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THE FINITE BASIS QUESTION FOR VARIETIES OF GROUPS—SOME RECENT RESULTS

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Abstract. We survey some recent results related to the finite basis question for varieties of groups.

1. Introduction

Let F be a free group on free generators $x_1, x_2, \ldots$. Let $v = v(x_1, \ldots, x_n)$ be a word (that is, $v \in F$) and let G be a group. We say that $v = 1$ is an *identity* (or a *law* or an *identical relation*) in G if $v(g_1, \ldots, g_n) = 1$ for all $g_1, \ldots, g_n \in G$. For instance, a group G satisfies the identity $[x_1, x_2] = 1$ if and only if it is abelian. The class of all groups satisfying a given set of identities is called a *variety* of groups. For example, the class $\mathbf{B}_n$ of all groups with exponent dividing n, the class $\mathbf{A}$ of all abelian groups and the class $\mathbf{N}_2$ of all nilpotent groups of class at most 2 are varieties which are defined by single identities $x_1^n = 1$, $[x_1, x_2] = 1$ and $[[x_1, x_2], x_3] = 1$, respectively.

The *finite basis question* for a variety can be stated as follows: *Can a given variety be defined by a finite set of identities?* If the answer is 'yes', we say that the variety is *finitely based* or *has a finite basis* for its identities. We refer to Hanna Neumann [34], Bakhturin and Olshanskii [2] and Olshanskii and Shmelkin [38] for further terminology and basic results related to varieties of groups.

In 1937 B. H. Neumann [33] asked whether *every* variety of groups is finitely based, or, equivalently, whether the answer to the finite basis question is affirmative for every variety. In 1970 Adian [1], Olshanskii [37] and Vaughan-Lee [46] answered Neumann's question in the negative showing that there are varieties of groups which are not finitely based. In the next few years further examples of systems of identities which define non-finitely based varieties were found. In particular, Bryant [6] and Kleiman [24] independently proved that the variety defined by the identities

$$x_1^8 = 1, (x_1^2 x_2^2)^4 = 1, \ldots, (x_1^2 x_2^2 \ldots x_k^2)^4 = 1, \ldots$$

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is not finitely based. This is probably the simplest example of such a variety.

However, for many varieties of groups the answer to the finite basis question is known to be affirmative. Oates and Powell [36] proved that each variety generated by a finite group is finitely based. Lyndon [31] showed that every nilpotent variety (that is, a variety which consists of nilpotent groups) is finitely based and Cohen [12] proved the same for metabelian varieties. McKay [32] proved the finite basis property for centre-by-metabelian varieties. Her result was generalized by Bryant and Newman [7] to varieties whose groups are extensions of a nilpotent group of class at most 2 by an abelian group and then by Krasilnikov [28] to arbitrary nilpotent-by-abelian varieties.

In the present article we survey some results related to the finite basis question which were published during the last decade. See [26], [30], [23] for references to earlier publications.

2. Centre-by-abelian-by-nilpotent varieties

The examples of non-finitely based varieties of groups constructed in [46], [6] and [24] satisfy the identity

$$[[x_1, x_2, x_3], [x_4, x_5, x_6], [x_7, x_8]] = 1. \tag{1}$$

N. D. Gupta and Levin (unpublished) observed that the construction in [46] in fact yields a non-finitely based variety which satisfies a stronger identity

$$[[x_1, x_2, x_3], [x_4, x_5, x_6], x_7, x_8] = 1.$$

However, it was unknown for some years if there exists a non-finitely based variety of groups which satisfies the identity

$$[[x_1, x_2, x_3], [x_4, x_5, x_6], x_7] = 1. \tag{2}$$

A group satisfying (2) is centre-by-abelian-by-(nilpotent of class at most 2), that is, its quotient group over the centre is an extension of an abelian group by a nilpotent group of class at most 2. The first example of a non-finitely based variety which satisfies (2) was constructed in [16].

For each $k \geq 1$, let

$$u_k = [x^8, [x_1, x_2], \dots, [x_{4k-3}, x_{4k-2}], x^8].$$

THEOREM 1 ([16]). *The variety of groups defined by the identity* (2) *and the identities* $u_k = 1$ $(k = 1, 2, \dots)$ *is not finitely based.*

In fact, it can be seen that the proof of Theorem 1 gives a stronger result: the variety defined by (2), the identity $x^{128} = 1$ and the identities $u_k = 1$ $(k = 1, 2, \dots)$ is non-finitely based. Later in [8] the following theorem was proved.

THEOREM 2 ([8]). *The variety of groups defined by* (2) *and the identity* $x^{32} = 1$ *has a non-finitely based subvariety.*

But Theorem 2, in contrast to Theorem 1, is an existence statement: no system of identities which defines that non-finitely based subvariety was explicitly found in [8].

Most recently in [29] a non-finitely based variety of groups which satisfy (2) and the identity $x^8 = 1$ was constructed. Let

$$v_k = [[x_1, x_2, x_3], x_1, x_1^2, x_4^2, \dots x_k^2, [x_1^{-1}, x_2^{-1}, x_3^{-1}]] \qquad (k = 3, 4, \dots).$$

THEOREM 3 ([29]). *The variety of groups defined by* (2)*, the identity* $x^8 = 1$ *and the identities* $v_k = 1$ $(k = 3, 4, \dots)$ *is not finitely based.*

Note that it is impossible to improve any further the exponent of a non-finitely based variety satisfying (2) because every soluble variety of groups of exponent 4 is nilpotent-by-abelian (Tobin [45]), and so finitely based [28].

3. Limit varieties

The results described in the previous section were motivated in particular by the problem of constructing of a *limit* variety of groups. It follows easily from Zorn's lemma that if a variety $\mathbf{V}$ is not finitely based then it contains a subvariety $\mathbf{V}^*$ such that all proper subvarieties of $\mathbf{V}^*$ are finitely based, but $\mathbf{V}^*$ itself is not. Any variety with these properties is called a *limit* or a *just non-finitely based* variety. In this sense limit varieties form a "border" between those which are finitely based and those which are not. It is known that infinitely many such varieties exist (Newman [35]) although no explicit examples are known.

Let $\mathbf{AN}_2$ be the variety defined by the identity

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 1,$$

that is, the variety of all groups which are extensions of abelian groups by nilpotent groups of class at most 2. In [18] a non-finitely based variety $\mathbf{W}$ satisfying the identity (1) was constructed whose intersection with $\mathbf{AN}_2$ has no non-finitely based subvarieties. The variety $\mathbf{W}$ comes closest to being limit.

Let $a^{(x,y)} = a^{xy}a^{-yx}$ and $a^{(x,y,z)} = a^{xyz}a^{-yxz}a^{-zxy}a^{zyx}$. Let $a^{u_1 \dots u_{k-1} u_k} = (a^{u_1 \dots u_{k-1}})^{u_k}$ for all $k > 1$. Let

$$w_k = [[x_1, x_2, x_3]^{[x_4, x_5] \dots [x_{2k+2}, x_{2k+3}]}, [x_1, x_2, x_3]] \qquad (k \in \mathbb{N}).$$

THEOREM 4 ([18]). *Let* $\mathbf{W}$ *be the variety of groups defined by the identity* (1)*, the identity*

$$[x_1, x_2, x_3]^{(x_4, x_5, x_6)(x_7, x_8, x_9)(x_{10}, x_{11})(x_{12}, x_{13})} = 1$$

and the identities $w_k = 1$ $(k \in \mathbb{N})$*. Then the variety* $\mathbf{W}$ *is non-finitely based but all the subvarieties of* $\mathbf{W} \cap \mathbf{AN}_2$ *are finitely based.*

Recently in [19] the first example of a limit variety of bigroups was constructed. *A bigroup* is a pair (H, π) consisting of a group H and an idempotent endomorphism (projection) π of H. One can consider π as a unary operation on H so a bigroup is a universal algebra. Clearly an idempotent endomorphism π of a group H determines, and is determined by, a splitting of H: $G = \pi(H)$ is a complement in H for $N = \ker \pi$, so $H = N \leftthreetimes G$. Thus, bigroups can be described by triples $\mathcal{H} = (H, N, G)$, where H is a group, N a normal subgroup of H and G a complement for N in H.

Bigroups were introduced by Bryce [10], [11]. He considered varieties of bigroups in order to classify some varieties of metabelian groups. Earlier Plotkin investigated varieties of integral group representations which are particular cases of varieties of bigroups (see Plotkin and Vovsi [39] or Vovsi [47]). There is a close similarity between varieties of bigroups and varieties of groups. In fact, precisely because of this similarity bigroups were introduced and investigated.

Let Z be the free group on the free generating set $\{x_i \mid i \in I\} \cup \{y_i \mid i \in I\}$. Let X be the subgroup of Z (freely) generated by $\{x_i \mid i \in I\}$ and let Y be the normal subgroup of Z generated by $\{y_i \mid i \in I\}$. Then it is straightforward to check that $\mathcal{Z} = (Z, Y, X)$ is the *free bigroup* on the free generators $z_i = x_i y_i$ $(i \in I)$.

Let $u = u(x_1, \ldots, x_n; y_1, \ldots, y_n) \in Z$. We say that a bigroup $\mathcal{H} = (H, N, G)$ satisfies the *identity* $u = 1$ if for every $h_1, \ldots, h_n \in H$ and for every (bigroup) homomorphism $\theta : \mathcal{Z} \to \mathcal{H}$ such that $\theta(z_i) = h_i$ $(i = 1, \ldots, n)$ we have $\theta(u) = 1$. Equivalently, $\mathcal{H} = (H, N, G)$ satisfies the identity $u = 1$ if for every $d_1, \ldots, d_n \in N$ and $g_1, \ldots, g_n \in G$ we have $u(g_1, \ldots, g_n; d_1, \ldots, d_n) = 1$.

Let $f_k = [y^{x_1^2 \cdots x_k^2}, y]$ for all $k \in \mathbb{N}$. Let $x = x_1$, $y = y_1$.

Theorem 5 ([19]). *Let $\mathbb{V}$ be the variety of bigroups defined by the identities*

$$[x_1, x_2, x_3] = 1, \qquad x^4 = 1, \tag{3}$$

$$[y_1, y_2, y_3] = 1, \qquad y^4 = 1, \tag{4}$$

$$[y^2, x] = 1, \tag{5}$$

$$[y_1^{[x_1,x_2]} y_1^{x_1^2 x_2^2} y_1^{-x_1^2} y_1^{-x_2^2}, y_2] = 1, \tag{6}$$

$$[y^{[x_1,x_2]} y^{x_1^2 x_2^2} y^{-x_1^2} y^{-x_2^2}, x_3] = 1, \tag{7}$$

$$\{f_k = 1 \mid k \in \mathbb{N}\}. \tag{8}$$

Then $\mathbb{V}$ is a limit variety.

Theorem 5 was obtained as a consequence of the following propositions.

Proposition 1. *The variety $\mathbb{V}$ is not finitely based.*

PROPOSITION 2. *Let $u = 1$ be an identity which is not a consequence of the identities* (3)–(8). *Then the system consisting of $u = 1$ and the identities* (3)–(8) *is finitely based.*

PROPOSITION 3. *The variety $\mathbb{V}$ satisfies the minimal condition on subvarieties (equivalently, every subvariety of $\mathbb{V}$ can be defined by the identities of $\mathbb{V}$ together with finitely many additional identities).*

We hope that Theorem 5 will be helpful in constructing a limit variety of groups (a problem which remains open). Let $\mathbf{V}$ be the variety of groups defined by

$$\mathbf{V} = \text{ var } \{H \mid \mathcal{H} = (H, \pi) \in \mathbb{V} \text{ for some projection } \pi : H \to H\}.$$

In other words, the variety of groups $\mathbf{V}$ is generated by all the bigroups $\mathcal{H} \in \mathbb{V}$ if we consider them as groups and "forget" about the additional operation π on $\mathcal{H}$.

PROBLEM 1. Is $\mathbf{V}$ a non-finitely based variety? Does it satisfy the minimal condition on subvarieties? Is $\mathbf{V}$ a limit variety of groups?

4. Varieties of prime power exponent

Let $\mathbf{B}_n$ be the variety of all groups of exponent dividing n.

4.1. The finite basis question for subvarieties of $\mathbf{B}_{p^3}$. It has been known since the beginning of nineteen-seventies that, for each prime p, the variety $\mathbf{B}_{p^3}$ contains a non-finitely based solvable subvariety. Such a subvariety can be defined, for example, by the identities

$$x_1^{p^3} = 1, (x_1^p x_2^p)^{p^2} = 1, \ldots, (x_1^p x_2^p \ldots x_k^p)^{p^2} = 1, \ldots$$

together with the identity

$$[[[x_1, x_2], x_3], [[x_4, x_5], x_6], \ldots, [[x_{3p+1}, x_{3p+2}], x_{3p+3}] = 1. \tag{9}$$

This has been proved by Bryant [6] and Kleiman [24] for $p = 2$ and by Kleiman [25] for $p > 2$. Another non-finitely based solvable subvariety of $\mathbf{B}_{p^3}$ can be defined (as a subvariety of $\mathbf{B}_{p^3}$) by the identities

$$[x_1, x_2]^{p^2} = 1, ([x_1, x_2][x_3, x_4])^{p^2} = 1, \ldots, ([x_1, x_2] \ldots [x_{2k-1}, x_{2k}])^{p^2} = 1, \ldots$$

together with the identity (9) (Kleiman [25]). Earlier more complicated examples of non-finitely based solvable subvarieties of $\mathbf{B}_{16}$ and $\mathbf{B}_{p^3}$ for each prime $p > 2$ have been constructed by Vaughan-Lee [46] and Newman [35], respectively.

4.2. Subvarieties of $\mathbf{B}_{p^2}$. For each "sufficiently large" prime p ($p > 10^{10}$) the subvariety of $\mathbf{B}_{p^2}$ defined by the identities

(10)
$$[x_1, x_2]^p = 1, ([x_1, x_2][x_3, x_4])^p = 1, \dots, ([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p = 1, \dots$$

is not finitely based. This has been proved by Olshanskii [37, Theorem 31.6]. However, to the best of our knowledge, the following problem is still open.

Problem 2. Is it true that for a large prime p the variety defined by the identities

$$x_1^{p^2} = 1, (x_1^p x_2^p)^p = 1, \dots, (x_1^p x_2^p \dots x_k^p)^p = 1, \dots \tag{11}$$

is not finitely based?

Further, the systems of identities (10) and (11) cannot be used to obtain solvable non-finitely based subvarieties of $\mathbf{B}_{p^2}$. Since every solvable variety of prime exponent p is nilpotent, it follows that every solvable variety satisfying the identities (10) is nilpotent-by-abelian and so finitely based [28]. It follows also from the same observation and a result of Higman [34, Theorem 34.23] that the variety defined by the identities (11) and the identity of solvability of some length is finitely based.

The systems (10) and (11) also define finitely based subvarieties of $\mathbf{B}_{p^2}$ for $p = 2, 3$. This can be easily checked using [34, Theorem 34.23 and Theorem 34.32].

However, the variety $\mathbf{B}_{p^2}$ does contain, for each prime $p > 2$, a solvable non-finitely based subvariety. Let $u_k = [x_4, x_5] \dots [x_{2k+2}, x_{2k+3}]$ and let

$$w_k = [[[x_1, x_2], x_3], [[x_1, x_2], x_3]^{u_k}, \dots, [[x_1, x_2], x_3]^{u_k^{p-1}}] \qquad (k \in \mathbb{N}).$$

Theorem 6 ([17]). *For each prime $p > 2$, the subvariety of $\mathbf{B}_{p^2}$ defined by the identities $w_k = 1$ ($k = 1, 2, \dots$) together with the identity (9) is not finitely based.*

For the subvariety of $\mathbf{B}_{p^3}$ defined by the identities $w_k = 1$ ($k = 1, 2, \dots$) together with the identity (9) a statement similar to Theorem 6 was proved earlier by Newman [35].

Note that, as we have mentioned before, every solvable variety of exponent 4 is nilpotent-by-abelian (Tobin [45]) and so finitely based [28]. On the other hand, it is still an open problem whether every subvariety of $\mathbf{B}_4$ is finitely based. See Quick [40], [41], [42] for partial results (and much more).

4.3. Subvarieties of $\mathbf{B}_p$. Clearly, we may assume $p > 3$ because the variety $\mathbf{B}_2$ does not contain proper subvarieties and the variety $\mathbf{B}_3$ is nilpotent of class 3 [34, Theorem 34.32] and its subvarieties are easy to describe. We can consider only non-solvable subvarieties of $\mathbf{B}_p$ because every solvable variety of prime exponent p is nilpotent and so finitely based.

Let $\mathbf{R}_n$ be the class of all locally finite groups of exponent dividing n. It follows from the positive solution of the Restricted Burnside Problem that $\mathbf{R}_n$ is a variety. Endimioni [13] proved that the variety $\mathbf{R}_5$ is finitely based and Zelmanov [48] proved the same for arbitrary prime $p > 3$.

Problem 3. Let p be a prime, $p > 3$. Does the variety $\mathbf{R}_p$ contain a non-finitely based subvariety?

Problem 4. Is the variety $\mathbf{R}_n$ finitely based if n is not a prime? Is $\mathbf{R}_n$ finitely based if $n = p^k$ for some prime p and some $k > 1$?

Kozhevnikov [27] proved that, for any 'sufficiently large' prime p, there are subvarieties of $\mathbf{B}_p$ which are not finitely based. These subvarieties are not contained in $\mathbf{R}_p$.

5. Miscellaneous results

A group G is called *nilpotent-by-nilpotent* or *metanilpotent* if it has a nilpotent normal subgroup N such that the quotient group G/N is also nilpotent. A variety of groups is metanilpotent if all its groups are. Many of the solvable non-finitely based varieties mentioned above are in fact metanilpotent. All these varieties have torsion in their relatively free groups and this fact was essentially used in the corresponding papers to prove that the varieties under consideration are not finitely based.

It can easily be proved that there are varieties of groups which are not finitely based and whose relatively free groups are torsion free. Indeed, let $\mathbf{A}$ be the variety of all abelian groups and let $\mathbf{V}$ be any non-finitely based variety. Then it is well-known that the product variety $\mathbf{AV}$ is not finitely based and its relatively free groups are torsion free.

However, one cannot obtain in this way any metanilpotent variety with the above properties. This led Kovács and Newman to ask ([26, Question 3]) whether there is a non-finitely based variety of metanilpotent groups whose relatively free groups are torsion free. The following theorem gives an answer to their question.

Theorem 7 ([17]). *There exists a non-finitely based variety* $\mathbf{V}$ *of (nilpotent of class* 2*)-by-(nilpotent of class* 2*) groups whose relatively free groups are torsion free.*

Theorem 7 was proved by constructing a free group of countably infinite rank of the variety $\mathbf{V}$. It is not known which identities define $\mathbf{V}$.

Problem 5. Find a system of identities which define the variety $\mathbf{V}$ mentioned in Theorem 7 or another variety with the same properties.

PROBLEM 6. Is the variety $\mathbf{V}$ constructed in Theorem 7 minimal with respect to the properties under consideration? Is there a non-finitely based centre-by-abelian-by-nilpotent variety with relatively free groups without torsion? Is there an abelian-by-nilpotent variety with such properties?

In the examples of non-finitely based metanilpotent varieties mentioned above there is no bound on the class of nilpotent groups contained in the variety. It seems to be an open problem whether a solvable variety, in which the nilpotent groups have bounded class, has a finite basis. In many simple cases the theorem below gives a positive answer to the finite basis question.

Let, for any positive integer n, $\mathbf{N}_{2,n}$ denote the variety of all groups which are nilpotent of class at most 2 and have exponent dividing n.

THEOREM 8 ([9]). *Let $\mathbf{V}$ be any subvariety of $\mathbf{N}_{2,m}\mathbf{N}_{2,n}$, where m and n are coprime positive integers. Then the variety $\mathbf{V}$ is finitely based.*

For subvarieties of the variety $\mathbf{A}_m\mathbf{N}_{2,n}$ with m, n coprime, Theorem 8 was proved earlier by Brady, Bryce and Cossey [5]. Here $\mathbf{A}_m$ denotes the variety of all abelian groups of exponent m.

The techniques developed in order to construct non-finitely based varieties of groups have been used in [20] and [21] to improve some results about varieties of associative algebras. Every variety of associative algebras over a field of characteristic 0 is finitely based: this is a celebrated result of Kemer [22]. On the other hand, over a field of a prime characteristic $p > 0$ there are non-finitely based varieties of associative algebras: this has been proved recently by Belov [3], Grishin [14] (for $p = 2$) and Shchigolev [43] (see also [4], [15], [44]). The systems of identities which define non-finitely based varieties in [3], [14] and [43] are relatively complicated. The following theorem gives a simpler example. Let $(x, y) = xy - yx$.

THEOREM 9 ([21]). *For any field $\mathbb{F}$ of characteristic 2 the identities*

$$\{(x, y^2)x_1^2x_2^2 \dots x_n^2(x, y^2)^3 = 0 \mid n = 0, 1, 2, \dots\}$$

define a non-finitely based variety of associative $\mathbb{F}$-algebras.

In [14], [15] Grishin constructed a non-finitely based variety of associative algebras (without 1) over a field of characteristic 2 which satisfies the identity $x^{32} = 0$. The following theorem improves his result. Let

$$w_n = ((y_1, z_1), t_1)x_1^2x_2^2 \dots x_n^2((y_2, z_2), t_2)((y_1, z_1), t_1)((y_2, z_2), t_2).$$

THEOREM 10 ([20]). *For any field $\mathbb{F}$ of characteristic 2 the identity $x^6 = 0$ together with the identities*

$$\{w_n = 0 \mid n = 0, 1, 2, \dots\}$$

define a non-finitely based variety of associative $\mathbb{F}$-algebras.

In the proofs of Theorems 9 and 10 the ideas of Belov [3] and Grishin [14] are used (and developed). On the other hand, most of the argument in those proofs is different from [3], [14], [43] and similar to one used in group theory (see Vaughan-Lee [46], Newman [35], Bryant [6], Kleiman [24], C. K. Gupta and Krasilnikov [16]).

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