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INDEFINITE BINARY QUADRATIC FORMS WITH MARKOV RATIO EXCEEDING 9

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To the memory of Reinhold Baer

Reinhold Baer's visits to the University of Chicago were memorable events. His enthusiasm was infectious, his wide knowledge of so many things was fully appreciated, and his lectures were inspiring. It is perhaps not widely known that his influence on John Thompson was crucial in John's student years. I have vivid memories of the fine times I had with him, his charming wife Marianna, and their son Klaus, a distinguished Egyptologist on the Chicago faculty.

This paper is not directly connected with any of the areas in which he worked (perhaps not all readers will be aware that his early work was in the field of topology). However, Markov chose the Mathematische Annalen for his ground breaking papers, and I think the German mathematical community appreciated the importance of these papers. The enthusiasm that Frobenius showed was impressive. So I believe that Reinhold would have thought it appropriate for this paper to be dedicated to him. I. K.

1. Introduction

A binary quadratic form is an expression of the form $px^2 + qxy + ry^2$. The variables x and y are to run over all integral values. In the literature the coefficients p, q, r are often allowed to be any real numbers, but we shall restrict them also to be integral; as long as there remain lots of unanswered questions in this case we are postponing the generalization.

The form is *indefinite* if it takes both positive and negative values. The discriminant D is $q^2 - 4pr$. As is customary, we exclude outright the case where D is a square. In particular, p and r are not 0. Whenever we say "form" we mean an indefinite binary quadratic form with discriminant not a square.

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A major role will be played by a class of forms which we shall call *R-forms*. (The “R” is intended to suggest “reduced”. We do not employ the term “reduced” because it is already used in the literature in various senses.) In an R-form the coefficient of y^2 is negative and we choose to absorb the sign in the notation. If $ax^2 + bxy - cy^2$ is an R-form (we abbreviate this to $a, b, -c$) the elements a and c are positive, $0 \le b \le a$, and a is the minimum of $|f(x, y)|$, where (x, y) run over all pairs of integers other than $(0, 0)$. If we allow ourselves to use linear changes of variable with determinant 1 or -1 and further allow ourselves to switch from f to $-f$, it is easy to see that any form can be converted into an R-form.

A form is *primitive* if the GCD (greatest common divisor) of its coefficients is 1. In many contexts one can assume that the forms under discussion are primitive but this is sometimes undesirable. The result of multiplying a form by a constant will be called a *scaling* of the form.

For an R-form $a, b, -c$ the discriminant is $D = b^2 + 4ac$. A key quantity is D/a^2. There seems to be no name for this in the literature. We propose the designation *Markov ratio*. Note that it is invariant under scaling.

2. Markov's results

In two fundamental papers [8], [9], Markov studied the case where the Markov ratio is less than 9. A first remarkable result is that (for his forms) $D = 9a^2 - 4$. Even more remarkable is the following: there is a bijection between these forms and the solutions (in positive integers) of the equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

We shall give a little detail, following the notation in [3, chapter II]. The smallest solution of (1) is 1,1,1. Then come 2,1,1 and 5,2,1. From any solution x, y, z of (1) one gets three solutions by replacing one of the variables. For example, if the chosen variable is z, it is replaced by $3xy - z$. In this way one builds the Markov tree

$$
\begin{array}{cccc}
 & & \text{1,1,1} & & \\
\hline
 & & \text{2,1,1} & & \\
\hline
 & & \text{5,2,1} & & \\
\hline
 & \text{13,5,1} & & \text{29,5,2} & \\
\hline
\text{34,13,1} & \text{194,13,5} & & \text{433,29,5} & \text{169,29,2} \\
\vdots & \vdots & & \vdots & \vdots
\end{array}
$$

(We are using the same notation for these triples and forms, but there should be no confusion.) Put aside the two top triples. For those below, one of the three replacements mentioned sends it to the triple above, and the other two send it to the two triples below. In this way one gets all solutions of (1).

There is a way of passing from each of these triples to a related form. Let the triple be m, m_1, m_2 with m the largest. We provisionally take $k \equiv m_2/m_1$

(mod m). By interchanging m_1 and m_2 if necessary we arrange that $0 \le k \le m/2$. Define l as $(k^2+1)/m$ (it is a fact that k^2+1 is divisible by m). Then the desired form is

$$m, 3m-2k, l-3k. \tag{2}$$

We wish to switch to an R-form. The change of variables $x \to x-y, y \to y$ accomplishes this and yields

$$m, m-2k, -(2m+k-l). \tag{3}$$

For both (2) and (3) the GCD of the coefficients is 1 or 2.

Our statement of Markov's theorem is as follows: any primitive R-form with Markov ratio less than 9 is a scaling of one of the forms (3).

There are available at least six accounts of Markov's theory: the original Markov papers, the combination of Frobenius [6] (or [7]) and Remak [11], Dickson [5, chapter VII], two expositions by Cassels ([2] and [3, chapter II]), and Cusick and Flahive [4].

Before proceeding we make a general remark about forms with Markov ratio exceeding 9. The situation is quite different. There are many such forms and it does not look hopeful to classify them. In what follows we get some information on some of them.

The literature on forms with Markov ratio above 9 is largely concerned with gaps in possible values of the Markov ratio. Many interesting things have been proved. The book [4] covers the state of the art as of 1989.

3. Enter the Fibonacci and Lucas numbers

Let us look at the farthest left "limb" of the Markov tree above; these triples all end in 1. The first six are

1,1,1
2,1,1
5,2,1
13,5,1
34,13,1
89,34,1.

Anyone familiar with the Fibonacci numbers will instantly recognize them. Let us recall the definition and notation: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Below we shall also need to go backwards to $F_0 = 0$ and $F_{-1} = 1$. Also relevant are the companion Lucas numbers L_n which satisfy the same recurrence but start with $L_1 = 1, L_2 = 3$.

The triples listed above are $F_n, F_{n-2}, 1$ with n odd. We are slightly surprised that, according to our search of the literature, this has apparently not been mentioned in print.

The corresponding forms in the versions (2) and (3) are

$$F_n, L_n, -F_n, \tag{4}$$

and

$$F_n, F_{n-3}, -L_n. \tag{5}$$

In both (4) and (5) the GCD of the coefficients is 1 for n prime to 3 and 2 if 3 divides n.

We plan to assign the notation G_n for the forms (5). The first six are:

$$\begin{aligned}
&G_1 : 1, 1, -1\\
&G_2 : 1, 1, -3\\
&G_3 : 2, 0, -4\\
&G_4 : 3, 1, -7\\
&G_5 : 5, 1, -11\\
&G_6 : 8, 2, -18.
\end{aligned}$$

Of course the ones with odd subscripts $(G_1, G_3, G_5, \ldots)$ are Markov forms and have discriminant $9a^2 - 4$ (as always, we are using a for the first coefficient of the form, in this case F_n). For even n, $(G_2, G_4, G_6, \ldots)$, the discriminant turns out to be $9a^2 + 4$. This is easily seen by using the version (4): the discriminant is $L_n^2 + 4F_n^2$ and this is $9F_n^2 + 4(-1)^n$ by the known identity $L_n^2 - 5F_n^2 = 4(-1)^n$.

The following question needs to be addressed: for even n is the minimum value of the form G_n equal to F_n (as it is for odd n)? We shall discuss this below (Section 7). Once this is established, we see that the G_{2n}'s furnish a sequence of forms with Markov ratio approaching 9 from above.

4. Discriminant $9a^2 + 4$

It is natural to seek further examples with discriminant $9a^2 + 4$. A short search conducted by hand found the first three of the H_n's listed below. It was then easy to guess a whole family. Define $u_0 = 0, u_1 = 1, u_n = 6u_{n-1} - u_{n-2}$. The u's can also be described as the y's in the solutions of the Pell equation $x^2 - 2y^2 = 1$. Then take the forms $u_n, u_{n-1}, -v_n$; here v_n is determined by the requirement that the discriminant is $9a^2 + 4$ and it turns out that $2v_n = 5u_n - 3u_{n-1}$. Here are the first six H_n's:

$$\begin{aligned}
&H_1 : 2, 0, -5\\
&H_2 : 12, 2, -27\\
&H_3 : 70, 12, -157\\
&H_4 : 408, 70, -915\\
&H_5 : 2378, 408, -5333\\
&H_6 : 13860, 2378, -31083.
\end{aligned}$$

Is u_n the minimum value assumed by H_n? As we did with G_{2n}, we postpone this to Section 7.

Do we now have all R-forms with discriminant $9a^2 + 4$? This has been verified for $b \leq 13,000,000$. The computer program written for this purpose changed each form to its equivalent Gauss-reduced form, as in [1, p. 21].

Since it is easy to see that $b \leq a/4$ after a few small exceptions, this covers a up to $a = 52,000,000$.

We repeat that, while discriminant $9a^2 - 4$ accounts for all forms with Markov ratio less than 9, those with discriminant $9a^2 + 4$ are a tiny fraction of the forms with Markov ratio larger than 9.

5. c versus $2a + b$ and $2c$ versus $5a - 3b$

For R-forms there are two inequalities for c. Each is valid with just one exception. They appear as the initial steps in the proof of Lemma 13 in [3, p. 37]. For the reader's convenience we repeat the proofs (Theorems 1 and 3) in the notation of R-forms.

The cases of equality (Theorems 2 and 4) can be pinpointed. Possibly these theorems will be useful in settling the question above concerning forms with discriminant $9a^2 + 4$.

THEOREM 1. *Let $f = a, b, -c$ be an R-form. Assume that f is not a scaling of* $1, 1, -1$. *Then $c \geq 2a + b$.*

Proof. We have $f(1, 1) = a + b - c$. Since a is the minimum of f, either $a + b - c \geq a$ or $a + b - c \leq -a$. In the first case $b \geq c$. Then $b \geq c \geq a \geq b$, so $a = b = c$. Hence the second case holds, i.e., $a + b - c \leq -a$ and $c \geq 2a + b$. □

THEOREM 2. *Let $f = a, b, -c$ be a primitive R-form satisfying $c = 2a + b$. Then f is one of the G_n's ($n \geq 2$), where, if n is a multiple of 3, we divide G_n by 2.*

Proof. It suffices to prove that $a/b = F_n/F_{n-3}$ for some $n \geq 2$. For suppose this is true. First assume n prime to 3. Then F_n and F_{n-3} are relatively prime. It follows that $a = tF_n, b = tF_{n-3}$ for some t. Then from $c = 2a + b$ we get $c = t(2F_n + F_{n-3})$. Now $L_n = 2F_n + F_{n-3}$ is a known identity. Hence $c = tL_n$. Since f is primitive, $t = 1$ and $f = G_n$. When n is divisible by 3, F_n and F_{n-3} have GCD 2. Then $a = tF_n/2, b = tF_{n-3}/2, c = tL_n/2$. Again $t = 1$ and f is G_n divided by 2.

So we may assume that a/b is not equal to any F_n/F_{n-3}. We evaluate f at $x = F_n, y = -F_{n-1}$ getting

$$aF_n^2 - bF_nF_{n-1} - cF_{n-1}^2. \tag{6}$$

Replace c by $2a + b$. Then (6) becomes

$$a(F_n^2 - 2F_{n-1}^2) - b(F_nF_{n-1} + F_{n-1}^2). \tag{7}$$

Since a is the minimum of f we have that (7) $\geq a$ or (7) $\leq -a$. The coefficient of b in (7) simplifies to $F_{n-1}F_{n+1}$. In processing the coefficient of a we use the following identities:

$$F_{n+1} - 2F_{n-1} = F_{n-2}, \tag{8}$$

$$F_{n-2}F_{n+2} - 2F_{n-1}^2 = F_{n-4}F_{n+1}, \tag{9}$$

$$\text{For } n \text{ even, } F_n^2 + 1 = F_{n-1}F_{n+1} \text{ and } F_n^2 - 1 = F_{n-2}F_{n+2}, \tag{10}$$

$$\text{For } n \text{ odd, } F_n^2 + 1 = F_{n-2}F_{n+2} \text{ and } F_n^2 - 1 = F_{n-1}F_{n+1}. \tag{11}$$

Using (8)-(11) we find, after some computation:

$$\text{For } n \text{ even, (7) } \geq a \text{ implies } aF_{n-4} \geq bF_{n-1}. \tag{12}$$

$$\text{For } n \text{ odd, (7) } \geq a \text{ implies } aF_{n-2} \geq bF_{n+1}. \tag{13}$$

$$\text{For } n \text{ even, (7) } \leq -a \text{ implies } aF_{n-2} \leq bF_{n+1}. \tag{14}$$

$$\text{For } n \text{ odd, (7) } \leq -a \text{ implies } aF_{n-4} \leq bF_{n-1}. \tag{15}$$

We make an ascent, first for odd values of n, starting with $n = 3$. Then (15) gives $a \leq b$ whence $a = b$ and we have the forbidden $a/b = F_2/F_{-1}$. Note that if (13) holds for $n = m$ then (15) cannot hold for $n = m + 2$, for we would have both inequalities and get $a/b = F_{m+1}/F_{m-2}$. Therefore (13) holds for all odd n. For even n we start with $n = 4$. Since (12) gives $b = 0$, a trivial case, we have (14) for $n = 4$. The ascent is now the same for even n as it was for odd. The upshot is that we have (13) for all odd n and (14) for all even n. In other words a/b lies above all members of the sequence

$$F_4/F_1, F_6/F_3, F_8/F_5, \ldots$$

and below all members of the sequence

$$F_5/F_2, F_7/F_4, F_9/F_6, \ldots$$

These two sequences have as common limit the cube of the golden ratio $(1 + \sqrt{5})/2$, an irrational number. But a/b is rational. This contradiction completes the proof. $\square$

Theorem 3. *Let $f = a, b, -c$ be an R-form. Assume that f is not a scaling of $1, 0, -2$. Then $2c \geq 5a - 3b$.*

Proof. We have that $f(3, -2) = 9a - 6b - 4c$ is either at least a or at most $-a$. The second alternative gives the desired conclusion, so we need only exclude the first. Thus we assume $9a - 6b - 4c \geq a$. The form $1, 1, -1$ satisfies $2c \geq 5a - 3b$ (even with equality) and so the conclusion of Theorem (1) is available. We deduce

$$8a - 6b \geq 4c \geq 4(2a + b) = 8a + 4b \tag{16}$$

and $-10b \geq 0, b \leq 0, b = 0$. Thus the two extremes of (16) are equal and there is equality throughout. Hence $8a = 4c, 2a = c$, and f is a scaling of $1, 0, -2$. □

In treating the case of equality in Theorem 3 we bring into play a second limb of the Markov tree. This limb appears on the right in the display above and consists of the triples that contain 2. Here are the usual first six, omitting 2,1,1 which does not quite fit:

5,2,1
29,5,2
169,29,2
985,169,2
5741,985,2
33461,5741,2.

We pass to the corresponding R-forms given by (3), labelling them with J's. It works nicely to start with $J_1 = 1, 1, -1$ and then label the subsequent forms with $J_2, J_3, \ldots$

$$\begin{aligned}
&J_1 : 1, 1, -1\\
&J_2 : 5, 1, -11\\
&J_3 : 29, 5, -65\\
&J_4 : 169, 29, -379\\
&J_5 : 985, 169, -2209\\
&J_6 : 5741, 985, -12875.
\end{aligned}$$

The coefficients satisfy the same recurrence as occurs in the H_n's ($t_n = 6t_{n-1} - t_{n-2}$) and the relation $2c = 5a - 3b$ also holds for the coefficients.

Theorem 4. *Let $f = a, b, -c$ be a primitive R-form satisfying $2c = 5a - 3b$. Then f is one of the H_n's or one of the J_n's .*

Proof. The proof is similar to that of Theorem 2, but a different sequence of test pairs x, y is required.

First we need some notation for $1, 5, 29, 169, \ldots$, the initial coefficients of the J_n's. We use r's: $r_1 = 1, r_2 = 5$, etc. Next we observe that if a/b is equal to some u_{n+1}/u_n or some r_{n+1}/r_n the proof is finished; the argument is essentially the same as in the proof of Theorem 2. So we assume the contrary.

The pairs to be used are the solutions of the Pell equation $x^2 - 2y^2 = 1$ and its negative version $x^2 - 2y^2 = -1$. Indeed, the u's are the y's for $x^2 - 2y^2 = 1$ (as noted above) and the r's are the y's for $x^2 - 2y^2 = -1$. For the x's we use w_n and p_n. Thus we have:

$w_n^2 - 2u_n^2 = 1$

w_n	u_n
3	2
17	12
99	70
577	408
3363	2378

$p_n^2 - 2r_n^2 = -1$

p_n	r_n
1	1
7	5
41	29
239	169
1393	985

It is routine to prove

$$2w_n + 3u_n = u_{n+1}, \qquad 2p_n + 3r_n = p_{n+1}. \tag{17}$$

We have

$$f(w_n, u_n) = aw_n^2 + bw_n u_n - cu_n^2. \tag{18}$$

In (18) replace $2c$ by $5a - 3b$, then use $w_n^2 - 2u_n^2 = 1$ and the first equation in (17):

$$2f(w_n, u_n) = a(2 - u_n^2) + bu_n u_{n+1}. \tag{19}$$

We know that (19) is either $\geq 2a$ or $\leq -2a$. In the first case we cancel $2a$ and divide by u_n, getting

$$a/b \leq u_{n+1}/u_n.$$

In the second case we use the identity

$$u_n^2 - 4 = u_{n-1}u_{n+1}$$

to get

$$a/b \geq u_n/u_{n-1}.$$

In sum:

$$a/b \qquad \geq u_n/u_{n-1} \qquad \text{or} \qquad \leq u_{n+1}/u_n. \tag{20}$$

To start an ascent we look at the bottom case:

$$\begin{aligned} f(3,2) &= 9a + 6b - 4c \\ &= 9a + 6b - 2(5a - 3b) \\ &= -a + 12b. \end{aligned}$$

If this is $\leq -a$ we get the trivial case $b = 0$. So $-a + 12b \geq a,\ a/b \leq 6$. Now (20) for $n = 2$ gives $a/b \geq 12/2 = 6$ or $a/b \leq 70/12$. But $a/b = 6$ is excluded, so $a/b \leq 70/12$. Hence $a/b <$ each u_n/u_{n-1}. The same reasoning works for the p's and r's. The identity $r_n^2 + 4 = r_{n-1}r_{n+1}$ is invoked at the proper moment. The upshot is that $a/b >$ each r_n/r_{n-1}. Now the sequences u_n/u_{n-1} and r_n/r_{n-1} converge to the same limit, and that limit is the irrational number $3+\sqrt{8}$, the larger root of $x^2 - 6x + 1 = 0$. This concludes the proof of Theorem 4. $\square$

6. Two trees of Markov type and two related families of forms

The first tree is attached to the equation $x^2 + y^2 + z^2 = 3xyz + 2$. The tree starts with the solution $1, 1, 0$ and grows to

$$
\begin{array}{cccc}
\multicolumn{4}{c}{1,1,0}\\
\hline
\multicolumn{4}{c}{3,1,1}\\
\hline
\multicolumn{4}{c}{8,3,1}\\
\hline
\multicolumn{2}{c}{21,8,1} & \multicolumn{2}{c}{71,8,3}\\
\hline
55,21,1 & 503,21,8 & \dots & \dots\\
\vdots & \vdots & &
\end{array}
$$

The procedure for passing from a triple to other triples is exactly the same as in Markov's case.

The left limb consists of triples $F_n, F_{n-2}, 1$ with n even. These have already played a role in our paper. What about the rest of the tree? We have detected a connection with binary quadratic forms only in the following triples: descending from the Fibonacci limb to the right instead of to the left. This gives the triples

71, 8,3
503, 21, 8
3464, 55, 21
23759, 144, 55
162863, 377, 144
...

A search was made for promising R-forms with these initial coefficients. The result:

71, 23, -167
503, 125, -1133
3464, 824 , -7754
23759, 5615, -53135
162863, 38453, -364181
...

The initial coefficient is, of course, $3F_nF_{n-2}-1 \quad (n \text{ even})$, starting with $n = 6$. In due course the middle coefficient was identified as $3F_nF_{n-5} - 1$. The final coefficient is determined by $2a + b - c = -2$.

Whether the initial coefficients are indeed the minima of their forms awaits investigation. This has been verified up to $n = 20$.

The second tree is similar. The equation is $x^2 + y^2 + z^2 = 3xyz + 8$, and the triples start with $2, 2, 0$.

2,2,0

12,2,2

70,12,2

408,70,2		2518,70,12	
2378,408,2	85678,408,72	...	...
$\vdots$	$\vdots$		

The left limb is familiar, needless to say. The same procedure was applied and yielded

2518, 420, -5666
85678, 14688, -192164
2910670, 499380, -6527606
98877238, 169646640, -221746136
3358915558, 576298788, -7532840714
...

The first coefficient is $3u_n u_{n-1} - 2$ and the middle coefficient is $3u_n u_{n-2}$. The final coefficient is determined by $5a - 3b - 2c = -2$.

Again the question as to whether the initial coefficients are minima for their forms is left to the future. This was checked for the five forms listed above.

7. Minima

For the forms G_{2n} and H_n above we have proved that the first coefficient is indeed the minimum. The details will appear in a second paper that we have prepared, not intended at present for publication (copies are available on request). Here we present a sketch.

With a form $a, b, -c$ we associate the quadratic equation $as^2 + bs - c = 0$. It has two roots: a positive one and a negative one. It is crucial to find the continued fraction expansion of the positive root. This was done as follows. A program was written, based on the "mqa" method presented in [10, p. 358]. This was used to compute several special cases. Repeatedly a pattern was detected. Then the correctness of the expansions thus guessed was verified. The expansions are as follows:

$$G_{2n} : [\ 1, 2, \overline{\underbrace{1, 1, \ldots, 1, 1}_{2n-3}, 2, 2}\ ],$$

$$H_n : [\ 1, 2, \overline{\underbrace{2, 2, \ldots, 2, 2}_{2n-3}, 1, 1, 2, 2}\ ].$$

With the continued fraction expansions at hand it is a standard matter to identify the minima of the forms.

8. Miscellaneous

8.1. Opposites and negatives. A form is ambiguous if it is equivalent to its opposite (i.e., if it admits an automorphism of determinant -1). In studying a form one usually wishes to know whether it is ambiguous and also whether it is equivalent to its negative. As regards the G_{2n}'s and the H_n's the answer is affirmative on both counts. The proof appears in the supplementary document.

The G's with odd subscripts are of course a special case of Markov's forms. It is standard that all of these are equivalent to their negatives. As regards ambiguity the answer is no, except for the first two forms (proof in the supplement). This result appears to be new.

8.2. A family based on the Fibonacci numbers. The family in question is the set of all $F_n, 0, -F_{n+2}$. That the minimum is F_n is proved by the method sketched in Section 7. It seems to be worth while to exhibit this specially simple family.

These forms are visibly ambiguous. On the other hand $F_n, 0, -F_{n+2}$ is equivalent to $-F_n, 0, F_{n+2}$ if and only if n is odd.

8.3. Other families. In various ways our investigation led to the discovery of numerous other infinite families of probable R-forms—about thirty of them. They are displayed, with comments, in the supplement.

Recall Markov's forms written as R-forms (3):

$$m, m-2k, -(2m+k-l).$$

We were able to prove that the following are R-forms:

$$m^2, m(m-2k), -2-m(2m+k-l).$$

The discriminant for the form beginning with m^2 is $D = 9m^4 + 4m^2$. The first few are $(1,1,-3), (4,0,-10), (25,5,-57), (169,39,-379), (841,145,-1887)$. The proof resembles [4, Theorem 2(ABC), pp. 20-22].

8.4. Two tables. We ran a program that compiled all primitive R-forms that have discriminant ≤ 10176245 and Markov ratio above 9 and ≤ 13. There are 750 of them. They are exhibited in two tables: by Markov ratio and by discriminant.

The supplement contains a number of observations and speculations suggested by the tables—details are available from the authors.

8.5. Equivalent R-forms. It is possible for two different R-forms to be equivalent. In the tables just mentioned there is only one such pair:

$$375, 65, -841 \qquad \text{and} \qquad 375, 85, -839.$$

But we have several examples with larger discriminants and/or Markov ratios.

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