in $Z_5(G)$ if there are no elements of order 2. The following example shows that this can not be improved to $B_1(G) \subseteq Z_4(G)$.

Let p be an odd prime and N an elementary abelian group of order p^{30} with generators $x_1, \ldots, x_5, y_1, \ldots, y_{12}, z_1, \ldots, z_{12}, v$.

Automorphisms a, b, c, d of N of order p are defined in the table below. Let $H = \langle a, b, c, d\rangle$ and $G = H \cdot N$, the semidirect product of N by H. The six commutators $[a, b], [a, c], [a, d], [b, c], [b, d], [c, d]$ are calculated, the results also being listed in the table. From this one can see that $[s, t]$ commutes with r for any $s, t, r \in \{a, b, c, d\}$. This proves that H has class 2 and order p^{10}. Each element $h \in H$ can then be written as

$$h = a^{i_1} b^{i_2} c^{i_3} d^{\,i_4} [a, b]^{j_1} [a, c]^{j_2} [a, d]^{j_3} [b, c]^{j_4} [b, d]^{j_5} [c, d]^{j_6}$$

with integers $i_1, \ldots, i_4, j_1, \ldots, j_6$ which are unique mod p. Since

$$[x_1, a, b, c, d] = [x_2, b, c, d] = [y_1, c, d] = [z_1, d] = v \neq 1,$$

we have $x_1 \notin Z_4(G)$. To show that $x_1 \in B_1(G)$, it suffices to prove $[x_1, g, h, g] = 1$ for $g, h \in H$, since N is abelian. The verification of $[x_1, g, h, g] = 1$ is straightforward but rather lengthy and omitted here.

	a	b	c	d	$[a,b]$	$[a,c]$	$[a,d]$	$[b,c]$	$[b,d]$	$[c,d]$
x_1	x_1x_2	x_1x_3	x_1x_4	x_1x_5	$x_1y_1y_4^{-1}$	$x_1y_2y_7^{-1}$	$x_1y_3y_{10}^{-1}$	$x_1y_5y_8^{-1}$	$x_1y_6y_{11}^{-1}$	$x_1y_9y_{12}^{-1}$
x_2	x_2	x_2y_1	x_2y_2	x_2y_3	x_2	x_2	x_2	$x_2z_1z_3^{-1}$	$x_2z_2z_4^{-1}$	$x_2z_5z_6^{-1}$
x_3	x_3y_4	x_3	x_3y_5	x_3y_6	x_3	$x_3z_3z_7$	$x_3z_4z_8$	x_3	x_3	$x_3z_{10}z_{11}^{-1}$
x_4	x_4y_7	x_4y_8	x_4	x_4y_9	$x_4z_1z_7^{-1}$	x_4	$x_4z_6z_9$	x_4	$x_4z_4z_{12}$	x_4
x_5	x_5y_{10}	x_5y_{11}	x_5y_{12}	x_5	$x_5z_2z_8^{-1}$	$x_5z_5z_9^{-1}$	x_5	$x_5z_{10}z_{12}^{-1}$	x_5	x_5
y_1	y_1	y_1	y_1z_1	y_1z_2	y_1	y_1	y_1	y_1	y_1	y_1v^2
y_2	y_2	y_2z_3	y_2	y_2z_5	y_2	y_2	y_2	y_2	y_2v^{-2}	y_2
y_3	y_3	y_3z_4	y_3z_6	y_3	y_3	y_3	y_3	y_3v^2	y_3	y_3
y_4	y_4	y_4	y_4z_7	y_4z_8	y_4	y_4	y_4	y_4	y_4	y_4v^{-2}
y_5	$y_5z_3^{-1}$	y_5	y_5	y_5z_{10}	y_5	y_5	y_5v^2	y_5	y_5	y_5
y_6	$y_6z_4^{-1}$	y_6	y_6z_{11}	y_6	y_6	y_6v^{-2}	y_6	y_6	y_6	y_6
y_7	y_7	$y_7z_7^{-1}$	y_7	y_7z_9	y_7	y_7	y_7	y_7	y_7v^2	y_7
y_8	$y_8z_1^{-1}$	y_8	y_8	y_8z_{12}	y_8	y_8	y_8v^{-2}	y_8	y_8	y_8
y_9	$y_9z_6^{-1}$	$y_9z_{11}^{-1}$	y_9	y_9	y_9v^2	y_9	y_9	y_9	y_9	y_9
y_{10}	y_{10}	$y_{10}z_8^{-1}$	$y_{10}z_9^{-1}$	y_{10}	y_{10}	y_{10}	y_{10}	$y_{10}v^{-2}$	y_{10}	y_9
y_{11}	$y_{11}z_2^{-1}$	y_{11}	$y_{11}z_{12}^{-1}$	y_{11}	y_{11}	$y_{11}v^2$	y_{11}	y_{11}	y_{11}	y_{11}
y_{12}	$y_{12}z_5^{-1}$	$y_{12}z_{10}^{-1}$	y_{12}	y_{12}	$y_{12}v^{-2}$	y_{12}	y_{12}	y_{12}	y_{12}	y_{12}
z_1	z_1	z_1	z_1	z_1v	z_1	z_1	z_1	z_1	z_1	z_1
z_2	z_2	z_2	z_2v^{-1}	z_2	z_2	z_2	z_2	z_2	z_2	z_2
z_3	z_3	z_3	z_3	z_3v^{-1}	z_3	z_3	z_3	z_3	z_3	z_3
z_4	z_4	z_4	z_4v	z_4	z_4	z_4	z_4	z_4	z_4	z_4
z_5	z_5	z_5v	z_5	z_5	z_5	z_5	z_5	z_5	z_5	z_5
z_6	z_6	z_6v^{-1}	z_6	z_6	z_6	z_6	z_6	z_6	z_6	z_6
z_7	z_7	z_7	z_7	z_7v^{-1}	z_7	z_7	z_7	z_7	z_7	z_7
z_8	z_8	z_8	z_8v	z_8	z_8	z_8	z_8	z_8	z_8	z_8
z_9	z_9	z_9v^{-1}	z_9	z_9	z_9	z_9	z_9	z_9	z_9	z_9
z_{10}	$z_{10}v^{-1}$	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}
z_{11}	$z_{11}v$	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}
z_{12}	$z_{12}v$	z_{12}	z_{12}	z_{12}	z_{12}	z_{12}	z_{12}	z_{12}	z_{12}	z_{12}
v	v	v	v	v	v	v	v	v	v	v

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